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Tangle decompositions of satellite knots.

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Abstract

We study when an essential tangle decomposition of a satellite knot gives an essential tangle decomposition of the companion knot, that is, when the decomposing sphere can be isotoped to intersect the knotted solid torus identified with the pattern in meridian disks.

1 Introduction

In this paper, we study on the next question.

Question. *Let K be a satellite knot contained in a companion solid torus V, and S be a sphere which gives an essential tangle decomposition of K. Suppose there is no essential tangle in (V, K). Does S give also an essential tangle decomposition of its companion knot ?*

The second author showed that Question is true for Whitehead double in [M].

In Sect. 2, we show that Question is true if $|S \cap K|$ is minimum. In Sect. 3, we consider the case where the wrapping number of K in V is 2, and give counterexamples to the Question. In Sect. 4, we show that Question is true when the pattern is a braided link.

More precisely, let V_0 be a solid torus embedded in the 3-sphere S^3 and K_0 a disjoint union of simple closed curves in int V_0. We say that K_0 is *essential* in V_0 and that the pair (V_0, K_0) is essential if K_0 is not ambient isotopic to the core of V_0 and $V_0 - K_0$ is irreducible (i.e., every 2-sphere in $V_0 - K_0$ bounds a ball in $V_0 - K_0$). Let V_1 be a tubular

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neighbourhood of a non-trivial knot K_1 in S^3 and $h: V_0 \to V_1$ a homeomorphism. Then $K = h(K_0)$ is called a *satellite link* if (V_0, K_0) is essential. The knot K_1 is called the *companion knot* of the satellite link K, and (V_0, K_0) the *pattern* of the satellite link K.

In general, let M be a 3-manifold, γ a 1-manifold properly embedded in M, and F a surface properly embedded in M transversely to γ. We say that F is *γ-compressible* if there is a disk D embedded in $M - \gamma$ such that $D \cap F = \partial D$ and ∂D does not bound a disk in $F - \gamma$. Otherwise, F is *γ-incompressible*.

Let B be a 3-ball and T a 1-manifold properly embedded in B. Then the pair (B, T) is called a *tangle*. If T is a disjoint union of n arcs $t_1, \cdots, t_n$, then the arcs are called *strings*, and the pair (B, T) an *n-string tangle*. An n-string tangle (B, T) with $T = t_1 \cup \cdots \cup t_n$ is *trivial* if there is a system of disjoint disks $D_1, \cdots, D_n$ embedded in B such that $D_i \cap T = \partial D_i \cap t_i = t_i$ and $\partial D_i - t_i \subset \partial B$ for $i = 1, \cdots, n$. A tangle (B, T) is *essential* if $B - T$ is irreducible, if ∂B is T-incompressible in (B, T) and if (B, T) is not a trivial 1-string tangle.

Let M be a 3-manifold, γ a 1-manifold properly embedded in M, and (B, T) a tangle such that $B \subset \operatorname{int} M$ and $T = B \cap \gamma$. Then (B, T) is *essential in* (M, γ) if ∂B is γ-incompressible in (M, γ) and (B, T) is an essential tangle.

A link K in S^3 admits an *essential tangle decomposition* if $(S^3, K) = (B_1, T_1) \cup (B_2, T_2)$, where $B_1 \cap B_2 = \partial B_1 = \partial B_2$ is a 2-sphere and (B_i, T_i) is an essential tangle for $i = 1$ and 2. A 2-sphere S in S^3 gives an *essential tangle decomposition of a knotted solid torus* V in S^3 if $S \cap V$ are meridian disk of V and S gives an essential tangle decomposition of the knot formed by the core of V.

Theorem 1.1. *Let K be a satellite link in S^3 with a pattern (V, K). Suppose there is no essential tangle in (V, K) and K admits an essential tangle decomposition. Then the decomposing 2-sphere S can be isotoped in (S^3, K) so that it gives an essential tangle decomposition of V if $|S \cap K|$ is minimum over all essential tangle decompositions of K.*

A *marked rational tangle* is a triple (B, T, C), where (B, T) is a trivial 2-string tangle with B oriented and C is a simple closed curve in $\partial B - \partial T$ separating the four points ∂T into two pairs. We assign a rational number or $\infty = 1/0$ to every marked rational tangle (B, T, C) as follows. The strings T is isotopic relative to ∂T to a union of two arcs, say α, on ∂B. Let F be a torus which is a double branch cover of ∂B with branch set ∂T. Cutting ∂B along α, we obtain an annulus. The torus F is obtained by gluing two copies of the annulus. Let $M \subset F$ be a component of the preimage of α. Note that M is a simple closed curve in the torus F. Let L be the circle component of the preimage of an arbitrary arc β in ∂B such that $\beta \cap \alpha = \partial\beta$ are two points in distinct components of α. Note that L is also a simple closed curve in F such that L intersects M transversely at a single point. Orient M and L so that the intersection number of $M \cdot L = 1$ with respect to the orientation of F induced from the orientation of ∂B. Then the preimage of C represents some element $q[L] + p[M]$ in $H_1(F)$, where p and q are coprime integers. We say that (B, T, C) is a *marked rational tangle of slope* p/q, and use $R(p/q)$ to denote it. Because of the ambiguity of the choice of β, $R(r) = R(r')$ if and only if $r \equiv r' \bmod \mathbf{Z}$.

Let V be a solid torus, and K a link in V. The pair (V, K) is a *rational pattern with slope* p/q if there is a meridian disk D of V such that $|D \cap K| = 2$ and it cuts (V, K) into a marked rational tangle (B, T, C) of slope p/q, where ∂D is isotopic to C in $\partial B - \partial T$. Whitehead's pattern is of slope $\pm 1/2$. Since the rational pattern of slope $1/0$ is the pair of a solid torus and an inessential loop in it, we assume that $p/q \neq 1/0$ throughout this paper.

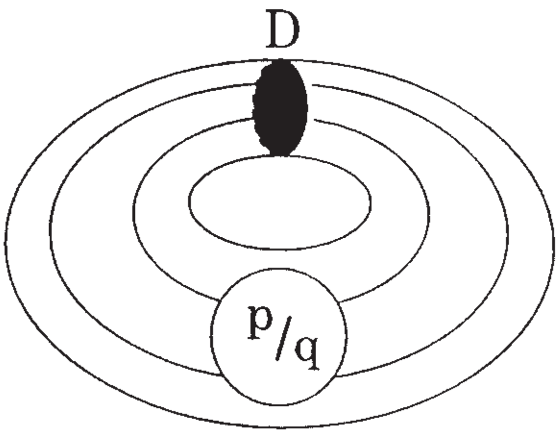

Figure 1: Rational pattern with slope p/q

As we will see later in Lemma 3.1, a pattern with wrapping number 2 contains an essential tangle if it is not rational. Hence we concentrate on rational patterns.

In general, let M be a 3-manifold, γ a 1-manifold properly embedded in M, and F a γ-incompressible surface in (M, γ). We say that F is *meridionally compressible* in (M, γ) if there is an embedded disk D transverse to γ in M such that $|D \cap \gamma| = |(\text{int}\, D) \cap \gamma| = 1$, $D \cap F = \partial D$ and ∂D does not bound a disk intersecting K at a single point in F. Otherwise, F is *meridionally incompressible* in (M, γ).

Theorem 1.2. *Let S be a 2-sphere which gives an essential tangle decomposition of a satellite link K with a rational pattern (V, K) of slope p/q. Then S can be isotoped to give an essential tangle decomposition of V if and only if S is meridionally incompressible in (S^3, K).*

Theorem 1.3. *Let K be a satellite link with a rational pattern of slope p/q. Then there is a meridionally compressible 2-sphere which gives an essential tangle decomposition of K if and only if $|q|$ is an odd integer greater than 1 and the companion knot admits an essential tangle decomposition.*

Let K be a satellite link in S^3 with a pattern (V, K). The pattern (V, K) is called an *m-braided link* if we can take a coordinate $V \cong D^2 \times S^1$ so that $D^2 \times \{p\}$ intersects K transversely in m points for all $p \in S^1$.

Theorem 1.4. *Let $K \subset S^3$ be a satellite link with an m-braided link pattern (V, K). Let S be a 2-sphere which gives an essential k-string tangle decomposition of K. Then S can be isotoped to give an essential (k/m)-string tangle decomposition of V.*

2 Proof of Theorem 1.1

In general, let M be a 3-manifold, γ a 1-manifold properly embedded in M, and F a surface properly embedded in M transversely to γ. We say F is *γ-boundary compressible* if there is a disk D embedded in $M - \gamma$ such that $D \cap F = \partial D \cap F = \alpha$ is an arc, $\partial D - \alpha \subset \partial M$ and α does not cut off a disk from $F - \gamma$. Otherwise, F is γ-boundary incompressible.

Lemma 2.1. *Let M be a 3-manifold, and γ a 1-manifold properly embedded in M. Let H be a γ-incompressible closed 2-manifold in (M, γ). Let M' be the 3-manifold obtained by cutting M along H, and $\gamma' = \gamma \cap M'$. Suppose that $M' - \gamma'$ is irreducible. Let F be a*

γ-incompressible 2-manifold in (M, γ) such that F intersects H transversely in minimal number of loops disjoint from γ up to isotopy of F in (M, γ). Then no loop of $F \cap H$ bounds a disk on $H - \gamma$, $M - \gamma$ is irreducible, and $F' = F \cap M'$ is γ'-incompressible in (M', γ'). Moreover, if F is orientable and γ-boundary incompressible, and if H is a disjoint union of tori disjoint from γ, then F' is γ'-boundary incompressible in (M', γ').

Proof. Suppose for a contradiction that there is a loop of $F \cap H$ bounding a disk D in $H - \gamma$. Since F is γ-incompressible in (M, γ), ∂D bounds a disk D_1 in $F - \gamma$. Let D_2 be an innermost disk bounded by an innermost loop of $F \cap H$ in D_1. That is, $D_2 \cap H = \partial D_2$. Since H is γ-incompressible, ∂D_2 bounds a disk D_3 in $H - \gamma$. Then the disks D_2 and D_3 cobound a ball in the irreducible manifold $M' - \gamma'$, and we isotope the disk D_2 along this ball onto D_3. After an adequate small isotopy of F, we obtain a contradiction to the minimality of $|F \cap H|$. Hence no component of $F \cap H$ bounds a disk in $H - \gamma$.

Suppose for a contradiction that $M - \gamma$ is reducible. Then there is a 2-sphere S which does not bound a ball in $M - \gamma$. We take S so that S intersects H transversely and so that $|S \cap H|$ is minimal up to isotopy of S in (M, γ). Since $M' - \gamma'$ is irreducible, $S \cap H \neq \emptyset$. Then there is an innermost loop of $S \cap H$ in S, and let R be the innermost disk. Since H is γ-incompressible, ∂R bounds a disk in $H - \gamma$, which contradicts the conclusion of the first paragraph of this proof.

For the proof of γ'-incompressibility of F', let P be an arbitrary disk in $M' - \gamma'$ such that $P \cap F' = \partial P$. Since F is γ-incompressible in (M, γ), ∂P bounds a disk P_1 in $F - \gamma$. If P_1 is disjoint from H, then we are done. If P_1 intersects H, there is an innermost loop of $P_1 \cap H$ bounding an innermost disk P_2 on P_1. Since H is γ-incompressible, ∂P_2 bounds a disk on $H - \gamma$, which contradicts the conclusion of the first paragraph of this proof. Hence F' is γ'-incompressible in (M', γ').

Suppose that F is orientable and γ-boundary incompressible, and that H is a disjoint union of tori disjoint from γ. For the proof of γ'-boundary incompressibility of F', let Q be a disk in $M' - \gamma'$ such that $Q \cap F = \partial Q \cap F = \alpha$ is an arc and $\beta = \partial Q - \alpha \subset \partial M'$. If $\beta \subset \partial M$, then α cuts off a disk Q_1 from $F - \gamma$ since F is γ-boundary incompressible. A standard innermost loop argument as in the above paragraphs shows that Q_1 is disjoint from H, and we are done. Hence we can assume that

$\beta \subset H$. If β connects two components of $F \cap H$, then we isotope a band neighbourhood of α on F along the disk Q slightly beyond β, to reduce the number $|F \cap H|$, which is a contradiction. Hence β has both endpoints in the same component of $F \cap H$. Since F is orientable and H is a disjoint union of tori disjoint from γ, β is isotopic into $F \cap H$ fixing its endpoints in H. This isotopy extends to that of Q, which implies that α cuts off a disk from $F' - \gamma'$ by the conclusion of the third paragraph of this proof. Hence F' is γ'-boundary incompressible in (M', γ').

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Lemma 2.2. *Let M be a 3-manifold, and γ a 1-manifold properly embedded in M. Let H be a γ-incompressible closed 2-manifold in (M, γ). Let M' be the 3-manifold obtained by cutting M along H, and $\gamma' = \gamma \cap M'$. Let F be a 2-manifold properly embedded in M transversely to γ. Suppose that $F' = F \cap M'$ is γ'-incompressible and γ'-boundary incompressible in (M', γ'). Then F is γ-incompressible in (M, γ).*

Proof. Suppose for a contradiction that F is γ-compressible. Let D be a γ-compressing disk of F. We isotope D slightly so that D is transverse to H.

If $D \cap H$ contains a loop component, then let D_1 be an innermost disk bounded by an innermost loop of $D \cap H$ on D. Since H is γ-incompressible, ∂D_1 bounds a disk D_2 in $H - \gamma$. Then we take $(D - D_1) \cup D_2$ as new D, and an adequate small isotopy reduces the number $|D \cap H|$.

Hence we can assume that $D \cap H$ does not contain a loop. If $D \cap H$ contains an arc component, then let α be an outermost arc on D, and D_3 the outermost disk, that is, $D_3 \cap H = \alpha$. Since F' is γ'-boundary incompressible in (M', γ'), $\mathrm{cl}\,(\partial D_3 - \alpha)$ cuts off a disk D_4 from $F' - \gamma'$. The boundary of the disk $D_3 \cup D_4$ bounds a disk D_5 on $H - \gamma$ because H is γ-incompressible. Let β be the outermost arc of $D \cap D_5$ on D_5 such that the outermost disk D_6 does not contain α. We perform a surgery on D along the disk D_6 and obtain two disks, one of which is a γ-compressing disk of F. We regard this disk as a new D and discard the other disk. Then an adequate small isotopy reduces the number $|D \cap H|$.

Repeating such operations, we can take D so that it is disjoint from H. Then ∂D bounds a disk in $F - \gamma$ since F' is γ'-incompressible in

(M', γ'). This is a contradiction.

■

Let S be a 2-sphere which gives an essential tangle decomposition of a satellite link K in S^3 with a pattern (V, K). Suppose that (V, K) does not contain an essential tangle. We take S so that S intersects the torus ∂V transversely in a minimum number of loops up to isotopy in (S^3, K).

Since (V, K) does not contain an essential tangle, S is not contained in V. Hence $S \cap \partial V \neq \emptyset$. The solid torus V does not contain a meridian disk disjoint from K. (Otherwise, we compress the torus along such a disk and obtain a 2-sphere bounding a ball containing K, which contradicts that K is essential in V.) In addition, since the solid torus V is knotted in S^3, the torus ∂V is K-incompressible.

Lemma 2.3. *For every innermost loop of $S \cap \partial V$ on S, the innermost disk is a meridian disk of V. Hence the loops $S \cap \partial V$ are meridian loops of ∂V.*

Proof. By Lemma 2.1, the loops $S \cap \partial V$ are essential in ∂V. Let D be an innermost disk bounded by an innermost loop of $S \cap \partial V$ in S. Since V is knotted, ∂V is incompressible in $\mathrm{cl}\,(S^3 - V)$. Hence D is contained in V, and it is a meridian disk of V.

■

We consider the regions in S separated by $S \cap \partial V$ and contained in $\mathrm{cl}\,(S^3 - V)$. Each region is homeomorphic to a sphere with holes. We choose a region Q whose number of holes n is minimum among these regions.

Lemma 2.4. *There is no component of $\mathrm{cl}(S - Q)$ such that it consists of at least two annulus regions and a single disk region, and such that the annulus regions are disjoint from K.*

Proof. Suppose for a contradiction that there is such a component R of $\mathrm{cl}\,(S - Q)$. Let R_A be the annulus region next to Q in R. Note that R_A is contained in V. Let R_1 be a small collar neighbourhood of ∂Q in R_A, R_2 a small collar neighbourhood of $\partial R_A - \partial Q$ in R_A, and A the closure of $R_A - (R_1 \cup R_2)$.

Since ∂R_A is meridional in ∂V, there are disks δ_1 and δ_2 embedded in V such that $\delta_i \cap R_A = \partial\delta_i = \partial R_i - \partial R_A$ and $R_i \cup \delta_i$ is a meridian disk of V for $i = 1$ and 2. We take δ_1 and δ_2 so that $|(\delta_1 \cup \delta_2) \cap K|$ is minimum over all such disks. Then $R_i \cup \delta_i$ is K-incompressible in (V, K) for $i = 1$ and 2. The two disks $R_1 \cup \delta_1$ and $R_2 \cup \delta_2$ cut V into two balls V_1 and V_2 such that $A \subset V_1$. We push the tangle $(V_1, K \cap V_1)$ slightly into int V. Then there is an annulus $R \subset \mathrm{cl}\,(V - V_1) - K$ one of whose boundary component is in ∂V and of meridional slope and the other is in ∂V_1 and separates δ_1 and δ_2.

Since (V, K) does not contain an essential tangle, either (1) there is a K-compressing disk D of ∂V_1, (2) $(V_1, K \cap V_1)$ is a trivial 1-string tangle or (3) $V_1 - K$ is reducible. In the case (3), we obtain a contradiction with the fact that $V - K$ is irreducible. In the case (2), we can isotope V onto $V_2 \cup N(K)$, and then an adequate small isotopy of S decreases the number $|S \cap \partial V|$, which is a contradiction.

We consider the case (1). When D is in $\mathrm{cl}\,(V - V_1)$, a standard innermost loop argument allows us to take D so that $D \cap R$ does not contain a loop since V does not contain a meridian disk disjoint from K. If $D \cap R$ contains arcs, then the arcs have endpoints in $R \cap V_1$. Let α be an outermost arc on R, and D_1 the outermost disk, that is, $D_1 \cap D = \alpha$. The arc α divides the disk D into two subdisks D_2 and D_3. Then one of the two disks $D_1 \cup D_2$ and $D_1 \cup D_3$ is a K-compressing disk of ∂V_1. We discard the other one. Repeating this operation, we can take a K-compressing disk of ∂V_1 disjoint from R. This contradicts the K-incompressibility of the disks $R_i \cup \delta_i$.

Hence D is in V_1. We move V_1 to be in the original position so that $V = V_1 \cup V_2$. The 2-sphere $\delta_1 \cup A \cup \delta_2$ bounds a ball B in V_1, and let $V_0 = \mathrm{cl}\,(V_1 - B)$. We can take D so that $D \cap A$ consists of essential arcs in A by a similar argument as above. By the K-incompressibility of the disks $R_i \cup \delta_i$, we have $D \cap A \neq \emptyset$. Let β be an outermost arc of $D \cap A$ on D, and D_4 be the outermost disk. Then $D_4 \subset V_0$. Since ∂D_4 intersects A in an essential arc in A, ∂D_4 is essential in the torus ∂V_0. We perform a surgery on ∂V_0 along the disk D_4 and obtain a 2-sphere bounding a ball in V_0. This ball is disjoint from K since $V - K$ is irreducible. Hence V_0 is a solid torus contractible to R_A and we can isotope V onto $V_2 \cup B$, and an adequate small isotopy of S decreases the number $|S \cap \partial V|$, which is again a contradiction. ∎

Proof of Theorem 1.1. Suppose that $|S \cap K|$ is minimum over all essential decomposing spheres of K.

We choose a meridian disk P of V whose geometric intersection number with K is minimum among all meridian disks of V. Then P is K-incompressible and K-boundary incompressible in (V, K). Remember that we chose a region Q right before Lemma 2.4. Let $l_1, \cdots, l_n$ be components of ∂Q, and $P_1, \cdots, P_n$ parallel copies of P in V. We isotope these disks near boundaries so that $\partial P_i = l_i$ for $i = 1, \cdots, n$. Let $S' = Q \cup P_1 \cup \cdots \cup P_n$. Then S' gives an essential tangle decomposition of V by Lemma 2.1. Moreover, S' gives an essential tangle decomposition of K by Lemma 2.2.

Suppose for a contradiction that $\mathrm{cl}\,(S - Q)$ does not consist of meridian disks of V. Then $|S' \cap K| < |S \cap K|$ by Lemma 2.4, which contradicts the minimality of $|S \cap K|$. Thus $\mathrm{cl}\,(S - Q)$ consists of meridian disks of V. Since $S \cap \mathrm{cl}\,(S^3 - V) = Q = S' \cap \mathrm{cl}\,(S^3 - V)$, S gives an essential tangle decomposition of V.

■

3 The case where the wrapping number is 2

In this section, we study essential tangle decompositions of satellite links whose patterns have the wrapping number equal to 2.

Lemma 3.1. *Let V be a solid torus, and K a link in V. Suppose that $V - K$ is irreducible and there is a K-incompressible meridian disk D of V intersecting K in two points. Then either (V, K) contains an essential tangle or (V, K) is a rational pattern.*

Proof. Let B' be the ball obtained by cutting V along D. We push slightly the 2-sphere $S = \partial B'$ into $\mathrm{int}\, B'$. Then S bounds a ball B in V and it intersects K in four points. There is an annulus A properly embedded in $\mathrm{cl}\,(V - B) - K$ connecting ∂V and ∂B such that $\partial A \cap \partial V$ is a meridian loop on ∂V. The annulus A is K-incompressible since V does not contain a meridian disk disjoint from K. Set $T = K \cap B$, and then $B - T$ is irreducible because $V - K$ is irreducible. If (B, T) is essential in (V, K), then we are done. If (B, T) is not essential, then there is a K-compressing disk Q of ∂B in (V, K). We can take Q so that it is disjoint from $\mathrm{int}\, A$ by a standard innermost loop and outermost arc argument. Then Q is contained in B since D is K-incompressible in (V, K). The disk Q divides the tangle (B, T) into two tangles whose boundaries intersect K in two points. If one of them is essential in (V, K), then we are done. If both of them are inessential, then they are trivial 1-string tangles. This implies that (B, T) is a trivial 2-string tangle, and hence (V, K) is a rational pattern.

∎

Hence we concentrate on rational patterns.

Similar arguments as in the proofs of Lemmas 2.1 and 2.2 show the next two lemmas. We omit the proofs.

Lemma 3.2. *Let M be a 3-manifold, and γ a 1-manifold properly embedded in M. Let H be a meridionally incompressible closed 2-manifold in (M, γ). Let M' be the 3-manifold obtained by cutting M along H, and $\gamma' = \gamma \cap M'$. Suppose that $M' - \gamma'$ is irreducible, and that every 2-sphere intersecting transversely γ' in two points bounds a trivial 1-string tangle in (M', γ'). Let F be a meridionally incompressible 2-manifold such that F intersects H transversely in a minimal number of loops disjoint from γ up to isotopy of F in (M, γ). Then $F' = F \cap M'$ is meridionally incompressible in (M', γ').*

Lemma 3.3. *Let M be a 3-manifold, and γ a 1-manifold properly embedded in M. Let H be a meridionally incompressible closed 2-manifold in (M, γ). Let M' be the 3-manifold obtained by cutting M along H, and $\gamma' = \gamma \cap M'$. Let F be a 2-manifold properly embedded in M in general position with respect to $\gamma \cup H$. Suppose that $F' = F \cap M'$ is meridionally incompressible and γ'-boundary incompressible in (M', γ'). Then F is meridionally incompressible in (M, γ).*

In general, let F_1, F_2 be embedded surfaces transverse to γ in M such that $\partial F_1 = F_1 \cap F_2 = \partial F_2$. We say that F_1 and F_2 are *γ-parallel*, if there is a submanifold N in M such that $(N, F_1 \cap F_2, N \cap \gamma)$ is homeomorphic to $(F_1 \times I, \partial F_1 \times \{1/2\}, P \times I)$ as a triple, where P is a union of finitely many points in $\operatorname{int} F_1$. We say that a surface F properly embedded in M and transverse to γ is *γ-boundary parallel* if there is a subsurface F' in ∂M such that F and F' are γ-parallel.

Lemma 3.4. *Let (V, K) a rational pattern with slope p/q, and F a meridionally incompressible surface in (V, K). Suppose that each component of ∂F is of the meridional slope in ∂V. Then either*

(0) F is a 2-sphere cutting off a ball disjoint from K or a trivial 1-string tangle from (V, K),
(1) F is an annulus which does not intersect K and K-boundary parallel in (V, K),
(2) F is a disk which is isotopic in (V, K) to the disk D in the definition of rational patterns, or
(3) F is a torus which does not intersect K and K-boundary parallel in (V, K).

Proof. We assume that F is not of type (0) to show that it is of type (1), (2) or (3).

We consider the three cases below simultaneously. (i) The surface F is a 2-sphere disjoint from K such that it does not bound a ball disjoint from K. (ii) There is no 2-sphere as in (i) and the surface F is a 2-sphere intersecting K in exactly two points such that it does not bound a trivial 1-string tangle. (iii) There is no surface as in (i) or (ii).

At the end of this proof, we have that neither (i) nor (ii) occurs.

Let D be the meridian disk of V in the definition of rational patterns. We cut (V, K) along D and obtain a trivial 2-string tangle (B', T'). Let $N \cong \partial B' \times I$ be a small neighbourhood of $\partial B'$ in B' such that $N \cap T'$ is composed of vertical arcs. Let $B = \text{cl}\,(B' - N)$ and $T = B \cap T'$. Then (B, T) is also a trivial 2-string tangle.

Let D' be a meridian disk of V such that $D' \cap B$ is a single disk, the annulus $A = \text{cl}\,(D' - B)$ is disjoint from K and vertical in $\partial B' \times I$.

We can isotope F so that $\partial F \cap \partial A \cap \partial V = \emptyset$ and $F \cap B$ is a parallel collection of disks each of which separates the two arcs of (B, T). We isotope F so that the boundary of disks $F \cap B$ intersect the loop $\partial A \cap \partial B$ in a minimal number of points. We isotope F so that it is transverse to A. Moreover we take F so that the pair of integers $(|F \cap B|, |F \cap A|)$ is minimal in lexicographical order, over all such 2-spheres in cases (i) and (ii), or up to isotopy in (V, K) in case (iii).

Claim. *$F \cap A$ is empty or consists of essential loops in A.*

Proof of Claim. By an innermost loop argument, $F \cap A$ does not contain an inessential loop in A. Suppose that $F \cap A$ contains an arc, then its endpoints are in $\partial A \cap B$ and it is inessential in A. Let α be an outermost one on A. We isotope F near α along the outermost disk in A. Then a band is attached to the collection of disks $F \cap B$. If the band connects two disks, then they are deformed into a disk T-boundary parallel in (B,T), which contradicts the minimality of $|F \cap B|$. Hence the band attached to a single disk Q, and Q is deformed into an annulus Q'. A component of $\partial Q'$ bounds a disk $P \subset \partial B$ which intersects K at a single point. In case (i), this is a contradiction since a solid torus does not contain a non-separating 2-sphere.

Then, since F is meridionally incompressible in (V,K), ∂P bounds a disk P' in F such that P' intersects K at a single point. In case (ii), we perform a surgery on F along P, and obtain two 2-spheres intersecting K in two points, one of which does not bound a trivial 1-string tangle. We discard the other 2-sphere. Then Q' is deformed into a disk intersecting T in exactly one point, and we can push it out of B. This contradicts the minimality of $|F \cap B|$. In case (iii), P' is isotopic to P in (V,K). We deform F as above and obtain a contradiction to the minimality of $|F \cap B|$. Thus $F \cap A$ is empty or consists of essential loops in A. This completes the proof of Claim.

■

Now F is disjoint from B, otherwise there is a disk in B separating two arcs of (B,T) such that it is disjoint from ∂A, and hence the slope of the rational pattern (V,K) is ∞.

We use the next result by F. Waldhausen.

Proposition 3.1 in [W]. *Let $M = F \times I$ be the product of the orientable surface F which is not the 2-sphere, and the interval $I = [0,1]$. Let $p : M \to F$ denote the projection onto the factor F. Let G be a system of incompressible surfaces in M. Suppose ∂G is contained in $F \times \{1\}$. Then G is isotopic, by a deformation that is constant on ∂M, to a system G' such that $p|_{G'}$ is homeomorphic on each component of G'.*

Note that a 2-sphere bounding a ball is called compressible in [W], but we call such a 2-sphere incompressible in this paper. We consider the surfaces $F \cap B'$. Their boundary loops are parallel to the meridian ∂D in $\partial B' - \partial T'$. Note that $M = \mathrm{cl}\,(V - (N(D') \cup B))$ is homeomorphic to $R \times I$, where R is a disk, two copies of $\partial D'$ are $\partial R \times \{0,1\}$ and $T_M = T' \cap M$ is vertical. Let $F_M = F \cap M$. The copies of the annulus A is T_M-incompressible since the arcs of T_M connect distinct components of $\partial B \cap \partial M$. Hence F_M is T_M-incompressible in (M, T_M), and $F_M \cap M'$ is incompressible in M', where $M' = \mathrm{cl}\,(M - N(T_M))$. We can isotope

$F_M \cap M'$ in M' into surfaces F'_M so that $\partial F'_M$ is in $R \times \{1\}$. Then by the result of F. Waldhausen, each component of F'_M is boundary parallel to either a disk with three holes or an annulus. A surface of the former type corresponds to a disk of F_M intersecting T_M twice and T_M-boundary parallel into a subdisk of $R \times \{0\}$ or $R \times \{1\}$. A surface of the latter type corresponds to an annulus of F_M, otherwise it would correspond to a 2-sphere bounding a trivial 1-string tangle in (V, T).

First we consider annuli of F_M. If an annulus has its boundary in $\partial R \times I$, then we obtain the conclusion (1). If an annulus has exactly one component of its boundary in $\partial R \times I$, then we can isotope it into $R \times \{0, 1\}$, and obtain a contradiction to the minimality of $|F \cap A|$ after an adequate small isotopy. If an annulus has its boundary entirely in $R \times \{0\}$ (resp. $R \times \{1\}$), then we can isotope it into $R \times \{0\}$ (resp. $R \times \{1\}$), and obtain a contradiction to the minimality of $|F \cap A|$ after an adequate small isotopy. Hence we can assume that every annulus of F_M connects $R \times \{0\}$ and $R \times \{1\}$.

We consider disks of F_M. If a disk has its boundary in $\partial R \times I$, then we obtain the conclusion (2). Hence we can assume that every disk of F_M has its boundary in $R \times \{0\}$ or $R \times \{1\}$.

When F_M contains an annulus, the outermost loops in $R \times \{0\}$ and $R \times \{1\}$ are glued, and F is a torus parallel to ∂V. This is the conclusion (3). When F_M does not contain an annulus, it consists of disks, the innermost loops in $R \times \{0\}$ and $R \times \{1\}$ are glued, and F is a sphere parallel to ∂B. This sphere bounds a trivial 2-string tangle, and hence is K-compressible, which is a contradiction.

Thus we cannot recover F by gluing the components of F_M in cases (i) and (ii), which is a contradiction.

∎

By Lemma 3.4, in a rational pattern (V, K) the wrapping number of K in V is 2, and hence ∂V is meridionally incompressible in (V, K).

Lemma 3.5. *Let (V, K) be a rational pattern of slope p/q. Then no tangle in (V, K) is essential. In addition, any K-incompressible meridian disk of V is isotopic in (V, K) to the meridian disk D in the definition of rational patterns.*

Proof. We prove only the first half of this lemma. The second one is proved by a similar argument, and we omit the proof.

Suppose for a contradiction that a 2-sphere S bounds a tangle which is essential in (V, K). Then S is meridionally compressible in (V, K) by Lemma 3.4. We perform meridional compressing on S, to obtain two 2-spheres which are K-incompressible. The 2-spheres intersect K at more than two points since they are yielded by a meridional compressing. We repeat such meridional compressing operations, and eventually obtain a meridionally incompressible 2-sphere intersecting K at more than two points in (V, K). This contradicts Lemma 3.4.

■

Lemma 3.6. *Let (V, K) be a rational pattern of slope p/q, where $|q|$ is an odd integer. Let D_1 and D_2 be disjoint disks which are K-parallel to the disk in the definition of rational patterns. Let F be the annulus obtained from D_1 and D_2 by a tubing operation outside the parallelism of D_1 and D_2. Then F is K-incompressible and K-boundary incompressible in (V, K) if and only if $|q|$ is greater than 1.*

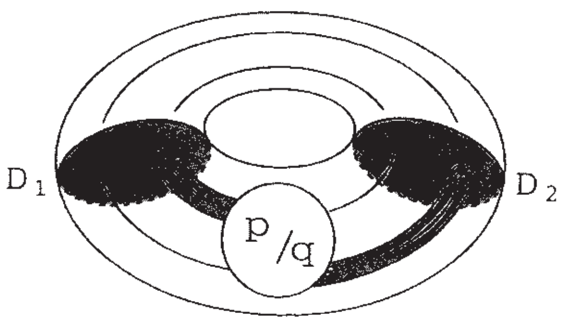

Figure 2: Annulus obtained from D_1 and D_2 by a tubing operation

Proof. Let B and B' be the balls obtained by cutting V along the disks D_1 and D_2, where $(B', K \cap B')$ is the parallelism of D_1 and D_2 and $(B, K \cap B, \partial D_1)$ is the marked rational tangle of slope p/q. The string $T = K \cap B$ consists of two components t_1 and t_2. Note that t_1 and t_2 connect D_1 and D_2 because q is an odd integer. We obtain F by performing a tubing operation on D_1 and D_2 along one of the arcs, say t_1. Let $N(t_i)$ be a tubular neighbourhood of t_i in B, $E(t_i) = \mathrm{cl}\,(B - N(t_i))$ and $E(T) = \mathrm{cl}\,(B - N(T))$. Then F is the union of the annuli

$\mathrm{cl}(D_i - N(t_1))$ and $N(t_1) \cap E(t_1)$. Let D be a cocore disk of the 1-handle $N(t_1)$ such that it intersects t_1 transversely in a single point and meridional compressing on F along D recovers the disks D_1 and D_2.

If $|q|$ is equal to 1, then (B, D_1, T) is homeomorphic to $(D_1 \times I, D_1, (K \cap D_1) \times I)$. Hence F is K-compressible in (V, K), and the 'only if' part follows.

On the other hand, we assume that F is K-compressible in (V, K) to show $|q| = 1$. Let Q be a K-compressing disk of F. If ∂Q is an essential loop in F, then by compressing F along Q, we obtain a meridian disk of V which intersects K at less than 2 points. This is a contradiction.

Hence ∂Q is an inessential loop in F. We can take Q disjoint from the interior of the cocore D of $N(t_1)$ by a standard innermost loop and outermost arc argument. Then Q is contained in $E(t_1)$ since D_1 and D_2 are K-incompressible. By compressing F along Q, we obtain an annulus F_1 which is isotopic to F in V (not in $V - K$), and a 2-sphere F_2. Since the solid torus V cannot contain a non-separating 2-sphere, F_2 intersects K in 2 points and F_1 does not intersect K. Because (B, t_1) is a trivial 1-string tangle, F_1 is K-parallel to the annulus $F_1' = B \cap \partial V$. Since (V, K) does not contain an essential tangle, the tangle (C, t_2) bounded by F_2 is the trivial 1-string tangle. We use the result by C. McA. Gordon below.

Theorem 2 in [G]. *Let C be a set of $n + 1$ disjoint simple loops in the boundary of a handlebody X of genus n. Suppose that for all proper subsets C' of C the 3-manifold obtained by attaching 2-handles along C' is a handlebody. Then $\cup C$ bounds a planar surface P in ∂X such that $(X, P) \cong (P \times [0,1], P \times \{0\})$.*

Let c_i be a core loop of the annulus $N(t_i) \cap E(t_i)$ for $i = 1$ and 2, and c_0 a core loop of F_1'.

If we attach 2-handles to $E(T)$ along $c_1 \cup c_2$, then we recover the 3-ball B. Since (B, t_1) and (B, t_2) are trivial 1-string tangles, we obtain a solid torus $E(t_2)$ or $E(t_1)$ if we attach a 2-handle to $E(T)$ along c_1 or c_2, and we obtain a 3-ball if we attach 2-handles to $E(T)$ along $c_i \cup c_0$ for $i = 1$ or 2. Since F_1 and F_1' cobound a solid torus V_1 contractible to F_1', we obtain a 3-ball when we attach a 2-handle to V_1 along c_0. This 3-ball and C share the disk Q. Hence we obtain a solid torus when we attach a 2-handle to $E(T)$ along c_0, since the tangle (C, t_2) is trivial. Thus (B, T) is homeomorphic to $(D_1, K \cap D_1) \times [0,1]$, and hence $|q|$ is equal to 1.

Suppose that F is K-boundary compressible. Let P be a K-boundary compressing disk. Let $\beta = \partial P \cap \partial V$ an arc, and A be the annulus $B \cap \partial V$ or $B' \cap \partial V$ containing β. If β connects two loops ∂D_1 and ∂D_2, then we perform a surgery on A along the disk P and obtain a K-compressing disk of the annulus F. (Note that $F \cap K \neq \emptyset$.) If β has both endpoints in the same loop, then we can isotope P so that $\partial P \subset F$ and again obtain a K-compressing disk of F. Hence in either case F is K-compressible, and $|q| = 1$ by the above argument.

■

Proposition 3.7. *Let K be a satellite link in S^3 with a rational pattern (V, K) of slope p/q. Suppose that the companion knot admits an essential tangle decomposition, and that $|q|$ is an odd integer greater than 1. Then there is a meridionally compressible 2-sphere S which gives an essential tangle decomposition of K and cannot be isotoped to give an essential tangle decomposition of V.*

Proof. Let Q be a 2-sphere which gives an essential tangle decomposition of V, and $P = Q \cap \mathrm{cl}\,(S^3 - V)$ the punctured sphere. We take a parallel copy P' of P in $\mathrm{cl}\,(S^3 - V)$. Let n be the number of components of ∂P. We take $2(n-1)$ parallel copies $D_1 \cup \ldots \cup D_{2(n-1)}$ of D which is the meridian disk of V in the definition of rational patterns. Note that these disks are K-incompressible and K-boundary incompressible since V does not contain a meridian disk intersecting K at less than 2 points. Let F be an annulus as in Lemma 3.6. We can take F to be disjoint from $D_1 \cup \ldots \cup D_{2(n-1)}$. We paste P, P', F, $D_1, \ldots, D_{2(n-1)}$ along their boundaries so that F connects P and P'. Then we obtain a 2-sphere S which is K-incompressible by Lemma 2.2.

Suppose for a contradiction that S can be isotoped in (S^3, K) to give an essential tangle decomposition of V. Then $S \cap V$ is a union of an even number, say $2m$, of K-incompressible disks in (V, K). By Lemma 3.5, these disks are isotopic in (V, K) to the meridian disk in the definition of rational patterns. Hence $|S \cap K| = 4m$. But $|S \cap K| = 4(n-1) + 2$ by the construction of S. This is a contradiction.

■

Example. Now we can give a counterexample to Question stated in Introduction. The satellite knot illustrated in Figure 3 has an essential tangle decomposing sphere consisting of P, P', F, D_1 and D_2. Proposition 3.7 guarantees that this sphere cannot be isotoped to give an essential tangle decomposition of the companion solid torus.

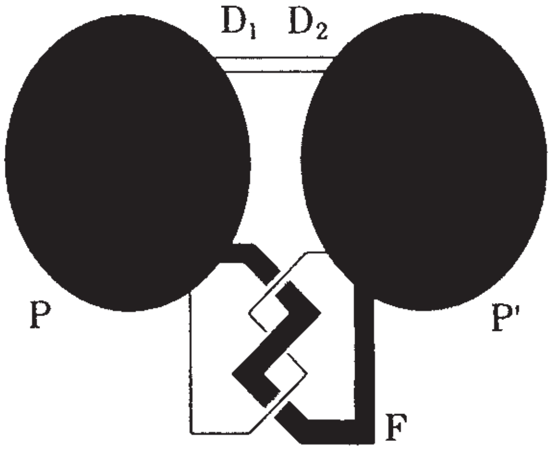

Figure 3: Counterexample to Question

Lemma 3.8. *Let (V, K) be a rational pattern of slope p/q, where $|q| = 1$ or an even integer. Let F be a K-incompressible and K-boundary incompressible planar surface with non-empty boundary in (V, K) such that $\partial F \cap \partial V$ is a union of meridian loops in ∂V. Then F is a union of meridian disks which are isotopic in (V, K) to the meridian disk in the definition of rational patterns.*

Proof. Suppose for a contradiction that F is not a disjoint union of such meridian disks of V. Then F is meridionally compressible in (V, K) by Lemma 3.4. We perform meridional compressings on F repeatedly to obtain a disjoint union of surfaces P which is meridionally incompressible. At each stage of the sequence of meridional compressing operations, we have a disjoint union of surfaces which is K-incompressible and K-boundary incompressible in (V, K). By Lemma 3.4, P is a union of disks $D_1, \cdots, D_n$ appearing in V in this order with the rational tangle between D_n and D_1. We can recover F from P by a sequence of tubing operations.

If the first tubing operation occurs between two adjacent disks D_i

and D_{i+1} for $1 \leq i \leq n-1$, then they are deformed into an annulus which is K-boundary compressible in (V, K). This is a contradiction. Hence the first tubing operation occurs in the rational tangle. If $|q|$ is an even integer, then the first tubing operation deform D_1 or D_n into a once punctured torus. This contradicts that F is planar. If $|q| = 1$, then the first tubing operation deform D_1 and D_n into an annulus which is K-boundary compressible in (V, K). This is also a contradiction.

∎

Proof of Theorem 1.3. The 'if' part is Proposition 3.7. Hence we show the 'only if' part. Let S be a 2-sphere which gives an essential tangle decomposition of K. We isotope S in (S^3, K) so that $|S \cap \partial V|$ is minimal up to isotopy in (S^3, K). Then by Lemma 2.1, $S \cap V$ is K-incompressible and K-boundary incompressible in (V, K) and $S \cap \mathrm{cl}\,(S^3 - V)$ is incompressible and boundary incompressible. Hence the companion knot admits an essential tangle decomposition. The loops $S \cap \partial V$ are meridional in ∂V by Lemma 2.3. When $|q| = 1$ or $|q|$ is an even number, $S \cap V$ is a union of meridian disks which are isotopic in (V, K) to the meridian disk in the definition of rational patterns by Lemma 3.8. The disks $S \cap V$ are meridionally incompressible and K-boundary incompressible in (V, K) since the wrapping number of K is 2 in V. Then S is meridionally incompressible in (S^3, K) by Lemma 3.3. Note that ∂V is meridionally incompressible in (S^3, K).

∎

Proof of Theorem 1.2. Let S be a 2-sphere giving an essential tangle decomposition of a satellite link K with a rational pattern (V, K).

Suppose that S can be isotoped to give an essential tangle decomposition of V. Then S intersects V in even number of meridian disks, and $S \cap \mathrm{cl}\,(S^3 - V)$ is incompressible and boundary incompressible. The meridian disks $S \cap V$ are K-incompressible, otherwise S would be K-compressible. Hence the disks $S \cap V$ are isotopic in (V, K) to parallel copies of the meridian disk D in the definition of rational patterns by Lemma 3.5. The disks $S \cap V$ are meridionally incompressible and K-boundary incompressible in (V, K) since the wrapping number of K is 2 in V. Then S is meridionally incompressible in (S^3, K) by Lemma 3.3. Note that ∂V is meridionally incompressible in (S^3, K).

On the other hand, suppose that S is meridionally incompressible in (S^3, K). We isotope S in (S^3, K) to intersect ∂V in minimal number of loops. Then S intersects V in K-incompressible surfaces and $S \cap \mathrm{cl}\,(S^3 - V)$ is incompressible and boundary incompressible by Lemma 2.1. The loops $S \cap \partial V$ are meridian loops of ∂V by Lemma 2.3. The surfaces $S \cap V$ are meridionally incompressible in (V, K) by Lemma 3.2. Hence the disks $S \cap V$ are isotopic in (V, K) to parallel copies of the meridian disk D in the definition of rational patterns by Lemma 3.4. Thus S gives an essential tangle decomposition of V.

■

4 Proof of Theorem 1.4

We will show that S can be isotoped in (S^3, K) so that every component of $S \cap V$ is a meridian disk of V which meets K transversely in m points. Then this completes the proof of Theorem 1.4.

We take S so that $|S \cap \partial V|$ is minimum among all 2-spheres isotopic to S in the pair $(S^3, \tilde{L})$. By Lemma 2.3, the loops of $S \cap \partial V$ are meridian loops on ∂V. Let $V_0 = \mathrm{cl}\,(V - N(K))$. Then the system of surfaces $S \cap V_0$ is incompressible in V_0. Let M be a meridian disk of V which meets K transversely in m points, and $M_0 = M \cap V_0$. We can isotope S so that $S \cap \partial V_0$ is disjoint from ∂M_0. Under such conditions we take M so that $S \cap M_0$ consists of a minimal number of loops up to isotopy of S in (S^3, K).

There is an innermost disk Δ bounded by an innermost loop of $\partial V \cap S$ in S. Then Δ is a meridian disk of V. We show first that $\Delta \cap M = \emptyset$. Suppose for a contradiction that $\Delta \cap M \neq \emptyset$. Then there is an innermost loop of $\Delta \cap M$ on Δ, and let δ be the innermost disk. When we cut V_0 along the punctured disk M_0, we obtain a 3-manifold V_0' homeomorphic to $M_0 \times [0, 1]$. Let $\delta_0 = \delta \cap V_0'$. We use F. Waldhausen's result [Proposition 3.1, W] whose statement is cited in the proof of Lemma 3.2 in this paper. We can isotope δ_0 along subannuli of $\partial N(K) \cap V_0'$ and slightly beyond the loops $\partial M_0 \cap \partial N(K)$ so that $\partial \delta_0$ is in $M_0 \times \{0\}$ or $M_0 \times \{1\}$, say $M_0 \times \{1\}$. Then the above result by Waldhausen implies that δ_0 is isotopic into $M_0 \times \{1\}$. We retake M to be $(M - M') \cup \delta$, where M' is the disk bounded by $\partial \delta$ on M. Then an adequate small isotopy decreases the number $|S \cap M|$. This contradicts the minimality of $|S \cap M|$.

Thus Δ is disjoint from the meridian disk M. We can isotope $\Delta_0 = \Delta \cap V_0'$ in V_0' along subannuli of $\partial N(K) \cap V_0'$ and slightly beyond the loops $\partial M_0 \cap \partial N(K)$ so that $\partial \Delta_0$ is in $M_0 \times \{1\}$. Hence Δ is isotopic to a fiber $M \times \{*\}$ again by the result of Waldhausen.

We retake M to be very close to Δ. Then $S \cap M = \emptyset$. We can show that every component of $S \cap V$ is a meridian disk isotopic to a fiber $M \times \{*\}$ by the same argument as in the previous paragraph.

∎

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