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Cyclic branched coverings of 2-bridge knots.

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Abstract

In this paper we study the connections between cyclic presentations of groups and the fundamental group of cyclic branched coverings of 2-bridge knots. Then we show that the topology of these manifolds (and knots) arises, in a natural way, from the algebraic properties of such presentations.

1 Cyclic Presentations

Several authors have recently remarked that cyclic presentations of groups are very interesting from a topological point of view. Connections between these types of presentations and the topology of cyclic branched coverings of knots and links can be found in [4], [5], [12], [13], [15], [18], [19], [22], [26], and [28]. The purpose of this paper is to study the cyclic branched coverings of (hyperbolic) 2-bridge knots. We show that these manifolds can be encoded by cyclically presented groups arising from Heegaard diagrams with a rotational symmetry (and so they correspond also to spines [14] of the considered manifolds). To state our main result, we recall some definitions on cyclic presentations of groups, and refer to [20] for a more detailed discussion on the topic. Let F_n be the free group on free generators $x_1, \ldots, x_n$, and let θ denote the automorphism of F_n defined by setting $\theta(x_i) = x_{i+1}$ (where the indices are taken mod n). For any reduced word w in F_n, let us consider the factor group $G_n(w) = F_n/R$, where R is the normal closure in F_n of the set

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$\{w, \theta(w), \ldots, \theta^{n-1}(w)\}$. A group G is said to have a *cyclic presentation* if G is isomorphic to $G_n(w)$ for some w and n. Of course, θ induces an automorphism of $G_n(w)$ which determines an action of the cyclic group $\mathbf{Z}_n =< \theta : \theta^n = 1 >$ on $G_n(w)$. The split extension group of $G_n(w)$ by $\mathbf{Z}_n$ admits a 2-generator presentation of type

$$H_n(v) =< \theta, x : \theta^n = 1, \quad v(\theta, x) = 1 >,$$

where

$$v(\theta, x) = w(x, \theta^{-1}x\theta, \ldots, \theta^{-(n-1)}x\theta^{n-1}).$$

The *polynomial* associated with $G_n(w)$ is defined to be

$$f_w(t) = \sum_{i=1}^{n} a_i t^i,$$

where a_i is the exponent sum of x_i in w.

The following questions are natural here:

1) Does $G_n(w)$ correspond to a spine (or a Heegaard diagram) of a closed orientable 3-manifold $M_n(w)$? (If so, then the presentation is also called *geometric*);

2) Which of the manifolds $M_n(w)$ admit a hyperbolic structure?;

3) Are the manifolds $M_n(w)$ homeomorphic to the cyclic branched coverings of some (hyperbolic) knots or links? In the first case, does $f_w(t)$ coincide (up to sign) with the Alexander polynomial of the knot?

We treat these questions for the cyclic branched coverings of 2-bridge knots (here we are principally interested in hyperbolic case, and refer to [4] for 2-bridge torus knots). A *2-bridge knot* is determined by a pair of coprime integers (α, β) satisfying $0 < \beta < \alpha$, and α odd. Following [16], let us denote by α/β the 2-bridge knot determined by (α, β). Two such knots α/β and α'/β' belong to the same knot type if and only if $\alpha = \alpha'$ and $\beta^{\pm 1} \equiv \beta' \pmod{\alpha}$. The knot group of α/β has the 2-generator presentation

$$< u, v : uw(\alpha, \beta) = w(\alpha, \beta)v >,$$

where

$$w(\alpha, \beta) = v^{\epsilon_1} u^{\epsilon_2} \cdots u^{\epsilon_{\alpha-3}} v^{\epsilon_{\alpha-2}} u^{\epsilon_{\alpha-1}},$$

and ϵ_i is the sign (± 1) of $i\beta$ reduced mod 2α in the interval $(-\alpha, \alpha)$. If $\beta > 1$ and $\beta \neq \alpha - 1$, then α/β is *hyperbolic*, i.e. its complement in the 3-sphere $\mathbb{S}^3$ admits a complete hyperbolic structure of finite volume. The conditions $\beta > 1$ and $\beta \neq \alpha - 1$ mean that α/β is not a torus knot (see [31]). Let $\mathcal{O}(\alpha/\beta; n)$ be the orbifold whose underlying space is $\mathbb{S}^3$, and its singular set is α/β with branch index $n \geq 1$. The geometric structures of the orbifolds $\mathcal{O}(\alpha/\beta; n)$ are well-known [31] (see also [7] and [16]). Finding a geometric structure for $\mathcal{O}(\alpha/\beta; n)$ immediately implies that the n-fold cyclic covering $M_n(\alpha/\beta)$ of the 3-sphere branched over α/β has a structure modelled on the same geometry [7].

Theorem 1.1. *Assume $\beta > 1$ and $\beta \neq \alpha - 1$. The manifolds $M_n(\alpha/\beta)$ are hyperbolic when $\alpha = 5$ and $n \geq 4$ or $\alpha \neq 5$ and $n \geq 3$. Furthermore, $M_2(\alpha/\beta)$ is homeomorphic to the lens space $L(\alpha, \beta)$ for any α, while $M_3(5/3)$ (i.e. the 3-fold cyclic branched covering over the figure eight knot) is Euclidean.*

Observe that all the results obtained in this paper on geometrical structures of the considered 3-manifolds are consequences of Theorem 1.1. Thus such results can be considered as remarks from other propositions proved here.

The following answers affirmatively the questions above for 2-bridge knots.

Theorem 1.2. *The fundamental group of $M_n(\alpha/\beta)$ admits a cyclic presentation $G_n(\alpha/\beta)$ arising from a Heegaard diagram with an n-rotational symmetry (and whence it corresponds to a spine of the manifold). The split extension group $H_n(\alpha/\beta)$ of $G_n(\alpha/\beta)$ is isomorphic to the fundamental group of $\mathcal{O}(\alpha/\beta; n)$. Furthermore, the polynomial associated with $G_n(\alpha/\beta)$ coincides (up to sign) with the Alexander polynomial of α/β.*

The fact about the polynomials can be seen in general as follows. Suppose $G_n(w)$ encodes the n-fold cyclic covering $M_n(K)$ of the 3-sphere branched over a knot K. Let $X = X(K)$ be the infinite cyclic cover of K. If $\mathbb{Z}[t, t^{-1}]$ denotes the ring of (finite) Laurent polynomials with integer coefficients, then the module $H_1(X)$ is the quotient of $\mathbb{Z}[t, t^{-1}]$ by the ideal generated by the Alexander polynomial $\Delta_K(t)$ of K (see for example [27]). So the module $H_1(M_n(K))$ is the quotient of $\mathbb{Z}[t, t^{-1}]$ by the ideal generated by $\Delta_K(t)$ and $t^n - 1$. But from [20] there is also a module isomorphism between the abelianized of $G_n(w)$ and the quotient of $\mathbb{Z}[t, t^{-1}]$ by the ideal generated by the polynomial $f_w(t)$, associated with $G_n(w)$, and $t^n - 1$. So $f_w(t)$ coincides (up to conjugation) with $\Delta_K(t)$, as required. In any case, we shall prove this directly for 2-bridge knots.

2 RR-systems

To prove that the group presentations, considered in the paper, are all geometric, we use the representation theory of closed orientable 3-manifolds by RR-systems, due to Osborne and Stevens ([23], [24], [25], and [30]). So we first recall some definitions and a basic result of the theory (for more details see the quoted papers). A *RR-system (railroad system)* is a simple planar graph-like object defined as follows. Let D be a regular hexagon in Euclidean plane E^2. For each pair of opposite faces construct a finite set (possibly empty) of parallel line segments, called *tracks*, through D with endpoints on these opposite faces. Any pair of opposite faces in the hexagon D is called a *station*. Let $\{D_i : i = 1, \ldots, n\}$ be a set of disjoint regular hexagons in E^2. A *route* is an arc whose interior lies in the complement of $\cup_i D_i$ in E^2, and connects endpoints of tracks. A *RR-system* is the union in $E^2 \subset \mathbb{S}^2 = E^2 + \underline{\infty}$ (the 2-sphere) of a finite set of disjoint routes in the complement of $\cup_i D_i$ in $\mathbb{S}^2$ such that each endpoint of every track intersects exactly one route in one of its endpoints. A RR-system gives rise to a family of group presentations in the following way. The generators x_i, for $i = 1, \ldots, n$, are in one-to-one correspondence with the hexagons D_i, and hence D_i can be labelled by x_i. In each D_i we start at some vertex of ∂D_i and proceed clockwise (according to an orientation of $\mathbb{S}^2$) along an edge. This edge corresponds to a station labelled by an integer ℓ_i. Orient the tracks corresponding to this station so that the positive direction is toward the above edge. Label the stations corresponding to the second and third edges of D_i encountered by integers k_i and $\ell_i + k_i$, respectively, and orient the tracks of these stations toward the corresponding edges. To illustrate the labelling of the stations we have depicted a simple RR-system with two hexagons in Figure 1. For each hexagon we have of course three stations corresponding to the three pairs of opposite faces in it. So moving clockwise along the boundary of the hexagon D_i in

Figure 1 ($i = 1, 2$) we label its stations by the integers ℓ_i, k_i, and $\ell_i + k_i$. Now we construct a set (possibly empty) of α_i parallel oriented line segments (tracks) with heads on the station labelled ℓ_i and ends (tails) on the opposite face of D_i (we have $\alpha_1 = \alpha_2 = 1$ in Figure 1). Next we construct a set (possibly empty) of β_i parallel oriented line segments with heads on the station labelled k_i and tails on the opposite side of D_i (we have $\beta_1 = 1$ and $\beta_2 = 0$ in Figure 1). Similarly, we construct a set (possibly empty) of γ_i tracks with heads on the station labelled $\ell_i + k_i$ and tails on the opposite side of D_i (we have $\gamma_1 = 1$ and $\gamma_2 = 2$ in Figure 1). Now we explain how to obtain the group presentation corresponding to a given RR-system. Beginning at some point on some route we write a word on generators x_i as follows. As we enter in each hexagon D_i we give the label of the station as exponent of x_i with sign $+1$ (resp. -1) if our direction of travel agrees with (resp. opposes to) the orientation of the tracks. When we have completed our travels on routes, we obtain the relations of the group presentations induced by the RR-system. The group presentation induced by the RR-system shown in Figure 1 has two generators x_1 and x_2 (which correspond to the hexagons D_1 and D_2), and two relations $x_1^{\ell_1+k_1}x_2^{\ell_2} = 1$ and $x_1^{\ell_1}x_2^{\ell_2+k_2}x_1^{k_1}x_2^{\ell_2+k_2} = 1$. To obtain these relations we proceed as follows. We trace out a simple closed curve by following a track through the hexagon D_1, then along an arc (route) connecting the head of this track to a track in the hexagon D_2. Next we follow the route connecting this last track with another one in D_1, and continue until we return to the starting point (i.e. the tail of the first track considered in D_1). This closed curve determines a relation of the group presentation in the following way. As we travel along the closed curve, we record a syllabe $x_1^{\ell_1}$ if we travel from tail to head along a track of the station labelled ℓ_1 in the hexagon D_1. Next we follow the closed curve along a route until we reach a track in the hexagon D_2. If we follow along this track from tail to head, then we record the syllabe $x_2^{\ell_2}$ (or $x_2^{k_2}$ or $x_2^{\ell_2+k_2}$) according to the label of the station. If we travel along a track from head to tail, then we record the syllabe with a minus sign preceding the exponent. We continue in this way until we return to the starting point. Beginning to the starting point labelled A (resp. B) in Figure 1 and travel along a closed curve we write the word $x_1^{\ell_1+k_1}x_2^{\ell_2}$ (resp. $x_1^{\ell_1}x_2^{\ell_2+k_2}x_1^{k_1}x_2^{\ell_2+k_2}$) as claimed.

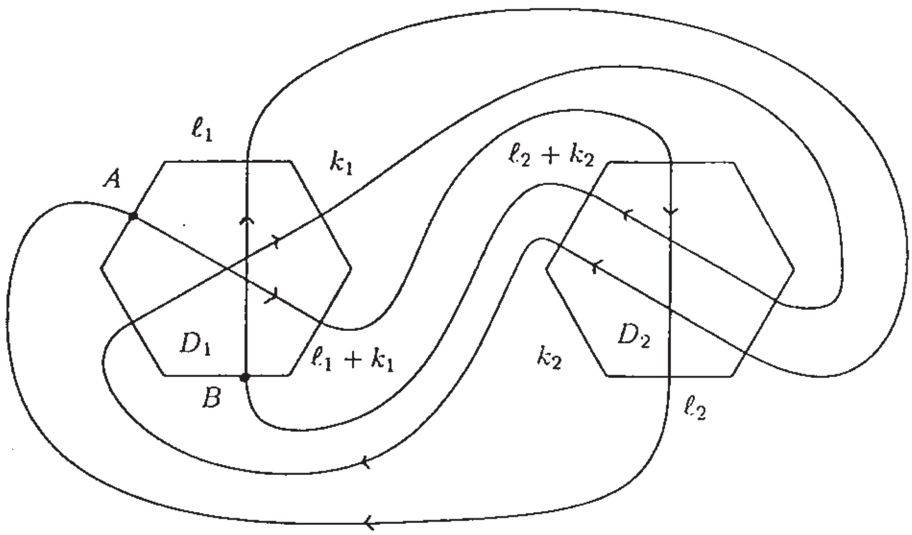

Figure 1: An example of RR–system

The following is a fundamental result in the theory of RR-systems:

Theorem 2.1. *Let $\mathcal{R}$ be a RR-system with n hexagons whose stations are labelled by integers ℓ_i, k_i, and $\ell_i + k_i$, and let $\Phi_{\mathcal{R}}$ be a group presentation induced by $\mathcal{R}$. If $(\ell_i, k_i) = 1$ for any $i = 1, \ldots, n$, then $\Phi_{\mathcal{R}}$ corresponds to a spine of a closed orientable 3-manifold M–i.e. $M\setminus$ (open 3-cell) collapses onto the canonical cell complex of dimension 2 given by the presentation $\Phi_{\mathcal{R}}$.*

Let us consider, for example, the RR-system depicted in Figure 2. One can easily verify that it induces a cyclically presented group with three generators x_i, and three relations (indices mod 3)

$$(x_i^{\ell_i} x_{i+1}^{\ell_{i+1}})^m x_{i+1}^{k_{i+1}} = (x_{i+1}^{-\ell_{i+1}} x_{i+2}^{\ell_{i+2}})^m.$$

If $(\ell_i, k_i) = 1$ for any $i = 1, \ldots, n$, then Theorem 2.1 implies that these presentations are geometric, and so correspond to spines of closed orientable 3-manifolds.

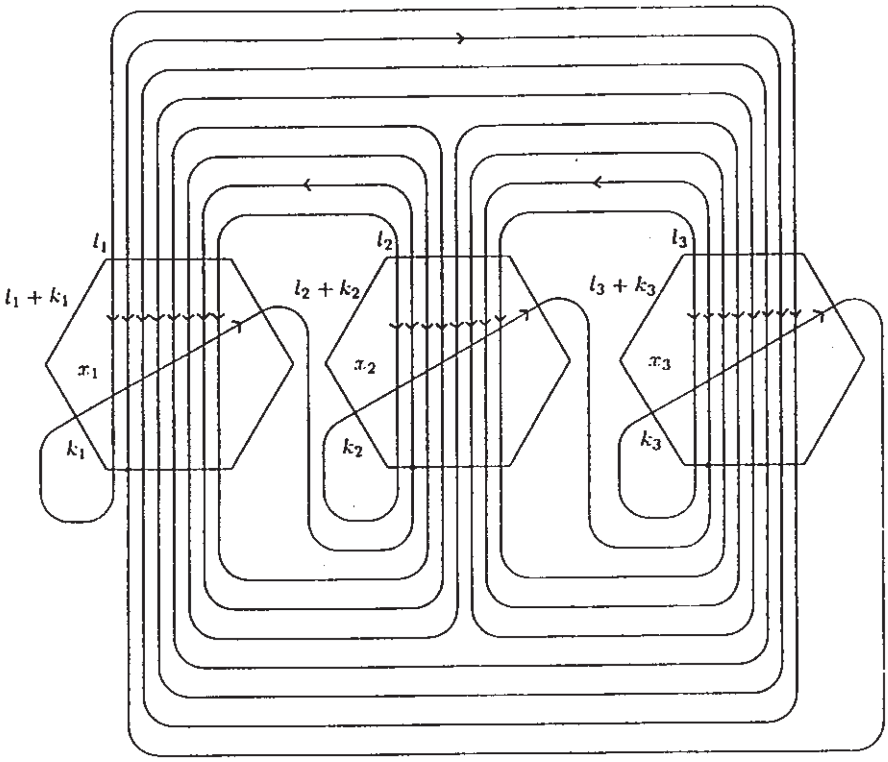

Figure 2: A RR–system inducing the Kim-Vesnin groups (case $n = 3$, $m = 2$, $l_i = l$, $k_i = \varepsilon$, $\varepsilon = \pm 1$)

3 The Kim-Vesnin Groups

We start with two interesting families of cyclically presented groups, due to Kim and Vesnin [19], which include the Fibonacci groups [12] [15] [22] and the Sieradski groups [5], respectively. These groups were proved to encode the cyclic branched coverings of the 2-bridge knots $(4m\ell \pm 1)/2\ell$ by using Dehn surgery on a certain chain of linked circles. We now present a simple alternative proof of the main theorem of [19]. It is immediate to see that our algebraic method works in general to study any cyclic branched covering of a 2-bridge knot by a cyclically presented group.

Let us consider the cyclically presented groups $G_n(m, \ell; \epsilon)$ with n generators x_i, and n relations ($\epsilon = \pm 1$)

$$(x_i^{-\ell} x_{i+1}^{\ell})^m x_{i+1}^{\epsilon} = (x_{i+1}^{-\ell} x_{i+2}^{\ell})^m \qquad \text{(indices mod } n\text{)}.$$

Figure 2 shows that these presentations are geometric (set $\ell_i = \ell$ and $k_i = \epsilon$ for any $i = 1, \ldots, n$). Here we have depicted the case $n = 3$ and $m = 2$, but one can easily obtain the general RR-system (with n generators) as follows. Take n hexagons $D_1, \ldots, D_n$ cyclically ordered in the plane, and label their stations by integers ℓ_i, k_i, and $\ell_i + k_i$, for any $i = 1, \ldots, n$. Then repeat the pictures of tracks and routes so that the resulting graph admits a rotational symmetry of order n (in fact, this symmetry is induced by the automorphism θ of the group presentation sending the generator x_i to x_{i+1}, where the indices are taken modulo n).

The following is the main result of [19]:

Theorem 3.1. *The cyclically presented group $G_n(m, \ell; \epsilon)$ is isomorphic to the fundamental group of the n-fold cyclic covering $M_n((4m\ell + \epsilon)/2\ell)$ of the 3-sphere branched over the 2-bridge knot $(4m\ell + \epsilon)/2\ell$, where $\epsilon = \pm 1$. The polynomial associated with $G_n(m, \ell; \epsilon)$ is the Alexander polynomial*

$$\Delta_{m,\ell}(t) = m\ell t^2 - (2m\ell + \epsilon)t + m\ell$$

of $(4m\ell + \epsilon)/2\ell$. The manifolds $M_n((4m\ell + \epsilon)/2\ell)$ are hyperbolic for all $n \geq 3$ if $m \geq 2$ or $\ell \geq 2$, and $M_n(5/2)$ are hyperbolic for all $n \geq 4$. In these cases, $G_n(m, \ell; \epsilon)$ are hyperbolic groups (hence infinite) which encode the corresponding manifolds.

Proof. For convenience, assume $\epsilon = +1$ (the other case is analogous), and simply denote $G_n(m, \ell; \epsilon)$ by $G_n(m, \ell)$. Let us consider the split extension group $H_n(m, \ell)$ of $G_n(m, \ell)$ by $\mathbb{Z}_n =< \theta : \theta^n = 1 >$, θ being the automorphism of F_n given by $\theta(x_i) = x_{i+1}$ (indices mod n). Then $H_n(m, \ell)$ has a finite presentation with generators θ and x, and relations $\theta^n = 1$ and $v(\theta, x) = 1$, where

$$\begin{aligned} v(\theta, x) &= w(x, \theta^{-1}x\theta, \theta^{-2}x\theta^2) \\ &= (\theta^{-2}x^{-\ell}\theta^2\theta^{-1}x^{\ell}\theta)^m(x^{-\ell}\theta^{-1}x^{\ell}\theta)^m\theta^{-1}x\theta \\ &= (\theta^{-2}x^{-\ell}\theta x^{\ell}\theta)^m(x^{-\ell}\theta^{-1}x^{\ell}\theta)^m\theta^{-1}x\theta. \end{aligned}$$

Setting $x = \theta\lambda$, we get the new presentation
$H_n(m, \ell) =< \theta, \lambda : \theta^n = 1$,

$$(\theta^{-2}(\lambda^{-1}\theta^{-1})^{\ell}\theta(\theta\lambda)^{\ell}\theta)^m((\lambda^{-1}\theta^{-1})^{\ell}\theta^{-1}(\theta\lambda)^{\ell}\theta)^m\theta^{-1}\theta\lambda\theta > .$$

The second relation can be simplified as follows:

$$\theta(\theta^{-2}(\lambda^{-1}\theta^{-1})^{\ell}\theta(\theta\lambda)^{\ell}\theta)^m = \lambda^{-1}(\theta^{-1}(\lambda^{-1}\theta^{-1})^{\ell}\theta(\theta\lambda)^{\ell})^m$$

and whence

$$(\theta^{-1}(\lambda^{-1}\theta^{-1})^{\ell}\theta(\theta\lambda)^{\ell})^m\theta = \lambda^{-1}(\theta^{-1}(\lambda^{-1}\theta^{-1})^{\ell}\theta(\theta\lambda)^{\ell})^m.$$

This relation can be written in the form $w\theta = \lambda^{-1}w$, where

$$\begin{aligned} w &= (\theta^{-1}(\lambda^{-1}\theta^{-1})^{\ell}\theta(\theta\lambda)^{\ell}\theta)^m \\ &= [\theta, (\theta\lambda)^{\ell}]^m \end{aligned}$$

(here we have set $[\theta, (\theta\lambda)^{\ell}] = \theta^{-1}(\lambda^{-1}\theta^{-1})^{\ell}\theta(\theta\lambda)^{\ell})$. Therefore the split extension group $H_n(v)$ has the presentation

$$H_n(v) =< \theta, \lambda : \theta^n = \lambda^n = 1, \quad w\theta = \lambda^{-1}w > .$$

The added relation $\lambda^n = 1$ is an immediate consequence of the other two relations in the group presentation. We have included it to give a better understanding of the fact that $H_n(v)$ is the fundamental group of a well-precised orbifold. Now we prove this claim. Because

$$w = \theta^{\epsilon_1}\lambda^{\epsilon_2}\cdots\lambda^{\epsilon_{4m\ell-2}}\theta^{\epsilon_{4m\ell-1}}\lambda^{\epsilon_{4m\ell}},$$

where the exponent ϵ_i is the sign (± 1) of $2i\ell$ reduced mod $8m\ell+2$ in the interval $(-4m\ell - 1, 4m\ell + 1)$, the word w corresponds to the 2-bridge knot $(4m\ell + 1)/2\ell$. In particular, the group $< \theta, \lambda : w\theta = \lambda^{-1}w >$ is isomorphic to the knot group of $(4m\ell + 1)/2\ell$, and the generator θ corresponds to a meridian of the knot. Comparing above presentations of $H_n(v)$ and of the knot group, we see that $H_n(v)$ is the fundamental group (in the sense of [11]) of the orbifold $\mathcal{O}((4m\ell+1)/2\ell; n)$ whose underlying space is the 3-sphere and whose singular set is the 2-bridge knot $(4m\ell+1)/2\ell$ with branch index equal to n. This orbifold is hyperbolic for any $n \geq 3$ if $m \geq 2$ or $\ell \geq 2$, and $\mathcal{O}(5/2; n)$ is hyperbolic for any $n \geq 4$. Furthermore, $\mathcal{O}((4m\ell+1)/2\ell; 2)$ is spherical, and $\mathcal{O}(5/2; 3)$ is Euclidean. Hence the orbifold group $H_n(v)$ is infinite when it is hyperbolic. The fact that the group $G_n(m, \ell)$ is a normal subgroup of $H_n(v)$ of index n implies that $G_n(m, \ell)$ is infinite for any $n \geq 3$ if $m \geq 2$ or $\ell \geq 2$, and for any $n \geq 4$ if $m = \ell = 1$.

■

As shown in Figure 3, the orbifold $\mathcal{O} = \mathcal{O}((4m\ell - 1)/2\ell; n)$ has a rotational symmetry of order 2 such that its fixed point set is disjoint from the 2-bridge knot $(4m\ell-1)/2\ell$. Factorizing by this symmetry yields an orbifold with underlying space $\mathbb{S}^3$ and singular set the 2-component link $L(m, \ell)$, depicted in Figure 3, with branch indices 2 and n on its components (which are equivalent). Let us denote this orbifold by $\mathcal{O}_L = \mathcal{O}(L(m, \ell); 2, n)$. Let $X = X((4m\ell - 1)/2\ell)$ be the universal covering space of $M_n = M_n((4m\ell - 1)/2\ell)$. The fundamental groups $G(m, \ell)$, $\Gamma(m, \ell)$, and $\Omega(m, \ell)$ of M_n, $\mathcal{O}$, and $\mathcal{O}_L$, respectively, act on X. We have a covering diagram

$$M_n \xrightarrow{n} \mathcal{O} \xrightarrow{2} \mathcal{O}_L$$

which corresponds to the subgroup embeddings $G(m, \ell) \triangleleft \Gamma(m, \ell) \triangleleft \Omega(m, \ell)$. To obtain $\Omega(m, \ell)$ we can use a Wirtinger presentation of the link group of $L(m, \ell)$ which has two generators a and b (corresponding to the arcs with the same labels in Figure 3), and one relator between them. According to [11], $\Omega(m, \ell)$ admits the group presentation obtained from that of the link group by adding relations $a^2 = b^n = 1$. Let us consider the natural epimorphism ψ from $\Omega(m, \ell)$ to $\mathbb{Z}_2 \oplus \mathbb{Z}_n$. By construction of the 2-fold covering $\mathcal{O} \to \mathcal{O}_L$, the loop a lifts to a trivial loop in $\Gamma(m, \ell)$

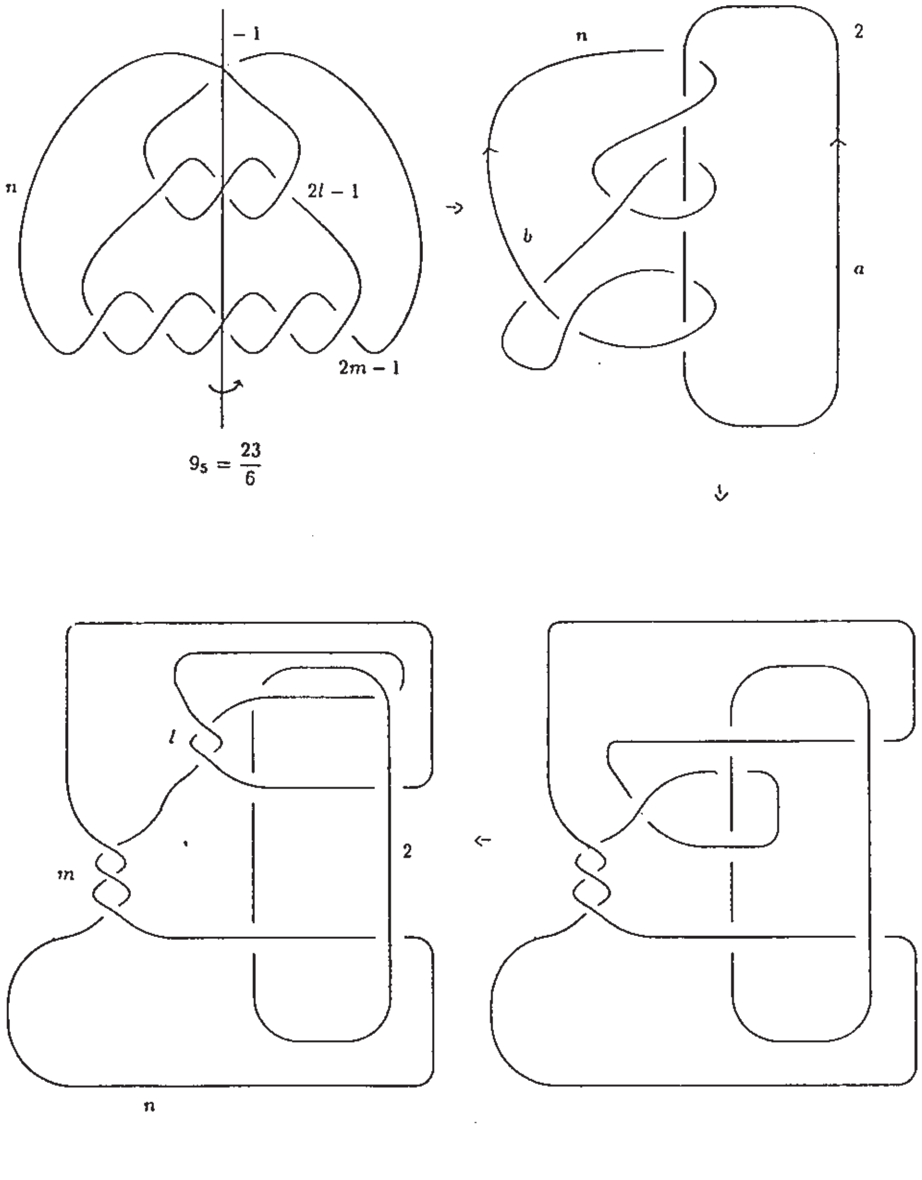

Figure 3: The orbifolds $\mathcal{O}((4ml-1)/2l;n)$ and $\mathcal{O}(L(m,l);2,n)$ (case $m=3$ and $l=2$)

while b lifts to a loop which generates a cyclic subgroup of order n. This implies that $\Gamma(m,\ell) = \psi^{-1}(\mathbb{Z}_n)$. For the $(2n)$-fold cyclic covering $M_n \to \mathcal{O}_L$, both loops a and b lift to trivial loops in $G(m,\ell)$, so $G(m,\ell) = \mathrm{Ker}\psi$. Setting $\Lambda(m,\ell) = \psi^{-1}(\mathbb{Z}_2)$, we have immediately the sequence of normal subgroups $G(m,\ell) \triangleleft \Lambda(m,\ell) \triangleleft \Omega(m,\ell)$. Since $\Lambda(m,\ell)$ acts on X by isometries, it defines an orbifold X/Λ. Then there is also a covering diagram

$$M_n \xrightarrow{2} X/\Lambda \xrightarrow{n} \mathcal{O}_L.$$

The second covering is cyclic and it is branched over the component with index n of the singular set of $\mathcal{O}_L$. It is evident from Figure 3 that there is an involution of $L(m,\ell)$ which changes the two components. Therefore the singular set of $\mathcal{O}_L$ is equivalent to the link diagram obtained from that in Figure 3 exchanging the branch indices. Of course, the map $X/\Lambda \to \mathcal{O}_L$ is a cyclic covering of $\mathbb{S}^3$ branched over an unknotted circle. Since the component with branch index 2 is just the closed 3-strings braid $\sigma_1^{1/\ell}\sigma_2^{1/m}$ (see for example [2]), it lifts in X/Λ to the n-periodic closed 3-strings braid $(\sigma_1^{1/\ell}\sigma_2^{1/m})^n$. Thus the orbifold X/Λ has underlying space $\mathbb{S}^3$ and singular set $(\sigma_1^{1/\ell}\sigma_2^{1/m})^n$ with branch index 2, i.e. $X/\Lambda = \mathcal{O}((\sigma_1^{1/\ell}\sigma_2^{1/m})^n; 2)$.

Summarizing, we have proved the following result (the statement for the 2-bridge knot $(4m\ell+1)/2\ell$ derives in the same way by substituting ℓ with $-\ell$):

Proposition 3.2. *There is a commutative diagram of cyclic branched coverings ($\epsilon = \pm 1$)*

$$\begin{array}{ccc}
M_n((4m\ell+\epsilon)/2\ell) & = & M_n((4m\ell+\epsilon)/2\ell) \\
2\downarrow & & \downarrow n \\
\mathcal{O}((\sigma_1^{\epsilon/\ell}\sigma_2^{1/m})^n; 2) & & \mathcal{O}((4m\ell+\epsilon)/2\ell; n) \\
n\downarrow & & \downarrow 2 \\
\mathcal{O}(L(m,\epsilon\ell); 2, n) & = & \mathcal{O}(L(m,\epsilon\ell); 2, n)
\end{array}$$

Furthermore, each manifold $M_n((4m\ell+\epsilon)/2\ell)$ has Heegaard genus ≤ 2.

Since the rational number $(4m\ell-1)/2\ell$ can be expressed by the continued fraction

$$\frac{4m\ell-1}{2\ell} = 2\ell - 1 + \frac{1}{1+\frac{1}{2m-1}},$$

the corresponding 2-bridge knot admits the Conway normal form $C(2\ell-1,1,2m-1)$, and hence it has the same knot type as the pretzel knot $P(-1,2\ell-1,2m-1)$ (see for example Theorem 2.3.1 of [17]). By [21], p. 57, a Seifert matrix for such a knot is

$$V = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} = \begin{pmatrix} \ell & \ell \\ \ell-1 & m+\ell-1 \end{pmatrix}.$$

We can now apply the procedure described in [10] to determine the homology characters of the cyclic branched coverings of our knots $(4m\ell-1)/2\ell$. Since $\det V = m\ell$, and

$$\begin{aligned} w &= \text{g.c.d.}(v_{11}, v_{12}+v_{21}, v_{22}) \\ &= \text{g.c.d.}(\ell, 2\ell-1, m+\ell-1) = 1, \end{aligned}$$

the homology groups of $M_n((4m\ell+\epsilon)/2\ell)$ can be completely computed from [10] (for $\epsilon = 1$ see the remark above).

Proposition 3.3. *Let $M_n = M_n((4m\ell+\epsilon)/2\ell)$ be the n-fold cyclic covering of the 3-sphere branched over the 2-bridge knot $(4m\ell+\epsilon)/2\ell$, where $\epsilon = \pm 1$. Then the first integral homology group of M_n is*

$$H_1(M_n) \cong \begin{cases} \mathbb{Z}_{|(4m\ell+\epsilon)a_n|} \oplus \mathbb{Z}_{|a_n|} & n \quad \textit{even} \\ \mathbb{Z}_{|b_n|} \oplus \mathbb{Z}_{|b_n|} & n \quad \textit{odd} \end{cases}$$

where

$$\begin{aligned} &a_1 = a_2 = 1, \quad a_{n+2} = a_{n+1} + \epsilon m\ell a_n \\ &b_1 = 1, \quad b_2 = 1 + 2\epsilon m\ell, \quad b_{n+2} = b_{n+1} + \epsilon m\ell b_n. \end{aligned}$$

4 Study of more cyclic presentations

Let us consider the cyclically presented groups $G_n^1(k)$, $k \geq 3$, with n generators x_i, and n relations (indices mod n)

$$\left(\prod_{j=1}^{k-1} x_{i+2k-2j}^{-2} x_{i+2k-2j-1}\right) x_i^{-1} = \prod_{j=1}^{k-1} x_{i+2k-2j-1} x_{i+2k-2j-2}.$$

These presentations correspond to spines of closed orientable 3-manifolds: in fact, they arise from the RR-system depicted in Figure 4 (case $n = 4$

and $k = 3$). One can easily obtain the general case taking n hexagons (which correspond to generators x_i) cyclically ordered in the plane, and preserving the rotational n-symmetry of the graph.

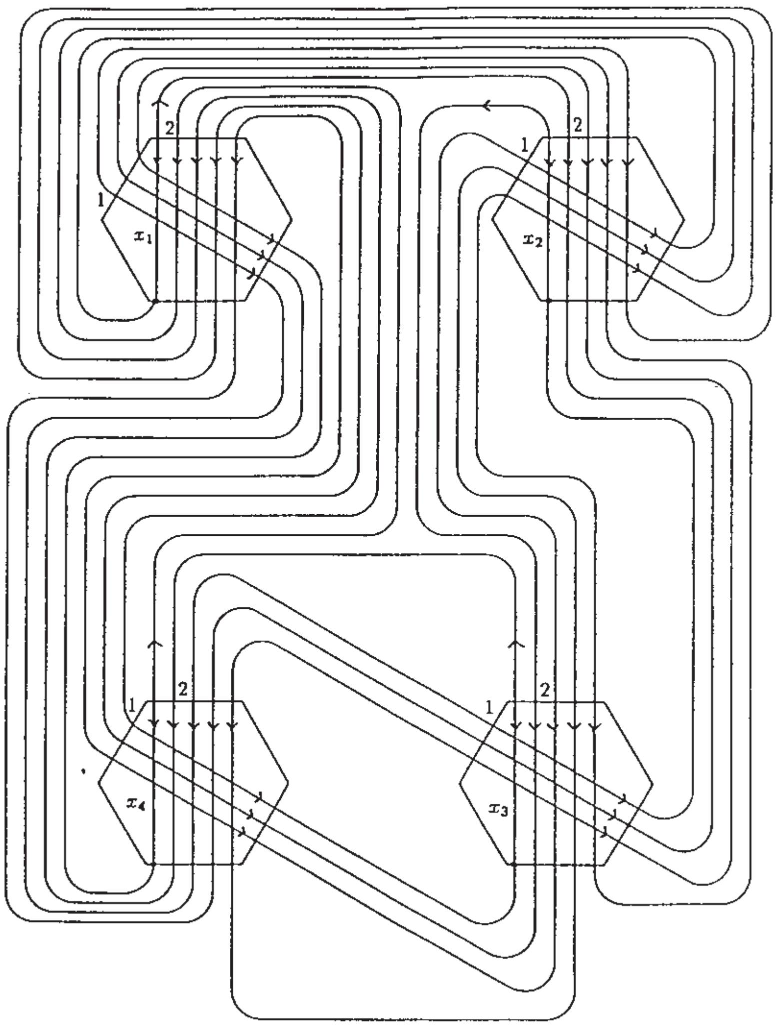

Figure 4: A RR–system inducing the cyclically presented group $G_n^1(k)$ (case $k = 3$ and $n = 4$)

The split extension group $H_n^1(k)$ of $G_n^1(k)$ has a presentation with generators θ and x (θ being the automorphism defined in Section 1), and relations $\theta^n = 1$ and $v(\theta, x) = 1$. The second relation is equivalent to

$$(\prod_{j=1}^{k-1} \theta^{-2k+2j} x^{-2} \theta^{2k-2j} \theta^{-2k+2j+1} x \theta^{2k-2j-1}) x^{-1}$$
$$= \prod_{j=1}^{k-1} \theta^{-2k+2j+1} x^{-2} \theta^{2k-2j-1} \theta^{-2k+2j+2} x \theta^{2k-2j-2},$$

hence

$$(x^{-2}\theta x \theta)^{k-1} x^{-1} = \theta (x^{-2}\theta x\theta)^{k-1}\theta^{-1}.$$

Setting $x = \theta\lambda$, we get the new presentation

$$H_n^1(k) =< \theta, \lambda : \theta^n = 1,$$
$$(\lambda^{-1}\theta^{-1}\lambda^{-1}\theta\lambda\theta)^{k-1}\lambda^{-1} = \theta(\lambda^{-1}\theta^{-1}\lambda^{-1}\theta\lambda\theta)^{k-1} > .$$

The second relation can be written in the form $w\lambda^{-1} = \theta w$, where

$$w = (\lambda^{-1}\theta^{-1}\lambda^{-1}\theta\lambda\theta)^{k-1}.$$

In particular, we obtain the relation $\lambda^n = 1$ as a direct consequence of the relations in the group presentation $H_n^1(k)$. Because

$$w = (\lambda^{-1})^{\epsilon_1}\theta^{\epsilon_2} \cdots \theta^{\epsilon_{6k-8}}(\lambda^{-1})^{\epsilon_{6k-7}}\theta^{\epsilon_{6k-6}},$$

where the exponent ϵ_i is the sign (± 1) of $i(2k-2)$ reduced mod $12k-10$ in the interval $(-6k+5, 6k-5)$, the word w corresponds to the 2-bridge knot $(6k-5)/(2k-2)$, which is $(2k+1)_3$, $k \geq 3$ according to notation of Rolfsen's book [27] (compare also [2]). So the group $< \theta, \lambda : w\lambda^{-1} = \theta w >$, where w is as above, is the knot group of $(2k+1)_3$, and the generator θ corresponds to a meridian of the knot. So $H_n^1(k)$ is the fundamental group of the orbifold $\mathcal{O}((2k+1)_3; n)$. Therefore, the cyclically presented group $G_n^1(k)$ corresponds to a spine of the n-fold cyclic covering $M_n^1(k)$ of the 3-sphere branched over $(2k+1)_3$.

The polynomial $f_w(t)$ associated with $G_n^1(k)$ is

$$2t^{2k-2} + 3\sum_{i=1}^{2k-3}(-1)^i t^i + 2,$$

which coincides with the Alexander polynomial of $(2k+1)_3$.

Let us consider the cyclically presented groups $G_n^2(k)$, $k \geq 2$, with n generators x_i and n relations (indices mod n)

$$\prod_{j=1}^{k} x_{i+2k-2j+1} x_{i+2k-2j}^{-2} = \left(\prod_{j=0}^{k-1} x_{i+2k-2j} x_{i+2k-2j-1}^{-2}\right) x_i^{-1}.$$

As above, these presentations are all geometric because it is immediate to construct a RR-system inducing them. Hence there are closed orientable 3-manifolds $M_n^2(k)$ whose fundamental groups are isomorphic to $G_n^2(k)$. The split extension group $H_n^2(k)$ of $G_n^2(k)$ has generators θ and x related by relations $\theta^n = 1$ and

$$\prod_{j=1}^{k} \theta^{-2k+2j-1} x \theta^{2k-2j+1} \theta^{-2k+2j} x^{-2} \theta^{2k-2j}$$
$$= \left(\prod_{j=0}^{k-1} \theta^{-2k+2j} x \theta^{2k-2j} \theta^{-2k+2j+1} x^{-2} \theta^{2k-2j-1}\right) x^{-1},$$

which is equivalent to

$$(\theta^{-1} x^2 \theta^{-1} x^{-1})^k (\theta x \theta x^{-2})^{k-1} \theta x \theta^{-1} x = 1.$$

Let λ be such that $x = \theta\lambda$. Then the last relation becomes

$$(\lambda\theta\lambda\theta^{-1}\lambda^{-1}\theta^{-1})^{k-1} \lambda\theta\lambda\theta^{-1}\lambda^{-1} (\theta\lambda\theta\lambda^{-1}\theta^{-1}\lambda^{-1})^{k-1} \theta\lambda\theta\lambda^{-1}\theta^{-1} = 1,$$

or equivalently

$$(\theta^{-1}\lambda^{-1}(\theta\lambda\theta\lambda^{-1}\theta^{-1}\lambda^{-1})^{k-1}\theta\lambda)\theta = \lambda^{-1}(\theta^{-1}\lambda^{-1}(\theta\lambda\theta\lambda^{-1}\theta^{-1}\lambda^{-1})^{k-1}\theta\lambda).$$

Thus it can be written in the form $w\theta = \lambda^{-1} w$, where

$$w = \theta^{-1}\lambda^{-1}(\theta\lambda\theta\lambda^{-1}\theta^{-1}\lambda^{-1})^{k-1}\theta\lambda.$$

One can now verify that such w coincides with the word

$$\theta^{\epsilon_1} (\lambda^{-1})^{\epsilon_2} \cdots (\lambda^{-1})^{\epsilon_{6k-4}} \theta^{\epsilon_{6k-3}} (\lambda^{-1})^{\epsilon_{6k-2}},$$

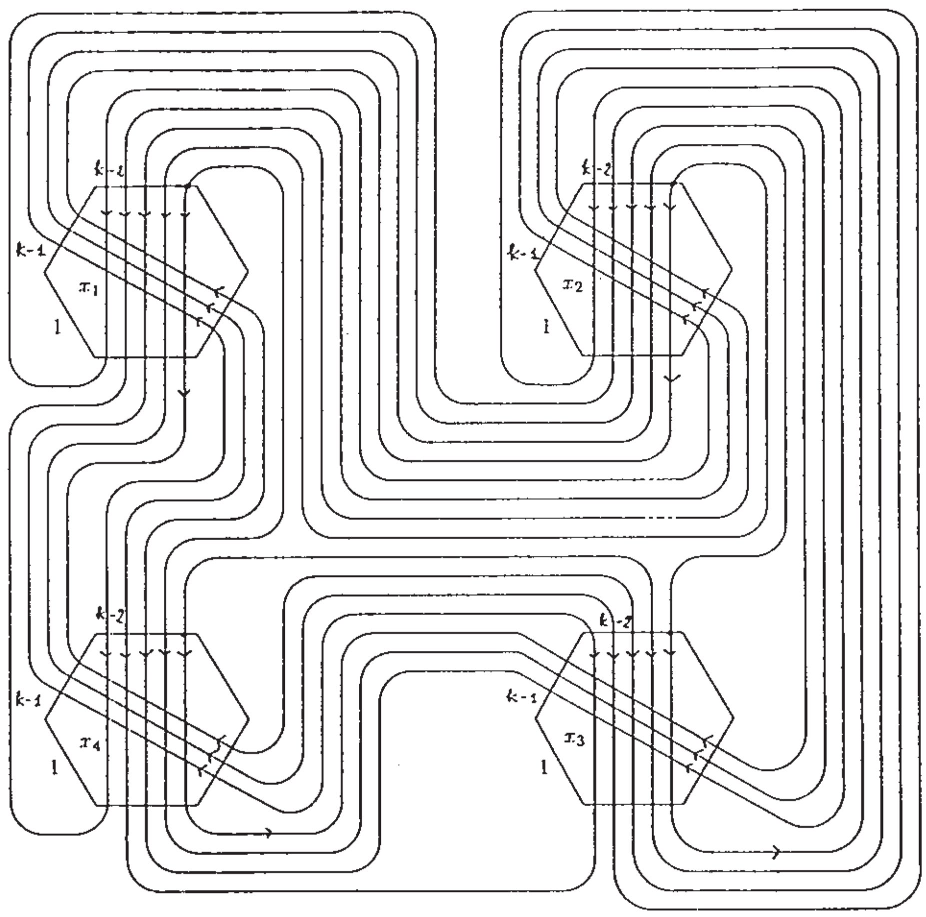

Figure 5: A RR–system inducing the cyclically presented group $G_n^3(k)$ (case $n = 4$)

where ϵ_i is the sign of $i(2k)$ reduced mod $12k-2$ in the interval $(-6k+1, 6k-1)$. This means that w corresponds to the 2-bridge knot $(6k-1)/2k$, $k \geq 2$, which is $(2k+2)_2$ in Rolfsen's notation. So the cyclically presented groups $G_n^2(k)$, $k \geq 2$, encode the n-fold cyclic coverings $M_n^2(k)$ of the 3-sphere branched over $(2k+2)_2$.

The polynomial $f_w(t)$ associated with $G_n^2(k)$ is

$$t^{2k} + 3\sum_{i=1}^{2k-3}(-1)^i t^i + 1$$

which coincides with the Alexander polynomial of $(2k+2)_2$.

Let $G_n^3(k)$, $k \geq 4$, be the group cyclically presented by generators $x_1, \ldots, x_n$ and n relations (indices mod n)

$$x_{i+1}^{-k+2} x_{i+2}^{k-1} x_{i+3}^{-k+2} x_{i+4}^{k-2} x_{i+3}^{-k+1} x_{i+2}^{k-2} x_{i+1}^{-k+1} x_i^{k-2} = 1.$$

These presentations are geometric since they are induced by the RR-system shown in Figure 5 (case $n=4$). One can easily obtain the general case taking n hexagons (which correspond to generators x_i) cyclically ordered in the plane, and preserving the rotational n-symmetry of the graph.

The split extension $H_n^3(k)$ of $G_n^3(k)$ is presented by generators θ and x, and relations $\theta^n = 1$ and

$$(\theta^{-1}x^{-k+2}\theta)(\theta^{-2}x^{k-1}\theta^2)(\theta^{-3}x^{-k+2}\theta^3)(\theta^{-4}x^{k-2}\theta^4)(\theta^{-3}x^{-k+1}\theta^3)$$
$$(\theta^{-2}x^{k-2}\theta^2)(\theta^{-1}x^{-k+1}\theta)x^{k-2} = 1,$$

which is equivalent to

$$\theta^{-1}x^{-k+2}\theta^{-1}x^{k-1}\theta^{-1}x^{-k+2}\theta^{-1}x^{k-2}\theta x^{-k+1}\theta x^{k-2}\theta x^{-k+1}\theta x^{k-2} = 1.$$

Setting $x = \theta\lambda^{-1}$ yields a new presentation for $H_n^3(k)$ with generators θ and λ, and relations $\theta^n = 1$ and

$$\theta^{-1}(\lambda\theta^{-1})^{k-2}\theta^{-1}(\theta\lambda^{-1})^{k-1}\theta^{-1}(\lambda\theta^{-1})^{k-2}\theta^{-1}(\theta\lambda^{-1})^{k-2}\theta(\lambda\theta^{-1})^{k-1}\theta$$
$$(\theta\lambda^{-1})^{k-2}\theta(\lambda\theta^{-1})^{k-1}\theta(\theta\lambda^{-1})^{k-2} = 1.$$

Since this relation is equivalent to

$$(\lambda\theta^{-1})^{k-2}\theta^{-1}(\theta\lambda^{-1})^{k-1}\theta^{-1}(\lambda\theta^{-1})^{k-2}\theta^{-1}(\theta\lambda^{-1})^{k-1}\lambda$$
$$= \theta(\lambda\theta^{-1})^{k-2}\theta^{-1}(\theta\lambda^{-1})^{k-1}\theta^{-1}(\lambda\theta^{-1})^{k-2}\theta^{-1}(\theta\lambda^{-1})^{k-1},$$

it can be written in the form $w\lambda = \theta w$, where

$$\begin{aligned} w &= (\lambda\theta^{-1})^{k-2}\theta^{-1}(\theta\lambda^{-1})^{k-1}\theta^{-1}(\lambda\theta^{-1})^{k-2}\theta^{-1}(\theta\lambda^{-1})^{k-1} \\ &= (\lambda\theta^{-1})^{k-2}\lambda^{-1}(\theta\lambda^{-1})^{k-2}\lambda\lambda^{-1}\theta^{-1}(\lambda\theta^{-1})^{k-2}\lambda^{-1}(\theta\lambda^{-1})^{k-2}\lambda\lambda^{-1}\theta^{-1}\theta \\ &= ([(\theta\lambda^{-1})^{k-2},\lambda]\lambda^{-1}\theta^{-1})^2\theta \end{aligned}$$

(here we have set $[(\theta\lambda^{-1})^{k-2},\lambda] = (\lambda\theta^{-1})^{k-2}\lambda^{-1}(\theta\lambda^{-1})^{k-2}\lambda)$. In particular, we obtain the relation $\lambda^n = 1$ as a direct consequence of the relations in the group presentation $H_n^3(k)$. One can now verify that the group $< \theta,\lambda : w\lambda = \theta w >$, where w is as above, is just the group of the 2-bridge knot $(8k-13)/(2k-3)$, $k \geq 4$, which is denoted by $(2k)_4$ in the appendix table of [27].

The polynomial associated with $G_n^3(k)$ is

$$(k-2)t^4 - (2k-3)t^3 + (2k-3)t^2 - (2k-3)t + k - 2,$$

which is the Alexander polynomial of $(2k)_4$.

Let $G_n^4(k)$, $k \geq 4$, be the cyclically presented group with n generators x_i, and n relations (indices mod n)

$$x_{i+1}^{-k+1}x_{i+2}^{k-2}x_{i+3}^{-k+1}x_{i+4}^{k-1}x_{i+3}^{-k+2}x_{i+2}^{k-1}x_{i+1}^{-k+2}x_i^{k-1} = 1.$$

One can proceed as above to show that these presentations are geometric, and to construct a presentation for the split extension $H_n^4(k)$ of type

$$< \theta,\lambda : \theta^n = \lambda^n = 1, \quad w\lambda = \theta w >,$$

where

$$\begin{aligned} w &= ((\theta\lambda^{-1})^{k-1}\lambda^{-1}(\lambda\theta^{-1})^{k-1}\lambda\lambda^{-1}\theta^{-1})^2\theta \\ &= ([(\lambda\theta^{-1})^{k-1},\lambda]\lambda^{-1}\theta^{-1})^2\theta. \end{aligned}$$

Since this word corresponds to the 2-bridge knot $(2k+1)_4$, $k \geq 4$, the cyclically presented group $G_n^4(k)$ encodes the n-fold cyclic covering $M_n^4(k)$ of the 3-sphere branched over such a knot.

The polynomial associated with $G_n^4(k)$ is

$$(k-1)t^4 - (2k-3)t^3 + (2k-3)t^2 - (2k-3)t + k - 1,$$

which is the Alexander polynomial of $(2k+1)_4$, $k \geq 4$.

Let us consider the cyclically presented group $G_n^5(k)$, $k \geq 4$, with n generators x_i, and n relations (indices mod n)

$$(\prod_{j=0}^{2k-8} x_{i+j}x_{i+j+1}^{-1}x_{i+j+2}x_{i+j+1}^{-1}x_{i+j+2}^{2}x_{i+j+3}^{-1})$$
$$(\prod_{j=0}^{2k-8} x_{i+2k-j-6}x_{i+2k-j-5}^{-1}x_{i+2k-j-6}x_{i+2k-j-7}^{-1}x_{i+2k-j-6}x_{i+2k-j-7}^{-2})$$
$$x_i x_{i+1}^{-1} = 1.$$

These presentations are all geometric; in fact, it is not difficult to construct a RR-system inducing them. As above, we have proved that $G_n^5(k)$ encodes the n-fold cyclic covering $M_n^5(k)$ of the 3-sphere branched over the 2-bridge knot $(2k)_6$.

The polynomial associated with $G_n^5(k)$ is

$$2t^{2k-4} - 6t^{2k-5} + 7\sum_{i=2}^{2k-6}(-1)^i t^i - 6t + 2,$$

which is the Alexander polynomial of $(2k)_6$.

Finally, let $G_n^6(k)$, $k \geq 2$, be the cyclically presented group with n generators x_i, and n relations (indices mod n)

$$x_i x_{i+1}^{-1} x_{i+3}^{-1}(x_{i+2}x_{i+1}^{-1}x_{i+3}^{-1})^k(x_{i+4}x_{i+2}x_{i+3}^{-1})^k$$
$$x_{i+4}x_{i+2}(x_{i+1}^{-1}x_{i+3}^{-1}x_{i+2})^k(x_i x_{i+1}^{-1}x_{i+2})^k = 1.$$

One can again verify that these presentations are geometric, and that they encode the cyclic coverings $M_n^6(k)$ of the 3-sphere branched over the 2-bridge knots $(12k+5)/(6k+1)$. Further, the polynomial associated with $G_n^6(k)$ is the Alexander polynomial of such a knot, i.e.

$$(k+1)t^4 - (3k+1)t^3 + (4k+1)t^2 - (3k+1)t + k + 1.$$

Summarizing, we have proved the following

Theorem 4.1. *The cyclically presented groups $G_n^i(k)$, defined in this section, correspond to spines of the n-fold cyclic coverings $M_n^i(k)$ of the*

3-sphere branched over the 2-bridge knots specified above. The split extension group $H_n^i(k)$ of $G_n^i(k)$ is isomorphic to the fundamental group of the orbifold with underlying space the 3-sphere and singular set the correspondent 2-bridge knot with branch index n. The polynomial associated with $G_n^i(k)$ is the Alexander polynomial of that knot. The manifolds $M_n^i(k)$ are hyperbolic for all $n \geq 3$. In these cases, the groups $G_n^i(k)$ are hyperbolic (hence infinite), i.e. they are isomorphic to properly discontinuous cocompact groups of isometries which act without fixed points on the hyperbolic 3-space.

5 Quotients

The pairwise identification of (oppositely oriented) boundary faces of a triangulated 3-ball is another standard method for constructing (orientable) closed 3-manifolds. Classical examples are given by the Poincaré homology sphere and the Weber-Seifert manifold (see for example [29] and [31]). More recently, many authors have studied interesting classes of closed 3-manifolds obtained as quotients of triangulated 3-balls (see references). Of course, not every pairing of boundary faces of a triangulated 3-ball Q yields a closed 3-manifold. The resulting quotient complex K triangulates a closed pseudomanifold M. The troublesome points of M may be only the vertices of K arising from those of ∂Q. In fact, they have regular neighborhoods that are cones over closed (possibly non spherical) surfaces.

The following criterion is well-known [29]:

Theorem 5.1. *Suppose Q a triangulated 3-ball endowed with a pairing of (oppositely oriented) boundary faces. Let M be the closed (orientable) 3-dimensional pseudomanifold obtained from Q by identifying the 2-cell pairs on the boundary of Q. Then M is a manifold if and only if its Euler characteristic vanishes.*

Let us consider a triangulated polyhedron $P_n(k)$, $k \geq 2$, which realizes the tessellation of the boundary of a 3-ball shown in Figure 6 (cases $k = 3$ and $k = 4$). It consists of $2n$ quadrilaterals, $2n$ $(2k)$-gons, $4n+2kn$ edges, and $2kn + 2$ vertices. The n quadrilaterals in the northern hemisphere (resp. in the equatorial zone) are labelled by F_i (resp. F_i') for any $i = 1, \ldots, n$. The n $(2k)$-gons in the southern hemisphere (resp. in the equatorial zone) are labelled by E_i (resp. E_i') for any $i = 1, \ldots, n$. The labelling of the edges and their orientations are as depicted in Figure 6.

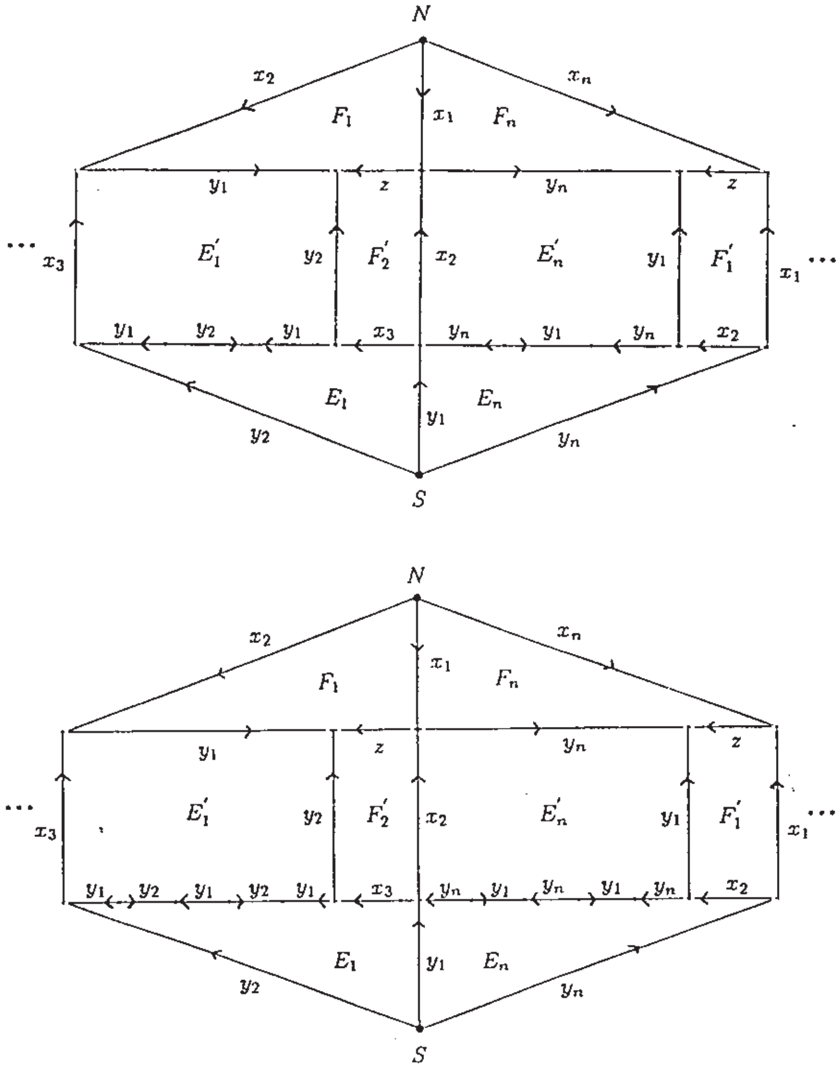

Figure 6: The polyhedral 3-ball $P_n(k)$ and the pairing of its boundary faces (cases $k = 3$ and $k = 4$)

Of course, to each face there corresponds precisely one distinct face with the opposite orientation. Identifying F_i with F_i' and E_i with E_i' for any i yields a 3-dimensional complex $K_n(k)$ which triangulates a closed orientable pseudomanifold $M_n(k)$. By construction, $K_n(k)$ consists of two vertices, $2n+1$ edges (denoted by x_i, y_i, and z in Figure 6), n quadrilaterals, n $(2k)$-gons, and one 3-cell. After the identification, the two equivalence classes of vertices can be represented by the north and south poles, respectively. Since the Euler characteristic of $M_n(k)$ vanishes, Theorem 5.1 implies that $M_n(k)$ is a closed orientable 3-manifold. Deforming the edge z to a point, we get a cellular decomposition of $M_n(k)$ with only one vertex, and $2n$ 1-cells (loops with that vertex as their base point), denoted again x_i and y_i. This permits to determine a finite presentation of the fundamental group $G_n(k)$ of $M_n(k)$. There are $2n$ generators x_i and y_i related by n relations of type (indices mod n)

$$x_{i+1}y_i x_i^{-1} = 1,$$

and n relations of type

$$y_i x_{i+2}(y_i y_{i+1}^{-1})^{k-1} = 1,$$

arising from the quadrilaterals $F_i \equiv F_i'$ and from the $(2k)$-gons $E_i \equiv E_i'$, respectively. Substituting $y_i = x_{i+1}^{-1}x_i$ in the relations of the second type, we get a cyclic presentation for $G_n(k)$ with n generators x_i, and n relations (indices mod n)

$$x_{i+1}^{-1}x_i x_{i+2}(x_{i+1}^{-1}x_i x_{i+1}^{-1}x_{i+2})^{k-1} = 1.$$

The polynomial associated with this presentation is

$$f_w(t) = kt^2 - (2k-1)t + k$$

which is again the Alexander polynomial of the 2-bridge knot $(4k-1)/2$, or equivalently $(2k+1)_2$ according to Rolfsen's notation. In fact, we are going to prove that $M_n(k)$ is just the n-fold cyclic covering of the 3-sphere branched over $(2k+1)_2$. In particular, $G_n(k)$ is isomorphic to the group $G_n(k,1;-1)$ defined in Section 3. For this, we first observe that the above presentations are all geometric since they arise from the Heegaard diagrams depicted in Figure 7 (case $k=3$: we have constructed them directly from a suitable RR-system).

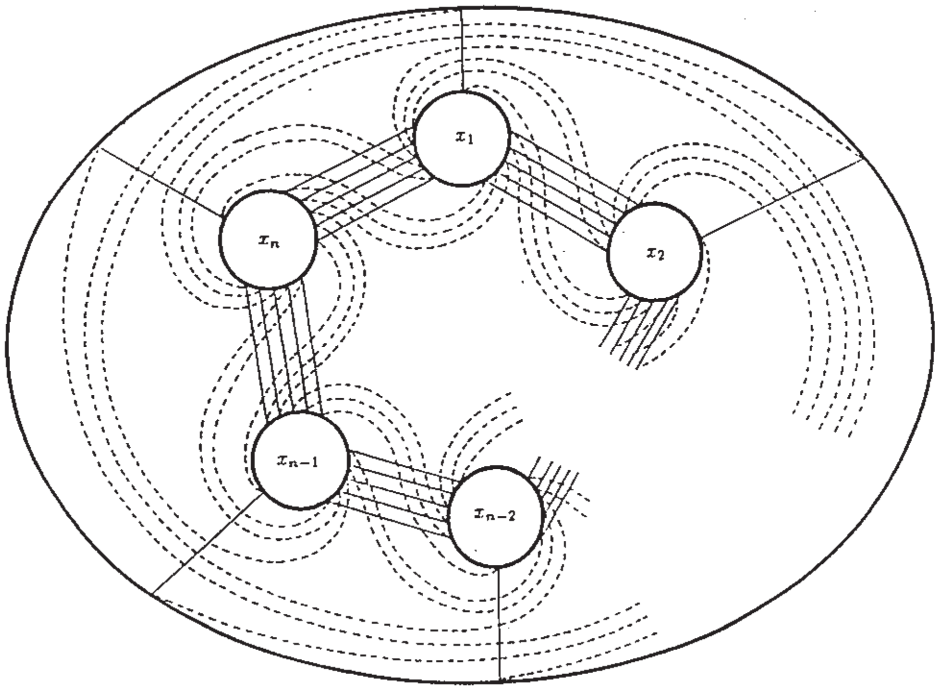

Figure 7: An $n-$symmetric Heegaard diagram of $M_n(k)$ (case $k = 3$)

These diagrams have an n-rotational symmetry which reproduces that of the polyhedron $P_n(k)$. Let $\theta = \theta_n$ be the clockwise rotation of $2\pi/n$ radians around the polar axis NS of the 3-ball $P_n(k)$. It is evident that the pairwise identifications of the polyhedral schemata are invariant under θ. Thus θ induces an orientation preserving homeomorphism, denoted again θ, of $M_n(k)$. The fixed point set of θ consists of the points arising from both the polar axis or the edge z. The quotient space $M_n(k)/\theta$ is homeomorphic to the 3-sphere $M_1(k) = \mathbb{S}^3$. Of course, the projection $M_n(k) \to \mathbb{S}^3$ is an n-fold cyclic branched covering. To determine the branch set it suffices to consider the 2-fold branched covering $M_2(k) \to \mathbb{S}^3$. We use the equivalence between the representation theories of closed orientable 3-manifolds via Heegaard diagrams and crystallizations (a special kind of colored graph), as proved in [3]. Colored graphs (resp. crystallizations) as a method of representation of simplicial (resp. contracted) pseudocomplexes, and a way for studying their topological properties from graph theory have been used by many authors (for details see the survey papers [1], [6], [9], and their references). By using an algorithm given in [8] we can immediately construct a 2-symmetric crystallization $\Gamma_2(k)$ of $M_2(k)$ which is equivalent to its 2-symmetric Heegaard diagram, discussed above. Figure 8 shows that $\Gamma_2(k)$ admits an involutory automorphism $\gamma = \gamma_k$ which interchanges 1-colored (resp. 3-colored) edges by 2-colored (resp. 4-colored) edges (in fact, γ is induced by the rotation of π radians around the x-axis). By [8] $\Gamma_2(k)$ represents the 2-fold covering of the 3-sphere branched over a knot or link. Moreover, there is a simple algorithm, described also in [8], for constructing the branch set of the covering. According to this algorithm, the branch set of the covering $M_2(k) \to \mathbb{S}^3$ is isomorphic to the orbit graph $\Gamma_2(k)/\gamma$. As one can see from Figure 8 (case $k = 3$), this is just the 2-bridge knot $(4k-1)/2$. The isomorphism $\Gamma_2(k)/\gamma \cong (4k-1)/2$ takes orbits of cycles, alternatively colored 1 and 2 (resp. 3 and 4) in $\Gamma_2(k)$, to the bridges (resp. arcs) of $(4k-1)/2$. Observe that one can proceed, as above, for constructing polyhedral schemata for all cyclic branched coverings of a 2-bridge knot. In fact, it suffices to draw Heegaard diagrams of those manifolds (with a rotational symmetry) corresponding to cyclic presentations of the fundamental group (for this, we can start, for example, from a suitable RR-system). Then we simply apply the procedure indicated in [15] to derive polyhedral 3-balls (endowed with a pairing of boundary faces) from those Heegaard diagrams.

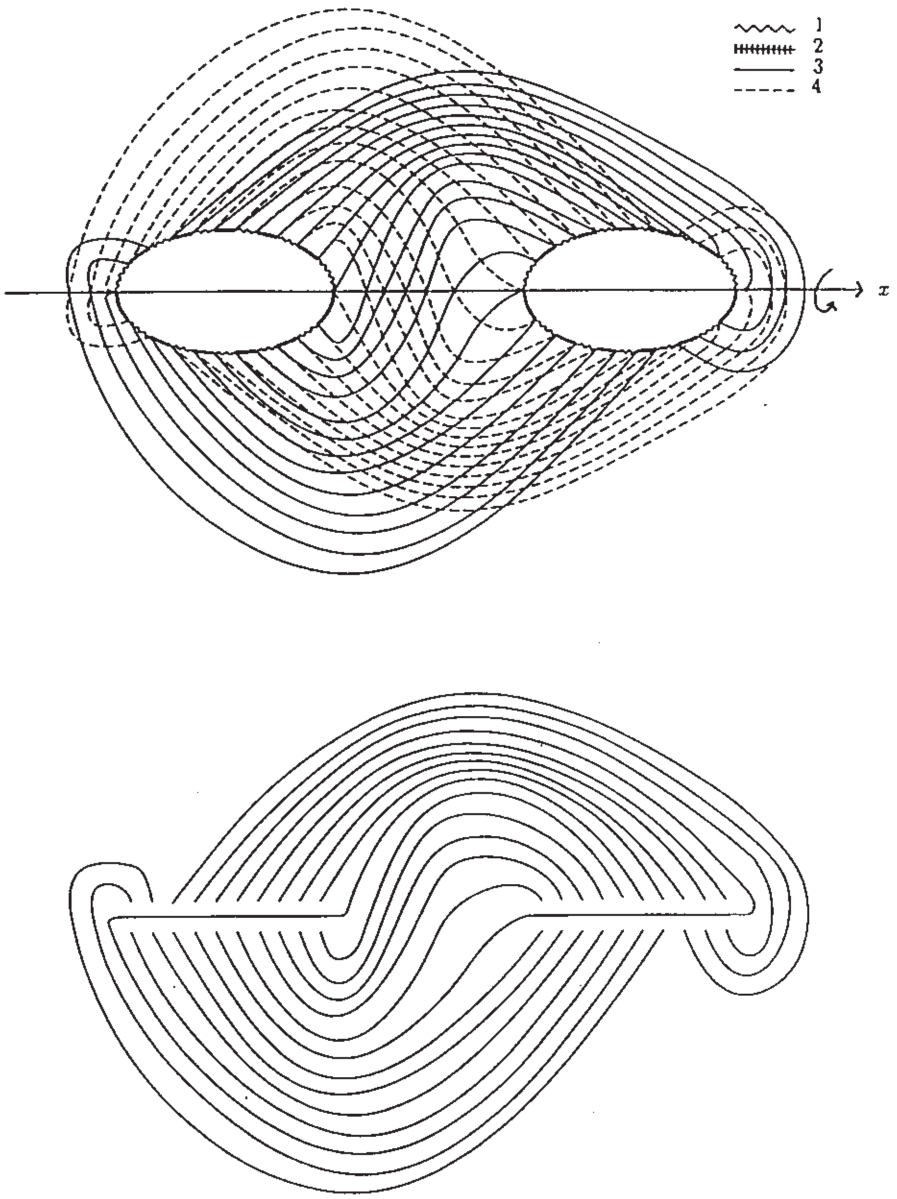

Figure 8: The 2–symmetric crystallization of $M_2(k)$ and the 2–bridge knot $(2k+1)_2$ (case $k=3$)

So we have proved

Theorem 5.2. *With the above notation, the triangulated 3-ball $P_n(k)$, $k \geq 2$, with the specified pairing of (oppositely oriented) boundary faces represents the n-fold covering $M_n(k)$ of the 3-sphere branched over the 2-bridge knot $(2k+1)_2$. The fundamental group $G_n(k)$ of $M_n(k)$ admits a cyclic presentation (which is alternative with respect to that of Section 3) with n generators x_i, and n relations (indices mod n)*

$$x_{i+1}^{-1}x_ix_{i+2}(x_{i+1}^{-1}x_ix_{i+1}^{-1}x_{i+2})^{k-1} = 1.$$

Furthermore, this presentation arises from a Heegaard diagram of $M_n(k)$ with an n-rotational symmetry.

6 Appendix

The following table shows a partial output of a computer program which generates cyclic presentations for the fundamental groups of the cyclic branched coverings of any 2-bridge knot up to nine crossings (we possess the results for 2-bridge knots with many more crossings but omit them as the table becomes too long). The first column contains also Rolfsen's notation of the considered 2-bridge knot α/β. The second column gives the word w which defines the cyclic group presentations encoding the cyclic branched coverings of α/β. The third column contains the polynomial $f_w(t)$ associated with the correspondent cyclic presentation (as one can verify, it coincides with the Alexander polynomial of the knot specified in the table).

2−bridge knots	*the words* w	*the polynomials* $f_w(t)$
$3_1 = \frac{3}{1}$	$x_0x_1^{-1}x_2$	$1-t+t^2$
$4_1 = \frac{5}{2}$	$x_0x_1^{-1}x_2x_1^{-2}$	$1-3t+t^2$
$5_1 = \frac{5}{1}$	$x_3x_4^{-1}x_2^{-1}x_0^{-1}x_1$	$1-t+t^2$ $-t^3+t^4$
$5_2 = \frac{7}{3}$	$x_1x_2^{-2}x_1x_0^{-2}x_1$	$2-3t+2t^2$
$6_1 = \frac{9}{4}$	$x_2^{-2}x_1^3x_0^{-2}x_1^2$	$2-5t+2t^2$
$6_2 = \frac{11}{4}$	$x_1x_2^{-1}x_4^{-1}x_3x_1x_2^{-1}x_3x_1x_0^{-1}x_2^{-1}x_3$	$1-3t+3t^2$ $-3t^3+t^4$
$6_3 = \frac{13}{5}$	$x_2x_3^{-1}x_4x_2x_3^{-1}x_2x_1^{-1}x_2x_0x_1^{-1}x_2x_1^{-1}x_3^{-1}$	$1-3t+5t^2$ $-3t^3+t^4$
$7_1 = \frac{7}{1}$	$x_5x_6^{-1}x_4^{-1}x_2^{-1}x_0^{-1}x_1x_3$	$1-t+t^2-t^3$ $+t^4-t^5+t^6$
$7_2 = \frac{11}{5}$	$x_1^{-3}x_2^3x_1^{-2}x_0^3$	$3-5t+3t^2$
$7_3 = \frac{13}{4}$	$x_3x_4^{-1}x_2^{-1}x_1x_3x_4^{-1}x_2^{-1}x_0^{-1}x_1x_3x_2^{-1}x_0^{-1}x_1$	$2-3t+3t^2$ $-3t^3+2t^4$
$7_4 = \frac{15}{4}$	$(x_1^2x_2^{-2})^2x_1x_0^{-2}x_1^2x_0^{-2}$	$4-7t+4t^2$
$7_5 = \frac{17}{7}$	$x_0x_1^{-1}x_3^{-1}x_2x_1^{-1}x_3^{-1}x_4x_2x_3^{-1}x_4x_2x_1^{-1}x_3^{-1}$ $x_2x_0x_1^{-1}x_2$	$2-4t+5t^2$ $-4t^3+2t^4$
$7_6 = \frac{19}{7}$	$x_0x_1^{-1}x_2^3x_3^{-1}x_4x_3^{-3}x_2x_1^{-1}x_2^2x_3^{-1}x_2x_1^{-3}$	$1-5t+7t^2$ $-5t^3+t^4$
$7_7 = \frac{21}{8}$	$x_1x_2^{-1}x_3^3x_4^{-1}x_3x_2^{-4}x_1x_0^{-1}x_1^3x_2^{-1}x_3x_2^{-3}$	$1-5t+9t^2$ $-5t^3+t^4$
$8_1 = \frac{13}{6}$	$x_2^{-3}x_1^4x_0^{-3}x_1^3$	$3-7t+3t^2$
$8_2 = \frac{17}{6}$	$x_4x_5^{-1}x_3^{-1}x_1^{-1}x_2x_4x_6x_5^{-1}x_3^{-1}x_1^{-1}x_2x_4x_5^{-1}$ $x_3^{-1}x_1^{-1}x_0x_2$	$1-3t+3t^2-3t^3$ $+3t^4-3t^5+t^6$
$8_3 = \frac{17}{4}$	$x_0^2x_1^{-2}x_2^2x_1^{-2}x_2^2x_1^{-3}x_0^2x_1^{-2}$	$4-9t+4t^2$
$8_4 = \frac{19}{5}$	$x_4^{-2}x_3^3x_2^{-2}x_1^3x_0^{-2}x_1^2x_2^{-3}x_3^2$	$2-5t+5t^2$ $-5t^3+2t^4$
$8_6 = \frac{23}{10}$	$(x_3x_1x_2^{-1})^2x_4^{-1}x_3x_2^{-1}x_4^{-1}(x_3x_1x_2^{-1})^2x_3$ $(x_1x_0^{-1}x_2^{-1})^2$	$2-6t+7t^2$ $-6t^3+2t^4$

$8_7 = \frac{23}{9}$	$x_3x_4^{-1}x_2^{-1}x_1x_3x_5x_4^{-1}x_2^{-1}x_3x_5x_6^{-1}x_4^{-1}x_2^{-1}$ $x_3x_5x_4^{-1}x_2^{-1}x_1x_3x_4^{-1}x_2^{-1}x_0^{-1}x_1$	$1-3t+5t^2-5t^3$ $+5t^4-3t^5+t^6$
$8_8 = \frac{25}{9}$	$x_2x_3^{-1}x_4x_2x_3^{-1}x_2x_1^{-1}x_3^{-1}x_2x_3^{-1}x_4x_2x_0$ $x_1^{-1}x_2x_1^{-1}x_3^{-1}x_2x_1^{-1}x_2x_0x_1^{-1}x_2x_1^{-1}x_3^{-1}$	$2-6t+9t^2$ $-6t^3+2t^4$
$8_9 = \frac{25}{7}$	$x_3x_4^{-1}x_6^{-1}x_5x_3x_4^{-1}x_5x_3x_2^{-1}x_4^{-1}x_3x_1x_2^{-1}$ $x_3x_1x_0^{-1}x_2^{-1}x_3x_1x_2^{-1}x_4^{-1}x_3x_2^{-1}x_4^{-1}x_5$	$1-3t+5t^2-7t^3$ $+5t^4-3t^5+t^6$
$8_{11} = \frac{27}{10}$	$x_2x_3^{-2}x_2x_1^{-2}x_2^2x_3^{-1}x_4^2x_3^{-2}x_2x_1^{-2}x_2^2x_3^{-2}x_2$ $x_1^{-2}x_0^2x_1^{-1}x_2$	$2-7t+9t^2$ $-7t^3+2t^4$
$8_{12} = \frac{29}{12}$	$x_1x_2^{-1}x_3^5x_4^{-1}x_3x_2^{-6}x_1x_0^{-1}x_1^5x_2^{-1}x_3x_2^{-5}$	$1-7t+13t^2$ $-7t^3+t^4$
$8_{13} = \frac{29}{11}$	$x_2^{-2}x_1^2x_2^{-1}x_3^2x_2^{-2}x_3^2x_4^{-2}x_3x_2^{-2}x_3^2x_2^{-2}x_1^2x_2^{-2}$ $x_1x_0^{-2}x_1^2$	$2-7t+11t^2-7t^3$ $+2t^4$
$8_{14} = \frac{31}{12}$	$x_2^2x_3^{-1}x_2x_1^{-1}x_2^2(x_3^{-1}x_4x_3^{-2}x_2)^2x_1^{-1}x_2^2x_3^{-1}$ $(x_2x_1^{-2}x_0x_1^{-1})^2$	$2-8t+11t^2-8t^3$ $+2t^4$
$9_1 = \frac{9}{1}$	$x_7x_8^{-1}x_6^{-1}x_4^{-1}x_2^{-1}x_0^{-1}x_1x_3x_5$	$1-t+t^2-t^3+t^4$ $-t^5+t^6-t^7$ $+t^8$
$9_2 = \frac{15}{7}$	$(x_0x_1^{-1})^4(x_2x_1^{-1})^3x_2$	$4-7t+4t^2$
$9_3 = \frac{19}{6}$	$x_5x_6^{-1}x_4^{-1}x_2^{-1}x_1x_3x_5x_6^{-1}x_4^{-1}x_2^{-1}x_0^{-1}x_1x_3$ $x_5x_4^{-1}x_2^{-1}x_0^{-1}x_1x_3$	$2-3t+3t^2-3t^3$ $+3t^4-3t^5+2t^6$
$9_4 = \frac{21}{5}$	$x_1^{-3}x_2^2x_3^{-3}x_4^3x_3^{-2}x_2^3x_1^{-2}x_0^3$	$3-5t+5t^2-5t^3$ $+3t^4$
$9_5 = \frac{23}{6}$	$x_1^{-6}x_2^6x_1^{-5}x_0^6$	$6-11t+6t^2$
$9_6 = \frac{27}{5}$	$x_3x_5x_6^{-1}x_4^{-1}x_2^{-1}x_3x_5x_6^{-1}x_4^{-1}x_2^{-1}x_1x_3x_5x_4^{-1}$ $x_2^{-1}x_0^{-1}x_1x_3x_4^{-1}x_2^{-1}x_0^{-1}x_1x_3x_5x_4^{-1}x_2^{-1}x_1$	$2-4t+5t^2-5t^3$ $+5t^4-4t^5+2t^6$
$9_7 = -\frac{29}{13}$	$x_2^{-3}x_1^3x_2^{-1}x_3^3x_2^{-3}x_3^3x_4^{-3}x_3x_2^{-2}x_1x_0^{-3}x_1^3$	$3-7t+9t^2-7t^3$ $+3t^4$
$9_8 = \frac{31}{11}$	$(x_1x_2^{-1}x_3)^3(x_2^{-1}x_4^{-1}x_3)^2(x_2^{-1}x_3x_1)^3$ $(x_2^{-1}x_1x_0^{-1})^2x_2^{-1}$	$2-8t+11t^2-8t^3$ $+2t^4$
$9_9 = \frac{31}{9}$	$x_3x_4^{-1}x_2^{-1}(x_1x_3x_5x_4^{-1}x_2^{-1})^2x_3x_5x_6^{-1}x_4^{-1}x_2^{-1}$ $x_3x_5x_6^{-1}x_4^{-1}x_3x_2^{-1}x_0^{-1}x_1x_3x_4^{-1}x_2^{-1}x_0^{-1}x_1$	$2-4t+6t^2-7t^3$ $+6t^4-4t^5+2t^6$
$9_{10} = \frac{33}{10}$	$(x_1x_2^{-1}x_4^{-1}x_3)^4x_2^{-1}(x_1x_0^{-1}x_2^{-1}x_3)^4$	$4-8t+9t^2-8t^3$ $+4t^4$

$9_{11} = \frac{33}{14}$	$x_4x_5^{-1}x_3^{-1}x_2x_4x_6x_5^{-1}x_3^{-1}x_2x_4x_5^{-1}x_3^{-1}x_1^{-1}$ $x_2x_4x_5^{-1}x_3^{-1}x_1^{-1}x_2x_4x_3^{-1}x_1^{-1}x_0x_2x_4x_3^{-1}$ $x_1^{-1}x_2x_4x_5^{-1}x_3^{-1}x_1^{-1}x_2$	$1-5t+7t^2-7t^3$ $+7t^4-5t^5+t^6$
$9_{12} = \frac{35}{13}$	$x_1^2x_2^{-2}x_1^2x_2^{-3}x_3^2x_2^{-2}x_3^2x_4^{-2}x_3^3x_2^{-2}x_3^2x_2^{-2}x_1^2$ $x_2^{-2}x_1^3x_0^{-2}$	$2-9t+13t^2$ $-9t^3+2t^4$
$9_{13} = \frac{37}{10}$	$(x_3^2x_4^{-2})^2x_3x_2^{-2}x_3^2x_2^{-2}x_1^2x_2^{-2}x_1(x_0^{-2}x_1^2)^2$ $x_2^{-2}x_1^2x_2^{-1}x_3^2x_2^{-2}$	$4-9t+11t^2$ $-9t^3+4t^4$
$9_{14} = \frac{37}{14}$	$x_2^2x_3^{-2}x_2^3x_1^{-2}x_2^2x_3^{-3}x_4^2x_3^{-2}x_2^3x_1^{-2}x_2^2x_3^{-2}x_2^3$ $x_1^{-2}x_0^2x_1^{-3}$	$2-9t+15t^2$ $-9t^3+2t^4$
$9_{15} = \frac{39}{16}$	$x_1x_2^{-1}x_3^3x_4^{-1}x_3x_2^{-1}x_3^3x_4^{-1}x_3x_2^{-4}x_1x_0^{-1}x_1^3$ $x_2^{-1}x_1x_0^{-1}x_1^3x_2^{-1}x_3x_2^{-3}x_1x_2^{-1}x_3x_2^{-3}$	$2-10t+15t^2$ $-10t^3+2t^4$
$9_{17} = \frac{39}{14}$	$x_1x_2^{-1}x_3^2x_4^{-1}x_5^2x_6^{-1}x_5x_4^{-2}x_3x_2^{-2}x_1x_2^{-1}x_3^2$ $x_4^{-1}x_5x_4^{-2}x_3x_2^{-2}x_1x_0^{-1}x_1^2x_2^{-1}x_3^2x_4^{-1}x_5x_4^{-2}$ $x_3x_2^{-2}$	$1-5t+9t^2-9t^3$ $+9t^4-5t^5+t^6$
$9_{18} = \frac{41}{17}$	$(x_3^2x_4^{-2}x_3x_2^{-2})^2x_1^2x_2^{-1}x_3^2x_2^{-2}x_1x_0^{-2}x_1^2x_2^{-2}x_1$ $x_0^{-2}x_1^2x_2^{-1}x_3^2x_2^{-2}x_1^2x_2^{-1}$	$4-10t+13t^2$ $-10t^3+4t^4$
$9_{19} = \frac{41}{16}$	$x_1x_2^{-1}x_3^3x_4^{-1}x_3x_2^{-3}x_1x_2^{-1}x_3^3x_4^{-1}x_3x_2^{-4}x_1$ $x_0^{-1}x_1^3x_2^{-1}x_3x_2^{-3}x_1x_0^{-1}x_1^3x_2^{-1}x_3x_2^{-3}$	$2-10t+17t^2$ $-10t^3+2t^4$
$9_{20} = \frac{41}{15}$	$x_1x_2^{-1}x_4^{-1}x_3x_1x_2^{-1}x_4^{-1}x_3x_2^{-1}x_4^{-1}x_5x_3x_2^{-1}$ $x_4^{-1}x_5x_3x_4^{-1}x_6^{-1}x_5x_3x_4^{-1}x_5x_3x_2^{-1}x_4^{-1}x_5$ $x_3x_2^{-1}x_4^{-1}x_3x_1x_2^{-1}x_4^{-1}x_3x_1x_2^{-1}x_3x_1$ $x_0^{-1}x_2^{-1}x_3$	$1-5t+9t^2-11t^3$ $+9t^4-5t^5+t^6$
$9_{21} = \frac{43}{18}$	$x_3^2x_4^{-2}x_3^5x_2^{-2}x_1^2x_2^{-3}x_3^2x_2^{-2}x_1^2x_2^{-3}x_3^2x_2^{-2}x_1^5$ $x_0^{-2}x_1^2x_2^{-5}$	$2-11t+17t^2$ $-11t^3+2t^4$
$9_{23} = \frac{45}{19}$	$x_2x_1^{-1}x_3^{-1}x_2x_3^{-1}(x_2x_1^{-1})^2x_3^{-1}x_2(x_3^{-1}x_4)^2$ $(x_2x_3^{-1})^2x_4x_3^{-1}x_4x_2x_3^{-1}(x_2x_1^{-1})^2(x_3^{-1}x_2)^2$ $x_1^{-1}x_2(x_0x_1^{-1})^2x_2x_1^{-1}x_2(x_0x_1^{-1})^2$	$4-11t+15t^2$ $-11t^3+4t^4$
$9_{26} = \frac{47}{18}$	$x_2x_3^{-1}x_5^{-1}x_4x_2x_3^{-1}x_4x_2x_1^{-1}x_3^{-1}x_4x_2x_3^{-1}$ $x_5^{-1}x_4x_3^{-1}x_5^{-1}x_6x_4x_3^{-1}x_5^{-1}x_4x_2x_3^{-1}x_4x_2$ $x_1^{-1}x_3^{-1}x_4x_2x_3^{-1}x_5^{-1}x_4x_2x_3^{-1}x_4x_2x_1^{-1}x_3^{-1}$ $x_2x_0x_1^{-1}x_3^{-1}x_2x_1^{-1}x_3^{-1}x_4$	$1-5t+11t^2-13t^3$ $+11t^4-5t^5+t^6$

$9_{27} = \frac{49}{19}$	$x_2x_3^{-1}x_4^2x_5^{-1}x_6x_5^{-2}x_4x_3^{-2}x_2x_3^{-1}x_4^2x_5^{-1}x_4$	$1-5t+11t^2-15t^3$
	$x_3^{-2}x_2x_1^{-1}x_2^2x_3^{-1}x_4x_3^{-2}x_2x_1^{-2}x_0x_1^{-1}x_2^2x_3^{-1}$	$+11t^4-5t^5+t^6$
	$x_4x_3^{-2}x_2x_1^{-1}x_2^2x_3^{-1}x_4^2x_5^{-1}x_4x_3^{-2}$	
$9_{31} = \frac{55}{21}$	$x_2x_3^{-1}x_4x_2x_3^{-1}x_2x_1^{-1}x_3^{-1}x_2x_3^{-1}x_4x_3^{-1}x_5^{-1}$	$1-5t+13t^2-17t^3$
	$x_4x_3^{-1}x_4x_2x_3^{-1}x_4x_3^{-1}x_5^{-1}x_4x_5^{-1}x_6x_4x_5^{-1}$	$+13t^4-5t^5+t^6$
	$x_4x_3^{-1}x_4x_2x_3^{-1}x_4x_3^{-1}x_5^{-1}x_4x_3^{-1}x_4x_2x_3^{-1}$	
	$x_2x_1^{-1}x_3^{-1}x_2x_3^{-1}x_4x_2x_3^{-1}x_2x_1^{-1}x_2x_0x_1^{-1}$	
	$x_2x_1^{-1}x_3^{-1}$	

References

[1] J. Bracho, L. Montejano, *The combinatorics of colored triangulations of manifolds*, Geometriae Dedicata **22** (1987), 308-328.

[2] G. Burde, H. Zieschang, *Knots*, Walter de Gruyter Inc., Berlin-New York, 1985.

[3] A. Cavicchioli, *Neuwirth manifolds and colourings of graphs*, Aequationes Math. **44** (1992), 168-187.

[4] A. Cavicchioli, F. Hegenbarth, A. C. Kim, *On cyclic branched coverings of torus knots*, J. of Geometry **64** (1999), 55-66.

[5] A. Cavicchioli, F. Hegenbarth, A. C. Kim, *A geometric study of Sieradski groups*, Algebra Colloquium **5** (1998), 203-217.

[6] A. Cavicchioli, D. Repovš, A. B. Skopenkov, *Open problems on graphs arising from geometric topology*, Topology and its Appl. **84** (1998), 207-226.

[7] W. Dunbar, *Geometric orbifolds*, Rev. Mat. Univ. Complut. Madrid **1** (1988), 67-99.

[8] M. Ferri, *Crystallizations of 2-fold branched coverings of* $\mathbb{S}^3$, Proc. Amer. Math. Soc. **73** (1979), 271-276.

[9] M. Ferri, C. Gagliardi, L. Grasselli, *A graph-theoretical representation of PL-manifolds. A survey on crystallizations*, Aequationes Math. **31** (1986), 121-141.

[10] R. H. Fox *The homology characters of the cyclic coverings of the knots of genus one*, Ann. of Math. **71** (1960), 187-196.

[11] A. Haefliger, N. D. Quach, *Une presentation de groupe fundamental d'une orbifold*, Asterisque **116** (1984), 98-107.

[12] H. Helling, A. C. Kim, J. L. Mennicke, *A geometric study of Fibonacci groups*, Diskrete Strukturen in der Mathematik, Preprint 90-343, Universität Bielefeld (1990).

[13] H. Helling, A. C. Kim, J. L. Mennicke, *Some honey-combs in hyperbolic 3-space*, Comm. in Algebra **23** (1995), 5169-5206.

[14] J. Hempel, *3-Manifolds*, Princeton Univ. Press, Princeton, N. J., 1976.

[15] H. M. Hilden, M. T. Lozano, J. M. Montesinos-Amilibia, *The arithmeticity of the figure eight knot orbifolds*, in 'Topology 90' (B. Apanasov, W. D. Neumann, A. W. Reid, and L. Siebenmann Eds.) Ohio State Univ., Math. Research Inst. Publ., Walter de Gruyter Ed., Berlin **1** (1992), 169-183.

[16] H. M. Hilden, M. T. Lozano, J. M. Montesinos-Amilibia, *On the arithmetic 2-bridge knots and link orbifolds and a new knot invariant*, J. of Knot Theory and Its Ramifications 4 (1995), 81-114.

[17] A. Kawauchi, *A survey of Knot Theory*, Birkhäuser Verlag, Berlin-New York, 1996.

[18] A. C. Kim, A. Vesnin, *The fractional Fibonacci groups and manifolds*, Siberian Math. J., to appear.

[19] A. C. Kim, A. Vesnin, *Cyclically presented groups and Takahashi manifolds as cyclic coverings*, Diskrete Strukturen in der Mathematik, Universität Bielefeld, Preprint 97-061 (1997).

[20] D. L. Johnson, *Topics in the theory of group presentations*, vol. **42**, London Math. Soc. Lect. Note Ser., Cambridge Univ. Press, Cambridge, U. K., 1980.

[21] W. B. R. Lickorish, *An introduction to knot theory*, vol. **175**, Graduate Texts in Math., Springer Verlag, New York-Berlin-Heidelberg, 1997.

[22] A. D. Mednykh, A. Yu. Vesnin, *Fibonacci manifolds as two-fold coverings of the 3-sphere and the Meyerhoff-Neumann conjecture*, Siberian Math. J. (1996).

[23] R. Osborne, *The simplest closed 3-manifolds*, Pacific J. Math. **74** (1978), 481-495.

[24] R. Osborne, R. S. Stevens, *Group presentations corresponding to spines of 3-manifolds, I*, Amer. J. Math. **96** (1974), 454-471.

[25] R. Osborne, R. S. Stevens, *Group presentations corresponding to spines of 3-manifolds, II, III*, Trans. Amer. Math. Soc. **234** (1977), 213-243; 245-251.

[26] L. Paoluzzi, B. Zimmermann, *On a class of hyperbolic 3-manifolds and groups with one defining relation*, Geometriae Dedicata **60** (1996), 113-123.

[27] D. Rolfsen, *Knots and Links*, Math. Lect. Series 7, Publish or Perish Inc., Berkeley, 1976.

[28] B. Ruini, F. Spaggiari, *On the structure of Takahashi manifolds*, Tsukuba Math. J. **22** (1998), 723-739.

[29] H. Seifert, W. Threlfall, *A Textbook of Topology; Topology of 3-dimensional fibered spaces*, Academic Press, New York-London, 1980.

[30] R. S. Stevens, *Classification of 3-manifolds with certain spines*, Trans. Amer. Math. Soc. **205** (1975), 151-166.

[31] W. P. Thurston, *The Geometry and Topology of 3-Manifolds*, Lect. Notes, Princeton Univ. Press, Princeton, N. J., 1980.

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