REVISTA MATEMÁTICA COMPLUTENSE
Volumen 12, número 2: 1999
http://dx.doi.org/10.5209/rev_REMA.1999.v12.n2.17118

Generalized Fourier expansions for distributions and ultradistributions.

S. N. MELIKHOV

Abstract

Let $D_\omega(\Pi_a)'$ be a space of distributions or ultradistributions of Beurling type on p-dimensional parallelepiped $\Pi_a := \prod_{j=1}^{p}[-a_j, a_j] \subset \mathbb{R}^p$. We investigate the following problems:
1) When can any element of $D_\omega(\Pi_a)'$ be expanded in absolutely convergent series in a system of generalized exponentials $(e_{\lambda_{(n)}})_{n\in\mathbb{N}^p}$ with special exponents $\lambda_{(n)}$, $n \in \mathbb{N}^p$. 2) When can a sequence of the coefficients $(c_n)_{n\in\mathbb{N}^p}$ in an expansions $u = \sum_{n\in\mathbb{N}^p} c_n e_{\lambda_{(n)}}$ be chosen so that it depends in a continuous and linear way on $u \in D_\omega(\Pi_b)'$, where $0 < b_j \leq a_j$ for all $1 \leq j \leq p$.

Introduction

The expansions of the distributions and the ultradistributions in generalized exponential series have been investigated by many authors (see Vladimirov [25] (Ch.II, § 7), Edvards [5] (12.5), Meise [14], Franken, Meise [6], Braun, Meise [2]). Here the elements of kernels of convolution operators have been expanded in the series of exponential solutions of the corresponding homogeneous convolution equations, in particular, the periodic distributions and ultradistributions. All these representations have the uniqueness property, i.e. any distribution or ultradistribution can be expanded in a unique way. This paper concerns the systems of

Mathematics Subject Classification: 46F05. 42C15.
Servicio Publicaciones Univ. Complutense. Madrid, 1999.

generalized exponents $(e_{\lambda_{(n)}})_{n\in\mathbb{N}^p}$ in the spaces of distributions or ultradistributions on the p-dimensional parallelepiped $\Pi_a := \prod_{j=1}^{p}[-a_j, a_j]$, where $p \in \mathbb{N}$ and $a_j > 0$ for any $1 \le j \le p$, that admit a nontrivial expansion of zero. First fundamental results for such systems of exponentials (in the spaces of analytic functions in convex domains in $\mathbb{C}$) were obtained by Leont'ev (see [13]).

In presented paper we solve the following problem. Let $D_\omega(\Pi_a)'_\beta$ be the space of the ultradistributions of Beurling type or the distributions on the parallelepiped Π_a with the strong topology. Let $\lambda_{(n)} := (\lambda_{n_j}^{(j)})_{j=1}^p$, $n \in \mathbb{N}^p$, be a sequence of the exponents where $(\lambda_m^{(j)})_{m\in\mathbb{N}}$ for any $1 \le j \le p$ are all zeros of an entire (in $\mathbb{C}$) function L_j and any zero of L_j is simple. In part one of the paper we show necessary and sufficient conditions that any ω-ultradifferentiable on Π_a function can be expanded in a generalized Fourier series in the system $(e_{\lambda_{(n)}})_{n\in\mathbb{N}^p}$ absolutely convergent in $D_\omega(\Pi_a)'_\beta$. These conditions are established in traditional terms of lower bounds of $|L_j|$ and $|L_j'(\lambda_m^{(j)})|$ for all $1 \le j \le p$. We show too that if these conditions are fulfilled then any $u \in D_\omega(\Pi_a)'_\beta$ can be expanded in an absolutely convergent series in $(e_{\lambda_{(n)}})_{n\in\mathbb{N}^p}$. In the terminology of Korobeinik [10] this means that the system $(e_{\lambda_{(n)}})_{n\in\mathbb{N}^p}$ is an absolutely representing system (ARS) in $D_\omega(\Pi_a)'_\beta$. Moreover we show that this system $(e_{\lambda_{(n)}})_{n\in\mathbb{N}^p}$ is an ARS in an ultradistribution or distribution space $D_\omega(K)'_\beta$ for any compact set $K \subset \Pi_a$.

In part two we study when a sequence of the coefficients $(c_n)_{n\in\mathbb{N}^p}$ in an expansions $u = \sum_{n\in\mathbb{N}^p} c_n e_{\lambda_{(n)}}$ can be chosen so that it depends in a continuous and linear way on $u \in D_\omega(\Pi_b)'_\beta$ where $0 < b_j \le a_j$ for all $1 \le j \le p$. In other words, we solve a problem when a corresponding representation operator $R : c \mapsto \sum_{n\in\mathbb{N}^p} c_n e_{\lambda_{(n)}}$ has a continuous linear right inverse. In addition for an entire function Q in $\mathbb{C}^{2p}$ with $Q(z, z) = \prod_{j=1}^{p} L_j(z_j)$ for all $z \in \mathbb{C}^p$ we introduce an interpolating functional $\Omega_Q : \mathbb{C}^{2p} \times D_\omega(\Pi_b)'_\beta \to \mathbb{C}$ such that for any $z, \mu \in \mathbb{C}^p$ the functional $\Omega_Q(z, \mu, \cdot)$ is continuous and linear on $D_\omega(\Pi_b)'_\beta$. If a continuous linear right inverse for the representation operator exists we show that one of them is the operator

$$u \mapsto \left((-i)^p \Omega_Q(\lambda_{(n)}, \lambda_{(n)}, u) / \prod_{j=1}^{p} L_j'(\lambda_{n_j}^{(j)}) \right)_{n \in \mathbb{N}^p} .$$

We note that a part of the the results of this paper which are related to the case of the distribution space were annonced in [21].

Acknowledgement. This paper was written during a stay of the author at University of Düsseldorf in the autumn 1997 by the support of Deutscher Akademischer Austauschdienst (DAAD). This support is gratefully acknowledged. The author wishes to thank Prof. Dr. R. Meise and Dr. S. Momm for their valuable remarks. The author thanks also the Russian Foundation for Basic Research (project 96-01-01041) for partial support.

1 Generalized Fourier series for ultradifferentiable functions in a space of ultradistributions

1.1 Definition. A continuous increasing function $\omega : [0, +\infty) \to [0, +\infty)$ is called a *weight function* if it satisfies the following conditions:

(α) $\omega(2t) = O(\omega(t))$ as $t \to +\infty$

(β) $\int\limits_1^\infty \frac{\omega(t)}{t^2} dt < \infty$

(γ) $\log t = o(\omega(t))$ as $t \to +\infty$

(δ) $\varphi = \omega \circ \exp$ is convex on $\mathbb{R}$.

We denote by φ^* the Young conjugate of φ, i.e. $\varphi^*(x) := \sup\limits_{y \geq 0}(xy - \varphi(y))$ for all $x \geq 0$.

Let ω be a weight function. For a compact set $K \subset \mathbb{R}^p$ we define a space

$$\mathcal{E}_\omega(K) := \{ f \in C^\infty(K) \mid \sup_{\alpha \in \mathbb{N}_0^p} \sup_{x \in K} |f^{(\alpha)}(x)| \exp(-m\varphi^*(|\alpha|/m)) < \infty \text{ for all } m \in \mathbb{N} \}$$

endowed with the natural topology of a Fréchet space.

For an open set $\Omega \subset \mathbb{R}^p$ let

$$\mathcal{E}_\omega(\Omega) := \operatorname*{proj}_{K \subset\subset \Omega} \mathcal{E}_\omega(K).$$

For a set $K \subset \mathbb{R}^p$ the elements of $\mathcal{E}_\omega(K)$ are called ω-ultradifferentiable functions of Beurling type on K.

For a compact set $K \subset \mathbb{R}^p$ we put

$$D_\omega(K) := \{f \in \mathcal{E}_\omega(\mathbb{R}^p) \,|\, \operatorname{supp}(f) \subset K\}$$

and endow $D_\omega(K)$ with the topology induced by $\mathcal{E}_\omega(\mathbb{R}^p)$.

For an open set $\Omega \subset \mathbb{R}^p$ we define

$$D_\omega(\Omega) := \operatorname*{ind}_{K \subset\subset \Omega} D_\omega(K).$$

For $\omega(t) = \log^+ t$ and a compact or open set $K \subset \mathbb{R}^p$ let

$$D_\omega(K) := D(K) := \{f \in C^\infty(\mathbb{R}^p) \,|\, \operatorname{supp}(f) \text{ is compact in } K\}$$

and

$$\mathcal{E}_\omega(K) := C^\infty(K).$$

The spaces $D(K)$ and $C^\infty(K)$ will be equipped with the natural topologies.

By $D_\omega(K)'_\beta$ (resp. $\mathcal{E}_\omega(K)'_\beta$) we denote the dual space of $D_\omega(K)$ (resp. $\mathcal{E}_\omega(K)$) endowed with its strong topology.

If ω is a weight function, the elements of $D_\omega(K)'$ are called ω-ultradistributions of Beurling type on K.

For a convex compact set $K \subset \mathbb{R}^p$ let H_K denote its support function, i.e.

$$H_K(y) := \sup_{x \in K} < x, y >, \;\; y \in \mathbb{R}^p,$$

where $< \lambda, z > := \sum_{j=1}^{p} \lambda_j z_j$ for all $\lambda, z \in \mathbb{C}^p$.

For some $r, q \in \mathbb{R}^p$ we write $r \geq q$ if $r_j \geq q_j$ for all $1 \leq j \leq p$ and $r > q$ if $r_j > q_j$ for all $1 \leq j \leq p$.

For any $r \in \mathbb{R}^p$ such that $r \geq 0$ we put $\Pi_r := \prod_{j=1}^{p}[-r_j, r_j]$ and $H_r := H_{\Pi_r}$. Note that $H_r(y) = \sum_{j=1}^{p} r_j|y_j|$ for all $y \in \mathbb{R}^p$. Hereafter,

$$|z| :=< z, \overline{z} >^{1/2}, \ \omega(z) := \omega(|z|), \ \operatorname{Im} z := (\operatorname{Im} z_j)_{j=1}^p \text{ for all } z \in \mathbb{C}^p.$$

For any $r > 0$ and $\varphi \in \mathcal{E}_\omega(\Pi_r)'$ its Fourier-Laplace transform $\widehat{\varphi}$ is defined as

$$\widehat{\varphi}(\lambda) := \varphi(\exp(-i < \lambda, \cdot >)), \ \lambda \in \mathbb{C}^p.$$

We call the following elements $e_\lambda \in D_\omega(\Pi_r)'$ the generalized exponentials:

$$e_\lambda(f) := \hat{f}(\lambda), \ f \in D_\omega(\Pi_r), \ \lambda \in \mathbb{C}^p.$$

We define for all $r \geq 0$ the following spaces of entire functions:

$$A_{\omega,n}(r,p) := \{f \in A(\mathbb{C}^p) \mid$$

$$\|f\|_n := \sup_{z \in \mathbb{C}^p} |f(z)| \exp(-H_r(\operatorname{Im} z) + n\omega(z)) < \infty\}, n \in \mathbb{N};$$

$$A_\omega(r,p) := \operatorname*{proj}_{\leftarrow n} A_{\omega,n}(r,p), \ \ A_\omega(p) := \operatorname*{ind}_{r>0} A_\omega(r,p);$$

$$B_{\omega,m}(r,p) := \{f \in A(\mathbb{C}^p) \mid$$

$$q_m(f) := \sup_{z \in \mathbb{C}^p} |f(z)| \exp(-H_r(\operatorname{Im} z) - m\omega(z)) < \infty\}, \ m \in \mathbb{N};$$

$$B_\omega(r,p) := \operatorname*{ind}_{m \rightarrow} B_{\omega,m}(r,p).$$

By Paley-Wiener-Schwartz theorem for ultradistributions and distributions (Braun, Meise, Taylor [3] (3.5 Proposition), Meise, Taylor [17] (3.6 Proposition) and Hörmander [8] (Theorem 7.3.1)) the following holds

1.2 Proposition. *Let ω be a weight function or $\omega(t) := \log^+ t$. The Fourier-Laplace transform $\mathcal{F} : \varphi \mapsto \widehat{\varphi}$ is an isomorphism of $D_\omega(\Pi_r)$ onto $A_\omega(r,p)$, of $\mathcal{E}_\omega(\Pi_r)'_\beta$ onto $B_\omega(r,p)$ for all $r > 0$ and of $D_\omega(\mathbb{R}^p)$ onto $A_\omega(p)$.*

1.3 Definition. As in Meise, Taylor [17] we call a weight function ω a strong weight function if in addition the following holds

(ε) there are $1 < C_1 < C$ such that $\omega(Ct) < C_1\omega(t)$ for large t.

Note that ω is a strong weight function if and only if the (equivalent) conditions of 1.3 Proposition of [17] are fulfilled.

1.4 Convention. For the sequel, let ω be a strong weight function or $\omega(t) := \log^+ t$. We fix $a \in \mathbb{R}^p$ such that $a > 0$ and put $E(a,p) := D_\omega(\Pi_a)'_\beta$. For any $1 \le j \le p$ let $L_j \in B_\omega(a_j, 1)$, $(\lambda_n^{(j)})_{n\in\mathbb{N}}$ denotes the set of all zeros of L_j and any zero of L_j is simple.

We write

$$\lambda_{(n)} := (\lambda_{n_j}^{(j)})_{j=1}^p,\ L(z) := \prod_{j=1}^p L_j(z_j),\ L'(\lambda_{(n)}) := \prod_{j=1}^p L_j'(\lambda_{n_j}^{(j)}),$$

$(z-\mu)^1 := \prod\limits_{j=1}^p (z_j - \mu_j)$ for all $n \in \mathbb{N}^p$ and $z, \mu \in \mathbb{C}^p$.

By 1.2 for any $n \in \mathbb{N}^p$ there exists a unique functional $\varphi_n \in \mathcal{E}_\omega(\Pi_a)'$ such that

$$\widehat{\varphi}_n(z)(z-\lambda_{(n)})^1 L'(\lambda_{(n)}) = L(z) \text{ for all } z \in \mathbb{C}^p \text{ and } n \in \mathbb{N}^p.$$

The system $(\varphi_n)_{n\in\mathbb{N}^p}$ is biorthogonal to $(e_{\lambda_{(n)}})_{n\in\mathbb{N}^p}$, i.e. $\varphi_n(e_{\lambda_{(k)}}) = \delta_{nk}$ for all $n, k \in \mathbb{N}^p$. We call for any $v \in \mathcal{E}_\omega(\Pi_a)$ the series $\sum\limits_{n\in\mathbb{N}^p} \varphi_n(v)\, e_{\lambda_{(n)}}$ the generalized Fourier series of v.

1.5 Remark. Note that the set of the functions $L_{(n)} := L/(\cdot - \lambda_{(n)})^1$, $n \in \mathbb{N}^p$, is bounded in $B_\omega(a,p)$, i.e. there is $s \in \mathbb{N}$ such that $\sup\limits_{n\in\mathbb{N}^p} q_s(L_{(n)}) < \infty$.

We put for any $r > 0$ and $m \in \mathbb{N}$

$$\|u\|_m^* := \sup_{\|f\|_m \le 1} |u(\mathcal{F}^{-1}(f))|,\ u \in D_\omega(\Pi_r)',$$

and

$$q_m^*(f) := \sup_{q_m(g)\le 1} |\mathcal{F}^{-1}(g)(f)|,\ f \in \mathcal{E}_\omega(\Pi_r).$$

If we identify the dual space of $A_\omega(r,p)$ with $D_\omega(\Pi_r)'$ by means of the bilinear form $< \cdot,\cdot >$, then $< f, e_\lambda > = f(\lambda)$ for all $\lambda \in \mathbb{C}^p$ and $f \in A_\omega(r,p)$. Therefore for any $\lambda \in \mathbb{C}^p$ and $m \in \mathbb{N}$

$$\|e_\lambda\|_m^* \le \exp(H_r(\operatorname{Im}\lambda) - m\omega(\lambda)).$$

To accurately desribe the space of the sequences of coefficients of all series in the system $(e_{\lambda_{(n)}})_{n\in\mathbb{N}^p}$ that are absolutely convergent in $D_\omega(\Pi_r)'_\beta$ lower estimates of $\|e_\lambda\|^*_m$ are necessary.

1.6 Definition. A weight function we call a (DN)-weight function if for it the (equivalent) conditions of 3.4 Theorem [16] are fulfilled.

By Meise, Taylor [16] (3.1 Proposition) ω is a (DN)-weight function if and only if

$$\text{for all } C > 1 \text{ there are } R_0 > 0 \text{ and } 0 < \delta < 1 \text{ such that}$$

$$\omega^{-1}(CR)\omega^{-1}(\delta R) \leq (\omega^{-1}(R))^2 \text{ for all } R \geq R_0. \tag{1}$$

1.7 Examples. The functions ω in (a)–(b) below are strong weight function and (DN)-weight functions.

(*a*) $\omega(t) := t^\alpha(\log(1+t))^\sigma$ where $0 < \alpha < 1$ and $\sigma \geq 0$.
(*b*) $\omega(t) := \exp(\alpha(\log(1+t))^\beta)(\log(1+t))^\sigma$ where $\alpha, \beta > 0$ and $\sigma \geq 0$.

1.8 Lemma. (*I*) *For any $r > 0$ there are the functions u_λ and v_λ, $\lambda \in \mathbb{C}^p$, which are plurisubharmonic on $\mathbb{C}^p$, such that $u_\lambda(\lambda) \geq 0$, $v_\lambda(\lambda) \geq 0$ and for any $k \in \mathbb{N}$ there are $m \in \mathbb{N}$ and $C > 0$ with*

$$u_\lambda(z) \leq H_r(\operatorname{Im} z) - H_r(\operatorname{Im}\lambda) - k\omega(z) + m\omega(\lambda) + C$$

and

$$v_\lambda(z) \leq H_r(\operatorname{Im} z) - H_r(\operatorname{Im}\lambda) + m\omega(z) - k\omega(\lambda) + C$$

for all $\lambda, z \in \mathbb{C}^p$.
(*II*) *The following assertions are equivalent:*

(*i*) *There are the functions u_λ, $\lambda \in \mathbb{C}^p$, which are plurisubharmonic on $\mathbb{C}^p$, such that $u_\lambda(\lambda) \geq 0$ and for any $k \in \mathbb{N}$ there are $m \in \mathbb{N}$ and a constant C with*

$$u_\lambda(z) \leq m\omega(z) - k\omega(\lambda) + C \text{ for all } \lambda, z \in \mathbb{C}^p.$$

(*ii*) *ω is a (DN)-weight function.*

Proof. The statement (I) holds by Langenbruch [12] (4.10 Lemma; see its proof too).

(II): $(i) \Rightarrow (ii)$: By Langenbruch [12] (3.1 Theorem b)) from (i) it follows that

> there is $\gamma > 0$ such that for any $C > 1$ and for any $n \in \mathbb{N}$ there is $I(n+1) \in \mathbb{N}$ with

$$(\omega^{-1}(nR/(2I(n+1))))^{C\gamma/n}\omega^{-1}(CR) \leq (\omega^{-1}(R))^{1+C\gamma/n} \tag{2}$$

for large R.

For any $C > 1$ we shall take $n \in \mathbb{N}$ such that $C\gamma/n \leq 1$, choose $I(n+1) \geq n$ according (2) and put $\delta := n/(2I(n+1))$. We have by (2) that for large R

$$\frac{\omega^{-1}(\delta R)}{\omega^{-1}(R)} \leq \left(\frac{\omega^{-1}(\delta R)}{\omega^{-1}(R)}\right)^{C\gamma/n} \leq \frac{\omega^{-1}(R)}{\omega^{-1}(CR)}.$$

Hence (1) holds and ω is a (DN)-weight function.

$(ii) \Rightarrow (i)$: We note that by Meise, Taylor [16] (3.1 Proposition) (ii) is equvalent to

> For any $d > 0$ and any $C > 1$ there exists R_0 and $0 < \delta < 1$ such that for all $R \geq R_0$ the following holds

$$\omega^{-1}(CR)(\omega^{-1}(\delta R))^d \leq (\omega^{-1}(R))^{1+d}. \tag{3}$$

By Langenbruch [12] (3.1 Theorem a)) (i) follows from

> there are $C > 1$ and $\gamma > 0$ such that for any $n \in \mathbb{N}$ there is $I(n) \in \mathbb{N}$ such that for large R

$$\left(\frac{\omega^{-1}(nR/I(n))}{\omega^{-1}(R)}\right)^{C\gamma/n} \leq \frac{\omega^{-1}(R)}{\omega^{-1}(CR)}. \tag{4}$$

For any $n \in \mathbb{N}$ and for $d := 2/n$, $C := 2$ we choose $0 < \delta < 1$ and R_0 according (3) and $I(n) \in \mathbb{N}$ such that $n/I(n) \leq \delta$. Then by (3) for all $R \geq R_0$

$$\left(\frac{\omega^{-1}(nR/I(n))}{\omega^{-1}(R)}\right)^{C/n} \leq \left(\frac{\omega^{-1}(\delta R)}{\omega^{-1}(R)}\right)^{C/n} \leq \frac{\omega^{-1}(R)}{\omega^{-1}(CR)}.$$

Hence (4) (with $\gamma = 1$) and (i) hold.

1.9 Corollary. *For any $r > 0$ there are functions f_λ, $\lambda \in \mathbb{C}^p$, which are entire on $\mathbb{C}^p$, such that $f_\lambda(\lambda) = 1$ and for any $k \in \mathbb{N}$ there are $m \in \mathbb{N}$ and $C > 0$ with*

$$|f_\lambda(z)| \leq C \exp(H_r(\operatorname{Im} z) - H_r(\operatorname{Im} \lambda) - k\omega(z) + m\omega(\lambda)) \text{ for all } \lambda, z \in \mathbb{C}^p.$$

Proof. This assertion follows from 1.8 Lemma and Hörmander [9] (4.4.2 Theorem).

1.10 Corollary. *For any $r > 0$ and $k \in \mathbb{N}$ there are $m \in \mathbb{N}$ and $c > 0$ such that for any $\lambda \in \mathbb{C}^p$ the following holds*

$$c \exp(H_r(\operatorname{Im} \lambda) - m\omega(\lambda)) \leq \|e_\lambda\|_k^* \leq \exp(H_r(\operatorname{Im} \lambda) - k\omega(\lambda)).$$

Proof. The upper estimate for $\|e_\lambda\|_k^*$ follows from the definition of $\|\cdot\|_k^*$.

To prove a lower estimate, we take by 1.9 the functions $f_\lambda \in A_\omega(r, p)$, $\lambda \in \mathbb{C}^p$, with $f_\lambda(\lambda) = 1$ such that for all $k \in \mathbb{N}$ there are $m \in \mathbb{N}$ und $C < \infty$ with

$$|f_\lambda(z)| \leq C \exp(H_r(\operatorname{Im} z) - H_r(\operatorname{Im} \lambda) - k\omega(z) + m\omega(\lambda)) \text{ for all } \lambda, z \in \mathbb{C}^p.$$

For the functions $g_\lambda := C^{-1} \exp(H_r(\operatorname{Im} \lambda) - m\omega(\lambda)) f_\lambda$ we have

$$< g_\lambda, e_\lambda >= C^{-1} \exp(H_r(\operatorname{Im} \lambda) - m\omega(\lambda)) \text{ and } \|g_\lambda\|_k \leq 1.$$

Consequently for $c := C^{-1}$ and for all $\lambda \in \mathbb{C}^p$

$$\|e_\lambda\|_k^* \geq c \exp(H_r(\operatorname{Im} \lambda) - m\omega(\lambda)).$$

1.11 Corollary. *For any $k \in \mathbb{N}$ there are $m \in \mathbb{N}$ and $B < \infty$ such that for any $\lambda, h \in \mathbb{C}^p$ with $|h| \leq p$ the inequality $\|e_{\lambda+h}\|_m^* \leq B\|e_\lambda\|_k^*$ holds.*

1.12 Sequence spaces. Representation operator. Now we introduce for all $r \in \mathbb{R}^p$ such that $r > 0$ the spaces of sequences corresponding to the system $(e_{\lambda_{(n)}})_{n \in \mathbb{N}^p}$:

$$K_m(r) := \{c = (c_n)_{n \in \mathbb{N}^p} \subset \mathbb{C} \mid$$

$$|c|_m := \sum_{n \in \mathbb{N}^p} |c_n| \exp(H_r(\operatorname{Im} \lambda_{(n)}) - m\omega(\lambda_{(n)})) < \infty\}, \ m \in \mathbb{N},$$

$$K(r) := \operatorname*{ind}_{m\to} K_m(r);$$

$$\Lambda_m(r) := \{c = (c_n)_{n\in\mathbb{N}^p} \subset \mathbb{C} \,|$$

$$\widetilde{|c|}_m := \sup_{n\in\mathbb{N}^p} |c_n| \exp(-H_r(\operatorname{Im}\lambda_{(n)}) + m\omega(\lambda_{(n)})) < \infty\}, m \in \mathbb{N};$$

$$\Lambda(r) := \operatorname*{proj}_{\leftarrow m} \Lambda_m(r).$$

The space $\Lambda(r)$ can be identified with the dual to $K(r)$ space by the bilinear form $< c, d >:= \sum_{n\in\mathbb{N}^p} c_n d_n, \quad c \in K(r), \quad d \in \Lambda(r)$.

It follows from 1.10 Corollary that a series $\sum_{n\in\mathbb{N}^p} c_n e_{\lambda_{(n)}}$, where $c_n \in \mathbb{C}$ for all $n \in \mathbb{N}^p$, is absolutely convergent in $D_\omega(\Pi_r)'_\beta$ if and only if $(c_n)_{n\in\mathbb{N}^p} \in K(r)$.

As representation operator R we define by $R(c) := \sum_{n\in\mathbb{N}^p} c_n e_{\lambda_{(n)}}$, $c \in K(r)$. It maps continuously and linearly $K(r)$ into $D_\omega(\Pi_r)'_\beta$. By Korobeinik [10] we call $(e_{\lambda_{(n)}})_{n\in\mathbb{N}^p}$ an absolutely representing system (ARS) in $D_\omega(\Pi_r)'_\beta$ if $R : K(r) \to D_\omega(\Pi_r)'_\beta$ is surjective.

In the first section we show that the system $(e_{\lambda_{(n)}})_{n\in\mathbb{N}^p}$ is an ARS in $E(a,p)$ under the natural (traditional) conditions for the functions L_j (as in Leont'ev [13], Korobeinik [10]).

At first we characterize those functions L_j as in 1.4 for which the generalized Fourier series of v converges absolutely in $E(a,p)$ to v for all $v \in \mathcal{E}_\omega(\Pi_a)$. This question originates from Leont'ev's study [13] for the functions analytic on a convex domain in $\mathbb{C}$.

We put $D_j(u) := \dfrac{\partial u}{\partial x_j}, \quad u \in E(a,p), 1 \le j \le p.$

1.13 Lemma. *If for some $\lambda \in \mathbb{C}^p$ and some $v \in E(a,p)$ the equality $D_j(v) = -i\lambda_j v$ holds for all $1 \le j \le p$, then there exists $\mu \in \mathbb{C}$ such that $v = \mu e_\lambda$.*

Proof. We give a brief proof of the well known fact. Since $D_j(v)+i\lambda_j v = 0$ for all $1 \le j \le p$, applying the Fourier-Laplace transform, we obtain, that $< (z_j - \lambda_j) f(z), v_z >= 0$ for all $f \in A_\omega(a,p)$ and $1 \le j \le p$. If $g \in A_\omega(a,p)$ and $< g, e_\lambda >= 0$, i.e. $g(\lambda) = 0$, for any $1 \le j \le p$ there is $g_j \in A_\omega(a,p)$ such that $g(z) = \sum_{j=1}^{p} (z_j - \lambda_j) g_j(z)$ for all $z \in \mathbb{C}^p$. Then

$< g, v >= \sum_{j=1}^{p} < (z_j - \lambda_j) g_j(z), v_z >= 0$. Consequently there exists $\mu \in \mathbb{C}$ with $v = \mu e_\lambda$.

1.14 Lemma. *If $M \subset \mathbb{C}^p$ is an uniqueness set for $A_\omega(a,p)$ then M is an uniqueness set for $B_\omega(r,p)$ for any $0 \leq r < a$.*

Proof. Let f be a function in $B_\omega(r,p)$ such that $f \equiv 0$ on M. We choose $z_0 \in M$ and a function $g \in A_\omega(a-r,p)$ with $g(z_0) \neq 0$. Since $fg \in A_\omega(a,p)$ and $fg \equiv 0$ on M, we obtain that $fg \equiv 0$ on $\mathbb{C}^p$ and consequently $f \equiv 0$ on $\mathbb{C}^p$.

We note that $D_\omega(\Pi_r)'_\beta$ for any $r > 0$ is a regular (LB)-spase. Hence by [18] (Theorem 5) a series $\sum_{n \in \mathbb{N}^p} u_n$ converges absolutely in $D_\omega(\Pi_r)'_\beta$ if and only if there exists $m \in \mathbb{N}$ with $\sum_{n \in \mathbb{N}^p} \|u_n\|_m^* < \infty$.

1.15 Theorem. *Let L_j, $1 \leq j \leq p$, $\varphi_n, n \in \mathbb{N}^p$, and ω as in 1.4. The following assertions are equivalent:*

(i) For any $v \in \mathcal{E}_\omega(\Pi_a)$ the series $\sum_{n \in \mathbb{N}^p} \varphi_n(v)\, e_{\lambda_{(n)}}$ converges absolutely in $E(a,p)$ to v.

(ii) For any $\lambda \in \mathbb{C}^p$ the series $\sum_{n \in \mathbb{N}^p} \frac{L(\lambda)}{L'(\lambda_{(n)})(\lambda - \lambda_{(n)})^1} e_{\lambda_{(n)}}$ converges absolutely in $E(a,p)$ to e_λ.

(iii) For any $1 \leq j \leq p$ the series $\sum_{m \in \mathbb{N}} (L_j'(\lambda_m^{(j)}))^{-1}\, e_{\lambda_m^{(j)}}$ converges absolutely in $E(a_j,1)$ to 0.

(iv) For any $1 \leq j \leq p$ and $z \in \mathbb{C}$ the series $\sum_{m \in \mathbb{N}} \frac{L_j(z)}{L_j'(\lambda_m^{(j)})(z - \lambda_m^{(j)})} e_{\lambda_m^{(j)}}$ converges absolutely in $E(a_j,1)$ to e_z.

(v) For any $1 \leq j \leq p$

1) there exists an increasing sequence $R_s > 0$, $s \in \mathbb{N}$, such that $\lim_{s \to \infty} R_s = +\infty$ and there is $c > 0$ with

$$|L_j(z)| \geq c^{-1} \exp(a_j |\mathrm{Im}\, z| - c\omega(z))$$

for all $s \in \mathbb{N}$ and $z \in \mathbb{C}$ satisfying $|z| = R_s$ and

2) $|L'(\lambda_m^{(j)})| \geq c^{-1}\exp(a_j|\operatorname{Im}\lambda_m^{(j)}| - c\omega(\lambda_m^{(j)}))$ *for all* $m \in \mathbb{N}$.

(vi) For all $f \in A_\omega(a,p)$ *and* $\lambda \in \mathbb{C}^p$ *the Lagrange's interpolation formula holds:*

$$f(\lambda) = \sum_{n\in\mathbb{N}^p} f(\lambda_{(n)}) \frac{L(\lambda)}{L'(\lambda_{(n)})(\lambda - \lambda_{(n)})^1},$$

where the series converges absolutely (in $\mathbb{C}$*).*

Proof. $(i) \Rightarrow (ii)$: This holds since by 1.4 for all $\lambda \in \mathbb{C}^p$ and $n \in \mathbb{N}^p$

$$\varphi_n(e_\lambda) = \widehat{\varphi_n}(\lambda) = L(\lambda)/(L'(\lambda_{(n)})(\lambda - \lambda_{(n)})^1).$$

$(ii) \Rightarrow (iii)$: We fix $\lambda \in \mathbb{C}^p$ with $\lambda_j \neq \lambda_m^{(j)}$ for all $m \in \mathbb{N}$ and $1 \leq j \leq p$. By (ii) $e_\lambda = \sum_{n\in\mathbb{N}^p} \frac{L(\lambda)}{L'(\lambda_{(n)})(\lambda - \lambda_{(n)})^1} e_{\lambda_{(n)}}$, where the series converges absolutely in $E(a,p)$. This implies that for all $f_j \in A_\omega(a_j, 1)$

$$\prod_{j=1}^{p} f_j(\lambda_j) = \left\langle \prod_{j=1}^{p} f_j(z_j), (e_\lambda)_z \right\rangle =$$

$$\sum_{n\in\mathbb{N}^p} \frac{L(\lambda)}{L'(\lambda_{(n)})(\lambda - \lambda_{(n)})^1} \prod_{j=1}^{p} f_j(\lambda_{n_j}^{(j)}) =$$

$$\prod_{j=1}^{p} \left(\sum_{m\in\mathbb{N}} \frac{L_j(\lambda_j)}{L_j'(\lambda_m^{(j)})(\lambda_j - \lambda_m^{(j)})} f_j(\lambda_m^{(j)}) \right). \tag{5}$$

We fix $1 \leq k \leq p$ and obtain by (ii)

$$0 = i\lambda_k e_\lambda + D_k(e_\lambda) =$$

$$i \sum_{n\in\mathbb{N}^p} \frac{L_k(\lambda_j)}{L_k'(\lambda_{n_k}^{(k)})} \left(\prod_{j=1, j\neq k}^{p} \frac{L_j(\lambda_j)}{L_j'(\lambda_{n_j}^{(j)})(\lambda_j - \lambda_{n_j}^{(j)})} \right) e_{\lambda_{(n)}}, \tag{6}$$

where the series converges absolutely in $E(a,p)$. Choose $g_j \in A_\omega(a_j, 1)$ for any $1 \leq j \leq p$ with $j \neq k$ such that $g_j(\lambda_j) \neq 0$. We have by (6) for all $f \in A_\omega(a_k, 1)$

$$0 = \left(\sum_{m\in\mathbb{N}} \frac{L_k(\lambda_k)}{L_k'(\lambda_m^{(k)})} f(\lambda_m^{(k)}) \right) \prod_{j=1, j\neq k}^{p} \left(\sum_{m\in\mathbb{N}} \frac{L_j(\lambda_j)}{L_j'(\lambda_m^{(j)})(\lambda_j - \lambda_m^{(j)})} g_j(\lambda_m^{(j)}) \right).$$

From the last equality by (5) it follows that for all $f \in A_\omega(a_k, 1)$

$$0 = \sum_{m\in\mathbb{N}} \frac{L_k(\lambda_k)}{L'_k(\lambda_m^{(k)})} f(\lambda_m^{(k)}) = \left\langle f, \sum_{m\in\mathbb{N}} \frac{L_k(\lambda_k)}{L'_k(\lambda_m^{(k)})} e_{\lambda_m^{(k)}} \right\rangle.$$

Since $L_k(\lambda_k) \neq 0$, we obtain $0 = \sum_{m\in\mathbb{N}} (L'_k(\lambda_m^{(k)}))^{-1} e_{\lambda_m^{(k)}}$ where the series converges absolutely in $E(a_k, 1)$.

$(iii) \Rightarrow (i)$: We use the idea of the proof of [20] (Thm 1, 1) $\Rightarrow$ 3)). By (iii) and 1.10 there is $k \in \mathbb{N}$ such that

$$A := \sum_{n\in\mathbb{N}^p} \frac{1}{|L'(\lambda_{(n)})|} \|e_{\lambda_{(n)}}\|_k^* < \infty. \tag{7}$$

By 1.5 the set $\{L/(\cdot - \lambda_{(n)})^1 = \widehat{\varphi}_n L'(\lambda_{(n)}) \mid n \in \mathbb{N}^p\}$ is bounded in $B_\omega(a,p)$. Consequently there are $s \in \mathbb{N}$ and $C_s > 0$ with

$$|\varphi_n(v)| \leq q_s^*(v) q_s(\widehat{\varphi_n}) \leq \frac{C_s}{|L'(\lambda_{(n)})|} q_s^*(v)$$

for all $v \in \mathcal{E}_\omega(\Pi_a)$ and $n \in \mathbb{N}^p$. Hence by (7) $\sum_{n\in\mathbb{N}^p} |\varphi_n(v)| \|e_{\lambda_{(n)}}\|_k^* \leq C_s A q_s^*(v)$ for all $v \in \mathcal{E}_\omega(\Pi_a)$ and a continuous linear operator

$$T(v) := \sum_{n\in\mathbb{N}^p} \varphi_n(v) e_{\lambda_{(n)}}, \quad v \in \mathcal{E}_\omega(\Pi_a),$$

from $\mathcal{E}_\omega(\Pi_a)$ into $E(a,p)$ is defined.

We prove that $T : \mathcal{E}_\omega(\Pi_a) \to E(a,p)$ is the embedding map. Let $v_\lambda := T(e_\lambda)$, $\lambda \in \mathbb{C}^p$. From $v_\lambda = \sum_{n\in\mathbb{N}^p} \widehat{\varphi}_n(\lambda) e_{\lambda_{(n)}}$ it follows that for any $1 \leq k \leq p$

$$(D_k(v_\lambda) + i\lambda_k v_\lambda)_x \left(\prod_{j=1}^p f_j(x_j) \right) =$$

$$i \left(\sum_{m\in\mathbb{N}} \frac{L_k(\lambda_k)}{L'(\lambda_m^{(k)})} \widehat{f}_k(\lambda_m^{(k)}) \right) \prod_{j=1, j\neq k}^p \left(\sum_{m\in\mathbb{N}} \frac{L_j(\lambda_j)}{L'(\lambda_m^{(j)})(\lambda_m - \lambda_m^{(j)})} \widehat{f}_j(\lambda_m^{(j)}) \right) = 0$$

for all $\lambda \in \mathbb{C}^p$ and $f_j \in D_\omega([-a_j, a_j])$. Since by Braun, Meise, Taylor [3](8.1 Theorem), Meyer [22] (if ω is a weight function) and [7] (Ch.II,

§ 3, Ex.4) (if $\omega(t) = \log^+ t$) the set of all functions $\prod_{j=1}^{p} f_j(x_j)$, where $f_j \in D_\omega([-a_j, a_j])$ for any $1 \le j \le p$, is total in $D_\omega(\Pi_a)$, we have $D_k(v_\lambda) + i\lambda_k v_\lambda = 0$ for all $\lambda \in \mathbb{C}^p$ and $1 \le k \le p$. Hence by 1.13 Lemma for any $\lambda \in \mathbb{C}^p$ there is $h(\lambda) \in \mathbb{C}$ with $v_\lambda = h(\lambda)e_\lambda$. By the definition of T we have $h(\lambda_{(n)}) = 1$ for all $n \in \mathbb{N}^p$. Let T' be an adjoint to T operator from $A_\omega(a,p)$ into $B_\omega(a,p)$. Then T' is the multiplication operator with the function h, i.e. $T'(f) = hf$ for all $f \in A_\omega(a,p)$. We prove that $h \in B_\omega(0,p)$. Since $T' : A_\omega(a,p) \to B_\omega(a,p)$ is continuous, by Grothendieck-Theorem there exists $l \in \mathbb{N}$ such that T' is continuous from $A_\omega(a,p)$ to $B_{\omega,l}(a,p)$. Consequently for any $k \in \mathbb{N}$ there is $B > 0$ such that for all $f \in A_\omega(a,p)$

$$\sup_{z\in\mathbb{C}^p} |h(z)f(z)| \exp(-H_a(\operatorname{Im} z) - l\omega(z)) \le \\ B \sup_{z\in\mathbb{C}^p} |f(z)| \exp(-H_a(\operatorname{Im} z) + k\omega(z)). \tag{8}$$

From (8) with $f := f_\lambda$, where f_λ are the functions from 1.9 Corollary, it follows that there are $m \in \mathbb{N}$ and $C > 0$ with $|h(\lambda)| \le BC\exp((m + l)\omega(\lambda))$ for all $\lambda \in \mathbb{C}^p$. Since $hf \in B_\omega(a,p)$ for all $f \in A_\omega(a,p)$, the function h is entire in $\mathbb{C}^p$. Hence $h \in B_\omega(0,p)$.

Since $h(\lambda)e_\lambda = \sum_{n\in\mathbb{N}^p} \widehat{\varphi}_n(\lambda)e_{\lambda_{(n)}}$ for all $\lambda \in \mathbb{C}^p$, where the series converges absolutely in $E(a,p)$, we have $h(\lambda)f(\lambda) = \sum_{n\in\mathbb{N}^p} \widehat{\varphi}_n(\lambda)f(\lambda_{(n)})$ for all $f \in A_\omega(a,p)$. Therefore for any $f \in A_\omega(a,p)$ with $f(\lambda_{(n)}) = 0$ for all $n \in \mathbb{N}^p$ it follows $hf \equiv 0$ and because $f \equiv 0$. Consequently $\{\lambda_{(n)} \,|\, n \in \mathbb{N}^p\}$ is the uniqueness set for $A_\omega(a,p)$. By 1.14 Lemma it is the uniqueness set for $B_\omega(0,p)$, too. Hence $h \equiv 1$ and T is the embedding map of $\mathcal{E}_\omega(\Pi_a)$ into $E(a,p)$.

$(iii) \Rightarrow (iv)$: This holds by $(iii) \Rightarrow (ii)$ for $p = 1$.

$(iv) \Rightarrow (iii)$: This holds by $(ii) \Rightarrow (iii)$ for $p = 1$.

$(iii) \Rightarrow (v)$: The proof of the conditions on L_j as in (v) goes from Leont'ev [13] (see Korobeinik [10] (p. 114,115) and [20], Note, p. 66, too). We fix $1 \le j \le p$. Since the series $\sum_{m\in\mathbb{N}} (L_j'(\lambda_m^{(j)}))^{-1} e_{\lambda_m^{(j)}}$ converges absolutely to 0 in $E(a_j, 1)$ and the operator D_1 is continuous in $E(a_j, 1)$, the series $\sum_{m\in\mathbb{N}} (L_j'(\lambda_m^{(j)}))^{-1} D_1^2(e_{\lambda_m^{(j)}})$ converges absolutely to 0 in $E(a_j, 1)$

too. Hence there is $k \in \mathbb{N}$ satisfying

$$K := \sum_{m\in\mathbb{N}} \frac{1}{|L_j'(\lambda_m^{(j)})|}(1+|\lambda_m^{(j)}|)^2 \|e_{\lambda_m^{(j)}}\|_k^* < \infty. \tag{9}$$

By $(iii) \Rightarrow (iv)$ we have $e_z = \sum\limits_{m\in\mathbb{N}} \dfrac{L_j(z)}{L_j'(\lambda_m^{(j)})(z-\lambda_m^{(j)})} e_{\lambda_m^{(j)}}$ for any $z \in \mathbb{C}$.

We put $B_m := \{z \in \mathbb{C} \mid |z-\lambda_m^{(j)}| < (1+|\lambda_m^{(j)}|)^{-2}\}$ for any $m \in \mathbb{N}$. By (9) for all $z \in \mathbb{C}\backslash(\cup_{m\in\mathbb{N}} B_m)$ the following holds

$$\|e_z\|_k^* \leq K|L_j(z)|. \tag{10}$$

Since L_j is an entire function of exponential type, we have $\sum\limits_{m\in\mathbb{N}}(1+|\lambda_m^{(j)}|)^{-2} < \infty$. Hence there is an increasing sequence $R_s > 0$ such that $\{z \in \mathbb{C} \mid |z| = R_s\} \cap (\cup_{m\in\mathbb{N}} B_m) = \emptyset$ and consequently (10) holds for all $s \in \mathbb{N}$ and $z \in \mathbb{C}$ with $|z| = R_s$. From here by 1.10 we obtain (v), 1). From (9) and 1.10 it follows (v), 2).

$(v) \Rightarrow (iii)$: From (v), 2) it follows that $\sum\limits_{m\in\mathbb{N}} (L_j'(\lambda_m^{(j)}))^{-1} e_{\lambda_m^{(j)}}$ converges absolutely in $E(a_j, 1)$ for all $1 \leq j \leq p$. By (v), 1) it converges to 0 (see the proof of Theorem 5 in [20]).

Since the Fréchet space $D_\omega(\Pi_a)$ is nuclear, by Pietsch [24] (4.4.2 Proposition) (ii) is equivalent to (vi).

From the proof of $(iii) \Rightarrow (v)$ in 1.15 Theorem it follows

1.16 Remark. Every of the assertions (i) – (vi) of 1.15 Theorem is equivalent to

(v') *For any* $1 \leq j \leq p$
1$'$) *there are a sequence of circles* $B_s := \{z \in \mathbb{C} \mid |z-\mu_s| < t_s\}$ *and a constant c such that* $\sum\limits_{s\in\mathbb{N}} t_s < \infty$ *and*

$$|L_j(z)| \geq c^{-1}\exp(H_a(\operatorname{Im} z) - c\omega(z)) \text{ for all } z \in \mathbb{C}\backslash(\cup_{s\in\mathbb{N}} B_s)$$

and
2) $\quad |L'(\lambda_m^{(j)})| \geq c^{-1}\exp(a_j|\operatorname{Im}\lambda_m^{(j)}| - c\omega(\lambda_m^{(j)}))$ *for all* $m \in \mathbb{N}$.

From Korobeinik [10] (Theorem 7) we recall

1.17 Lemma. *For any sequence $(\mu_{(l)})_{l\in\mathbb{N}^p}$ in $\mathbb{C}^p$ with $|\mu_{(l)}|\to\infty$ the system $(e_{\mu_{(l)}})_{l\in\mathbb{N}^p}$ is an ARS in $E(a,p)$ if and only if for any $k\in\mathbb{N}$ there are $m\in\mathbb{N}$ and a constant C such that*

$$\|f\|_k \le C\sup_{l\in\mathbb{N}^p}|f(\mu_{(l)})|\exp(-H_a(\operatorname{Im}\mu_{(l)})+m\omega(\mu_{(l)})) \text{ for all } f\in A_\omega(a,p).$$

1.18 Theorem. *If the assumptions of 1.15 Theorem are fullfilled and one of its statements (i) – (vi) holds, then $(e_{\lambda_{(n)}})_{n\in\mathbb{N}^p}$ is an ARS in $E(a,p)$.*

Proof. To apply 1.17 Lemma, we use a method of Napalkov [23]. We fix $k\in\mathbb{N}$. By 1.5 Remark and 1.15 Theorem, (v) there are $m_1\in\mathbb{N}$ and a constant C_1 such that for all $\lambda\in\mathbb{C}^p$ and $n\in\mathbb{N}^p$

$$\frac{|L(\lambda)|}{|(\lambda-\lambda_{(n)})^1|}\le C_1\exp(H_a(\operatorname{Im}\lambda)+m_1\omega(\lambda))$$

and

$$|L'(\lambda_{(n)})|\ge C_1^{-1}\exp(H_a(\operatorname{Im}\lambda_{(n)})-m_1\omega(\lambda_{(n)})).$$

Note that there exists $K\in\mathbb{N}$ such that

$$\omega(\sum_{j=1}^p t_j)\le K(\sum_{j=1}^p\omega(t_j)+1) \text{ for all } t_1,\dots,t_j\ge 0.$$

Since ω is a subharmonic function of finite order on $\mathbb{C}$, by Yulmukhametov [26] (Theorem 5) there are a function $g_0\in A(\mathbb{C})$, a sequence of the circles $B_s:=\{z\in\mathbb{C}\,|\,|z-\mu_s|<t_s\}$ and a constant C_2 satisfying $\sum_{s\in\mathbb{N}}t_s<\infty$ and

$$|K(m_1+k+1)\omega(z)-\log|g_0(z)||\le C_2\log(1+|z|) \text{ for all } z\in\mathbb{C}\backslash(\cup_{s\in\mathbb{N}}B_s).$$

(For $\omega(t):=\log^+t$ we put $g_0(z):=z^{K(m_1+k+1)}$. It follows $C_2=0$.) Let

$$\mathcal{B}:=\{\lambda\in\mathbb{C}^p\,|\,|\lambda_j-\mu_s|\ge t_s \text{ for all } s\in\mathbb{N} \text{ and } 1\le j\le p\}.$$

For a function $g(\lambda):=\prod_{j=1}^p g_0(\lambda_j)$ there is a constant C_3 such that for all $\lambda\in\mathcal{B}$

$$(1+|\lambda|)^{-C_3}\exp((m_1+k+1)\omega(\lambda)) \le |g(\lambda)| \le$$
$$(1+|\lambda|)^{C_3}\exp(Kp(m_1+k+1)\omega(\lambda)).$$

(For $\omega(t) = \log^+ t$ we have $C_3 = 0$.) In a standart way (with the help of the maximum principle) we deduce from the last inequalities that there are a constant C_4 and $m_2 \in \mathbf{N}$ such that $|g(\lambda)| \le C_4 \exp(m_2\omega(\lambda))$ for all $\lambda \in \mathbb{C}^p$. By 1.15 (vi) for any $h \in A_\omega(a,p)$

$$h(\lambda) = \sum_{n\in\mathbb{N}^p} h(\lambda_{(n)}) \frac{L(\lambda)}{L'(\lambda_{(n)})(\lambda - \lambda_{(n)})^1} \quad \text{for all } \lambda \in \mathbb{C}^p. \tag{11}$$

Since $fg \in A_\omega(a,p)$ for all $f \in A_\omega(a,p)$, by (11) we have for all $\lambda \in \mathcal{B}$

$$f(\lambda) = \sum_{n\in\mathbb{N}^p} f(\lambda_{(n)})g(\lambda_{(n)}) \frac{L(\lambda)}{g(\lambda)L'(\lambda_{(n)})(\lambda - \lambda_{(n)})^1}$$

and

$$\sup_{\lambda\in\mathcal{B}} |f(\lambda)| \exp(-H_a(\operatorname{Im}\lambda) + k\omega(\lambda)) \le$$
$$C_1^2 C_4 \sum_{n\in\mathbb{N}^p} |f(\lambda_{(n)})| \exp(m_2\omega(\lambda_{(n)}) + m_1\omega(\lambda_{(n)}) - H_a(\operatorname{Im}\lambda_{(n)}))$$
$$\sup_{\lambda\in\mathbb{C}^p} \exp(C_3 \log(1+|\lambda|) - \omega(\lambda))$$
$$\le C_5 \sup_{n\in\mathbb{N}^p} |f(\lambda_{(n)})| \exp(-H_a(\operatorname{Im}\lambda_{(n)}) + m_0\omega(\lambda_{(n)})),$$

where

$$C_5 := C_1^2 C_4 \exp\left(\sup_{\lambda\in\mathbb{C}^p} (C_3\log(1+|\lambda|) - \omega(\lambda))\right) \sum_{n\in\mathbb{N}^p} \exp(-2\omega(\lambda_{(n)})) < \infty \quad \text{and} \quad m_0 := m_1 + m_2 + 2.$$

By the maximum principle there are $C > C_5$ and $m > m_0$ such that for all $f \in A_\omega(a,p)$ we have

$$\|f\|_k \le C \sup_{n\in\mathbb{N}^p} |f(\lambda_{(n)})| \exp(-H_a(\operatorname{Im}\lambda_{(n)}) + m\omega(\lambda_{(n)})).$$

By 1.17 Lemma $(e_{\lambda_{(n)}})_{n\in\mathbb{N}^p}$ is an ARS in $E(a,p)$.

Because statement (v) of 1.15 Theorem is valid for the functions $L_j(z) := \sin(a_j z)$, $1 \le j \le p$, the following corollary holds, where $n/a := (n_j/a_j)_{j=1}^p$ for all $n \in \mathbf{Z}^p$. [1]

1.19 Corollary. *The system $(e_{\pi n/a})_{n\in\mathbf{Z}^p}$ is an ARS in $E(a,p)$. Every function $v \in \mathcal{E}_\omega(\Pi_a)$ can then be expanded into a Fourier series absolutely convergent in $E(a,p)$: $v = \sum_{n\in\mathbf{Z}^p} \varphi_n(v) e_{\pi n/a}$, where the system $(\varphi_n)_{n\in\mathbf{Z}^p} \subset \mathcal{E}_\omega(\Pi_a)'$ is such that*

$$\widehat{\varphi}_n(z)(z-n)^1 = (-1)^{|n|} \prod_{j=1}^{p} a_j^{-n_j} \sin(a_j z_j) \text{ for all } z \in \mathbb{C}^p \text{ and } n \in \mathbf{Z}^p.$$

1.20 Corollary. *Let K be a compact set in $\mathbb{R}^p$. For all $a > 0$ such that $K \subset \Pi_a$ and for all functions L_j, $1 \le j \le p$, satisfying the statement (v) of 1.15 Theorem the system $(e_{\lambda_{(n)}})_{n\in\mathbb{N}^p}$ is an ARS in $D_\omega(K)'_\beta$. Every function $v \in \mathcal{E}_\omega(\Pi_a)$ in addition can then be expanded into a generalized Fourier series absolutely convergent in $D_\omega(K)'_\beta$: $v = \sum_{n\in\mathbb{N}^p} \varphi_n(v) e_{\lambda_{(n)}}$, where the system $(\varphi_n)_{n\in\mathbb{N}^p}$ in $\mathcal{E}_\omega(\Pi_a)'$ as in 1.4.*

1.20 Corollary follows from Hahn-Banach theorem, 1.15 Theorem and 1.18 Theorem.

1.21 Remark. As in [19] we can prove:

---

[1] ADDED IN PROOF. The condition (v) of 1.15 Theorem is satisfied for the entire functions L_j of exponential type such that for each $1 \le j \le p$ there is a constant K with

$$0 < \inf_{|\mathrm{Im}\, z| > K} |L_j(z)| \exp(-a_j |\mathrm{Im}\, z|) \le \sup_{|\mathrm{Im}\, z| > K} |L_j(z)| \exp(-a_j |\mathrm{Im}\, z|) < +\infty$$

and $\inf_{k \ne n} |\lambda_k^{(j)} - \lambda_n^{(j)}| > 0$, where $(\lambda_m^{(j)})_{m\in\mathbb{N}}$ is the set of all zeros of L_j and any zero of L_j is simple. The functions as above are called the functions of sine type; the class of such functions (for $a_j = \pi$) was introduced by B. Ya. Levin. Other examples of the functions of sine type, besides $\sin(a_j z)$, may be found in the paper of B. Ya. Levin and Yu. I. Lyubarskii "Interpolation by entire functions of special classes and expansions in exponential series connected with it", Izv. Akad. Nauk USSR Ser. Mat. 39 (1975), No 3, 657–702 (Russian); English trans. in "Math. USSR Izv." 9 (1975). In particular, the condition (v) of 1.15 Theorem is satisfied for the functions $L_j(z) := A_j \exp(i a_j z) + B_j \exp(-i a_j z) + C_j$, $z \in \mathbb{C}$, with $A_j, B_j, C_j \in \mathbb{C}$, $A_j B_j \ne 0$, $C_j^2 \ne 4A_j B_j$, $1 \le j \le p$.

Let $(\mu_{(k)})_{k\in\mathbb{N}}$ be a sequence in $\mathbb{C}^p$ such that $|\mu_{(k)}| \to \infty$ and the system $X := (e_{\mu_{(k)}})_{k\in\mathbb{N}}$ is an ARS in the space $\mathcal{E}_\omega(\Pi_a)$ if it endowed with induced topology from $E(a,p)$. Then X is an ARS in $E(a,p)$ too.

2 A right inverse for a representation operator and a formula for it

In this part we solve the following problem: Assume that a sequence of exponents $(\lambda_{(n)})_{n\in\mathbb{N}^p}$ as in 1.5 is such that the statements $(i)-(vi)$ of 1.15 Theorem are valid. We fix $b \in \mathbb{R}^p$ with $0 < b \leq a$. By 1.20 Corollary the system $(e_{\lambda_{(n)}})_{n\in\mathbb{N}^p}$ is an ARS in $D_\omega(\Pi_b)'_\beta$. When does the surjective representation operator $R : K(b) \to D_\omega(\Pi_b)'_\beta$ admit a continuous linear right inverse (in the sequel, a right inverse)? As in the first section we put $E(b,p) := D_\omega(\Pi_b)'_\beta$.

For $u \in E(b,p)$ and $1 \leq j \leq p$ let $u_j^{(-1)}$ denote an antiderivatives of u as in Bremermann [4] (2.11) such that $D_j(u_j^{(-1)}) = u$.

We choose $\varphi_j \in D_\omega([-b_j, b_j])$ with $\int_{-\infty}^{\infty} \varphi_j(t)dt = 1$ (the function φ_j exists by Braun, Meise, Taylor [3], 2.6 Corollary). For any $f \in D_\omega(\Pi_b)$ and $x \in \mathbb{R}^p$ we put

$$f_j(x) := \int_{-\infty}^{x_j} (f(x_1, ..., x_{j-1}, t, x_{j+1}, ..., x_p) -$$

$$\varphi_j(t) \int_{-\infty}^{\infty} f(x_1, ..., x_{j-1}, y, x_{j+1}, ..., x_p)dy)dt \quad \text{and} \quad u_j^{(-1)}(f) := -u(f_j).$$

The map $u \mapsto u_j^{(-1)}$ is continuous und linear in $E(b,p)$.

For example ,

$$-i\lambda_j(e_\lambda)_j^{(-1)} = e_\lambda - \widehat{\varphi_j}(\lambda_j)e_{(\lambda_1,...,\lambda_{j-1},0,\lambda_{j+1},...,\lambda_p)} \text{ for any } \lambda \in \mathbb{C}^p. \quad (12)$$

For any $u \in E(b,p)$ we put $u^{(-1)} := \left(... \left(u_1^{(-1)} \right)_2^{(-1)} ... \right)_p^{(-1)}$.

The map $u \mapsto u^{(-1)}$ is continuous and linear in $E(b,p)$.

To derive the formulas for a right inverse for the representation operator, we will use the following definition with goes back to Leont'ev [13].

2.1 Definition. Let Q be an entire function in $\mathbb{C}^{2p}$ such that for any $\mu \in \mathbb{C}^p$ the function $Q(\cdot, \mu)$ belong to $A_\omega(p)$. The functional

$$\Omega_Q(z, \mu, u) := \left(e_z(ue_{-z})^{(-1)}\right)\left(\mathcal{F}^{-1}(Q(\cdot, \mu))\right), \quad z, \mu \in \mathbb{C}^p, \ u \in D_\omega(\mathbb{R}^p)',$$

we call a Q-interpolating functional.

2.2 Lemma. *Let Q be an entire function in $A(\mathbb{C}^{2p})$ such that $Q(\cdot, \mu) \in A_\omega(b, p)$ for any $\mu \in \mathbb{C}^p$.*

(i) For all $z, t \in \mathbb{C}^p$

$$(-i)^p(t-z)^1\Omega_Q(z, z, e_t) = Q(t, z) +$$

$$\sum_{k=1}^{p}(-1)^k \sum_{1 \le s_1 < s_2 < \ldots < s_k \le p}\left(\prod_{m=1}^{k}\widehat{\varphi}_{s_m}(t_{s_m} - z_{s_m})\right)Q(s(z,t), z), \quad (13)$$

where $s(z,t) \in \mathbb{C}^p$ and $s(z,t)_j := z_j$ if $j = s_m$ for some $1 \le m \le k$ and $s(t,z)_j := t_j$ if $j \ne s_m$ for all $1 \le m \le k$.

(ii) For all $z, \mu \in \mathbb{C}^p$ the functional $\Omega_Q(z, \mu, \cdot)$ is continuous and linear on $E(b, p)$.

(iii) We assume that $Q(z, \mu) = \prod_{j=1}^{p} Q_j(z_j, \mu_j)$ for all $z, \mu \in \mathbb{C}^p$ where Q_j are entire functions in $\mathbb{C}^2$ such that $Q_j(z_j, z_j) = L_j(z_j)$ for all $z_j \in \mathbb{C}$ and for all $1 \le j \le p$ and for any $k \in \mathbb{N}$ there are $m \in \mathbb{N}$ and a constant C satisfying

$$|Q(z, \mu)| \le C\exp(H_b(\operatorname{Im} z) + H_{a-b}(\operatorname{Im} \mu) - k\omega(z) + m\omega(\mu))$$

for all $z, \mu \in \mathbb{C}^p$. Then for any $k \in \mathbb{N}$ there are $s \in \mathbb{N}$ and $B < \infty$ such that

$$|\Omega_Q(\lambda_{(n)}, \lambda_{(n)}, u)| \le B\exp(H_{a-b}(\operatorname{Im} \lambda_{(n)}) + s\omega(\lambda_{(n)}))\|u\|_k^*$$

for all $u \in E(b, p)$ and $n \in \mathbb{N}^p$.

Proof. (i) follows from (12).

(*ii*): The map $u \mapsto e_z(ue_z)^{(-1)}$ is continuous and linear in $E(b,p)$. Since $\mathcal{F}^{-1}(Q(\cdot,\mu)) \in D_\omega(\Pi_b)$, the linear functional $v \mapsto v(\mathcal{F}^{-1}(Q(\cdot,\mu)))$ is continuous on $E(b,p)$. Hence $\Omega(z,\mu,\cdot) \in E(b,p)'$.

(*iii*): By (13) for all $t \in \mathbb{C}^p$ and $n \in \mathbb{N}^p$ we have $(-i)^p(t-\lambda_{(n)})^1\Omega_Q(\lambda_{(n)},\lambda_{(n)},e_t) = Q(t,\lambda_{(n)})$ and $\Omega_Q(\lambda_{(n)},\lambda_{(n)},e_t) = (-i)^pQ(t,\lambda_{(n)})/(t-\lambda_{(n)})^1$. We fix $k \in \mathbb{N}$ and choose m for k by 1.11 Corollary. From the estimates from above for $|Q|$ and 1.10 it follows that there are $l,s \in \mathbb{N}$ and $C_1,C_2 < \infty$ such that for all $t \in \mathbb{C}^p$ with $|t_j - \lambda_r^{(j)}| \geq 1$ for all $r \in \mathbb{N}$ and $1 \leq j \leq p$ the following holds

$$\begin{aligned}&|\Omega_Q(\lambda_{(n)},\lambda_{(n)},e_t)| \leq |Q(t,\lambda_{(n)})| \leq \\ &C_1\exp(H_b(\operatorname{Im} t) + H_{a-b}(\operatorname{Im}\lambda_{(n)}) - l\omega(t) + s\omega(\lambda_{(n)})) \leq \\ &C_2\exp(H_{a-b}(\operatorname{Im}\lambda_{(n)}) + s\omega(\lambda_{(n)}))\|e_t\|_m^* \text{ for all } n \in \mathbb{N}^p.\end{aligned}$$

By the maximum principle applying to the entire in $\mathbb{C}^p$ function $t \mapsto \Omega_Q(\lambda_{(n)},\lambda_{(n)},e_t)$ there is $C_3 < \infty$ such that for all $t \in \mathbb{C}^p$ and $n \in \mathbb{N}^p$

$$|\Omega_Q(\lambda_{(n)},\lambda_{(n)},e_t)| \leq C_3\exp(H_{a-b}(\operatorname{Im}\lambda_{(n)}) + s\omega(\lambda_{(n)}))\|e_t\|_k^*.$$

We put $h_n(t) := \Omega_Q(\lambda_{(n)},\lambda_{(n)},e_t)$ for all $t \in \mathbb{C}^p$ and $n \in \mathbb{N}^p$. From the estimates from above for $|Q|$ and (13) it follows that $h_n \in A_\omega(b,p)$. Since the linear functionals $u \mapsto \Omega_Q(\lambda_{(n)},\lambda_{(n)},u)$ (by 2.2 (*ii*)) and $u \mapsto \langle h_n,u\rangle$ are continuous on $E(b,p)$, for all $t \in \mathbb{C}^p$ we have $\langle h_n,e_t\rangle = \Omega_Q(\lambda_{(n)},\lambda_{(n)},e_t)$ and the set $\{e_t \,|\, t \in \mathbb{C}^p\}$ is total in $E(b,p)$, the equality $\langle h_n,u\rangle = \Omega_Q(\lambda_{(n)},\lambda_{(n)},u)$ holds for all $u \in E(b,p)$ and $n \in \mathbb{N}^p$. Hence $|\Omega_Q(\lambda_{(n)},\lambda_{(n)},u)| \leq \|u\|_k^*\|h_n\|_k$ for all $n \in \mathbb{N}^p$ and $u \in E(b,p)$. By 1.10 Corollary

$$\begin{aligned}\|h_n\|_k = \sup_{t\in\mathbb{C}^p}|h_n(t)|\exp(-H_b(\operatorname{Im} t) + k\omega(t)) \leq \sup_{t\in\mathbb{C}^p}\frac{|h_n(t)|}{\|e_t\|_k^*} \leq \\ C_3\exp(H_{a-b}(\operatorname{Im}\lambda_{(n)}) + s\omega(\lambda_{(n)})) \text{ for all } n \in \mathbb{N}^p.\end{aligned}$$

The proof of the following lemma is based on an idea of S. Momm.

2.3 Lemma. *Let $a_j > b_j$ for some $1 \leq j \leq p$ or ω is a (DN)-weight function (see 1.6). Then there is a function $Q_j \in A(\mathbb{C}^2)$ such that $Q_j(z,z) = L_j(z)$ and for any $k \in \mathbb{N}$ there are $m \in \mathbb{N}$ and a constant C with*

$$\log|Q_j(z,t)| \leq b_j|\operatorname{Im} z| + (a_j-b_j)|\operatorname{Im} t| - k\omega(z) + m\omega(t) + C \text{ for all } z,t \in \mathbb{C}.$$

Proof. By 1.8 Lemma there are the subharmonic on $\mathbb{C}$ functions u_λ and v_λ, $\lambda \in \mathbb{C}$, such that $u_\lambda(\lambda) \geq 0$, $v_\lambda(\lambda) \geq 0$ and for any $k \in \mathbb{N}$ there are $m \in \mathbb{N}$ and a constant C with

$$u_\lambda(z) \leq b_j|\operatorname{Im} z| - b_j|\operatorname{Im} \lambda| - k\omega(z) + m\omega(\lambda) + C$$

and

$$v_\lambda(t) \leq (a_j - b_j)|\operatorname{Im} t| - (a_j - b_j)|\operatorname{Im} \lambda| + m\omega(t) - k\omega(\lambda) + C$$

for all $z, t, \lambda \in \mathbb{C}$. The upper semicontinuous regularization $w(z,t)$ of the function $\sup\limits_{\lambda \in \mathbb{C}}(u_\lambda(z)+v_\lambda(t)+b_j|\operatorname{Im} \lambda|+(a_j-b_j)|\operatorname{Im} \lambda|)$ is plurisubharmonic on $\mathbb{C}^2$ and such that $w(z,z) \geq a_j|\operatorname{Im} z|$ and for any $k \in \mathbb{N}$ there are $m \in \mathbb{N}$ and $C < \infty$ with

$$w(z,t) \leq b_j|\operatorname{Im} z| + (a_j - b_j)|\operatorname{Im} t| - k\omega(z) + m\omega(t) + C \text{ for all } z, t \in \mathbb{C}.$$

By 1.4 there is $s \in \mathbb{N}$ such that $|L_j(z)| \leq s\exp(a_j|\operatorname{Im} z| + s\omega(z))$ and consequently $|L_j(z)| \leq s\exp(w(z,z) + s\omega(z))$ for all $z \in \mathbb{C}$. From a modification of 4.4.3 [8] the existence of a function Q_j follows.

2.4 Lemma. *There is no a family of the convex functions f_t, $t \geq 0$, on $[0, +\infty)$ such that $f_t(t) \geq 0$ and for any $k \in \mathbb{N}$ there are $m \in \mathbb{N}$ and a constant C with*

$$f_t(x) \leq C - kt + mx \text{ for all } t, x \geq 0.$$

Proof. We assume that such functions f_t exist. We put $g_t(x) := \limsup\limits_{n\to\infty} f_{nt}(nx)/n$ for all $t, x \geq 0$. Then g_t are the convex functions satisfying $g_t(t) \geq 0$ and for any $k \in \mathbb{N}$ there is $m \in \mathbb{N}$ such that

$$g_t(x) \leq -kt + mx \text{ for all } t, x \geq 0.$$

Since $g_1(0) \leq -k$ for all $k \in \mathbb{N}$, we have $g_1(0) = -\infty$ and hence a contradiction.

2.5 Lemma. *From (i) follows (ii):*

(i) There is a right inverse for the representation operator $R : K(b) \to E(b,p)$.

(ii) For any $1 \le s \le p$ *there are functions* v_λ, $\lambda \in \mathbb{C}$, *which are subharmonic on* $\mathbb{C}$, *such that* $v_\lambda(\lambda) \ge 0$ *and for any* $k \in \mathbb{N}$ *there are* $m \in \mathbb{N}$ *and a constant* C *with*

$$v_\lambda(t) \le (a_s - b_s)|\operatorname{Im} t| - (a_s - b_s)|\operatorname{Im} \lambda| + m\omega(t) - k\omega(\lambda) + C$$

for all $t, \lambda \in \mathbb{C}$.

Proof. We assume that there is a right inverse for R. Then there is a continuous linear left inverse κ for $R' : A_\omega(b,p) \to \Lambda(b)$. We put $f_n := \kappa(e_n^*)$, where $e_n^* := (\delta_{mn})_{m\in\mathbb{N}^p}$ for all $n \in \mathbb{N}^p$. Since $\kappa : \Lambda(b) \to A_\omega(b,p)$ is continuous, for any $k \in \mathbb{N}$ there are $m_1 \in \mathbb{N}$ and $C_1 < \infty$ such that for all $n \in \mathbb{N}^p$ and $z \in \mathbb{C}^p$

$$|f_n(z)| \le C_1 \exp(H_b(\operatorname{Im} z) - H_b(\operatorname{Im} \lambda_{(n)}) - k\omega(z) + m_1\omega(\lambda_{(n)})). \quad (14)$$

We fix $1 \le s \le p$ and choose $z_j \in \mathbb{C}$ and $g_j \in D_\omega([-b_j, b_j])$ for any $1 \le j \le p$ with $j \ne s$ such that $L_j(z_j) \ne 0$ and $\widehat{g}_j(z_j) \ne 0$. We put for any $z_s, \mu_s \in \mathbb{C}$

$$T_s(z_s, \mu_s) :=$$

$$\left(\Pi' L_j(z_j)\right) \sum_{n\in\mathbb{N}^p} \frac{L_s(\mu_s)}{\mu_s - \lambda_{n_s}^{(s)}} f_n(z)(z_s - \lambda_{n_s}^{(s)}) \left(\Pi' \widehat{g}_j(\lambda_{n_j}^{(j)})\right) e_{\lambda_{n_s}^{(s)}}, \quad (15)$$

where $\prod'$ denotes $\prod\limits_{j=1, j\ne s}^{p}$. By 1.5, 1.10, (14) the last series converges absolutely in $E(b_s, 1)$ for all $z_s, \mu_s \in \mathbb{C}$. Since κ is a continuous linear left inverse for $R' : A_\omega(b,p) \to \Lambda(b)$, for all $g \in A_\omega(b,p)$ and $\lambda \in \mathbb{C}^p$ the following holds

$$g(\lambda) = \sum_{n\in\mathbb{N}^p} g(\lambda_{(n)}) f_n(\lambda). \quad (16)$$

Hence

$$e_\lambda = \sum_{n\in\mathbb{N}^p} f_n(\lambda) e_{\lambda_{(n)}} \text{ for all } \lambda \in \mathbb{C}^p,$$

where the series converges absolutely in $E(b,p)$. This implies that in $E(b,p)$

$$0 = i\lambda_s e_\lambda + D_s(e_\lambda) = i \sum_{n\in\mathbb{N}^p} f_n(\lambda)(\lambda_s - \lambda_{n_s}^{(s)}) e_{\lambda_{(n)}} \text{ for all } \lambda \in \mathbb{C}^p. \quad (17)$$

By (15) and (17) we have for all $g_s \in D_\omega([-b_s, b_s])$ and $z_s, \mu_s \in \mathbb{C}$

$$< g_s, i\mu_s T_s(z_s,\mu_s) + D_1(T_s(z_s,\mu_s)) >=$$

$$i\left(\Pi' L_j(z_j)\right) \sum_{n\in\mathbb{N}^p} L_s(\mu_s) f_n(z)(z_s - \lambda_{n_s}^{(s)}) \left(\prod_{j=1}^{p} \widehat{g}_j(\lambda_{n_j}^{(j)})\right) = 0.$$

By 1.13 for any $z_s \in \mathbb{C}$ there exists a function $h_{z_s} : \mathbb{C} \to \mathbb{C}$ such that

$$T_s(z_s,\mu_s) = h_{z_s}(\mu_s) e_{\mu_s} \text{ for all } \mu_s \in \mathbb{C}. \tag{18}$$

From (15) and(18) it follows that for all $z_s \in \mathbb{C}$ the function h_{z_s} is entire in $\mathbb{C}$. By (15), (16) and (18) we have for all $g_s \in D_\omega([-b_s, b_s])$ and $z_s \in \mathbb{C}$

$$\begin{aligned} h_{z_s}(z_s) g_s(z_s) &= \left(\prod_{j=1}^{p} L_j(z_j)\right) \sum_{n\in\mathbb{N}^p} f_n(z) \left(\prod_{j=1}^{p} \widehat{g}_j(\lambda_{n_j}^{(j)})\right) = \\ & L_s(z_s)\left(\Pi' L_j(z_j) g_j(z_j)\right) g_s(z_s). \end{aligned}$$

Consequently, there exists a constant $B := \Pi' L_j(z_j) g_j(z_j) \neq 0$ such that

$$h_{z_s}(z_s) = B L_s(z_s) \text{ for all } z_s \in \mathbb{C}. \tag{19}$$

From (14), (18) and 1.10 it follows that for any $k \in \mathbb{N}$ there are $m_2 \in \mathbb{N}$ and a constant C_2 such that for all $z_s, \mu_s \in \mathbb{C}$

$$\log|h_{z_s}(\mu_s)| \leq b_s|\operatorname{Im} z_s| + (a_s - b_s)|\operatorname{Im} \mu_s| - k\omega(z_s) + m_2\omega(\mu_s) + C_2.$$

By 1.16 Remark and (19) there is a sequence of the circles $B_l := \{z_s \in \mathbb{C} \,|\, |z_s - \nu_l| < t_l\}$ with $\sum_{l\in\mathbb{N}} t_l < \infty$ and such that there are $m_3 \in \mathbb{N}$ and $C_3 < \infty$ with

$$\log|h_{z_s}(z_s)| = \log|L_s(z_s)| + \log B \geq a_s|\operatorname{Im} z_s| - m_3\omega(z_s) - C_3$$

for all $z_s \in \mathbb{C}\backslash(\cup_{l\in\mathbb{N}} B_l)$. We put $t_0 := \sum_{l\in\mathbb{N}} t_l$ and

$$P_{z_s}(\mu_s) := \sup_{|w|\leq t_0} \log|h_{z_s}(\mu_s + w)|, \quad z_s, \mu_s \in \mathbb{C}.$$

The function P_{z_s} is subharmonic in $\mathbb{C}$ and such that there are $m_4 \in \mathbb{N}$ and a constant C_4 with

$$P_{z_s}(z_s) \geq a_s|\mathrm{Im}\, z_s| - m_4\omega(z_s) - C_4 \text{ for all } z_s \in \mathbb{C}$$

and for any $k \in \mathbb{N}$ there are $m \in \mathbb{N}$ and a constant C with

$$P_{z_s}(\mu_s) \leq b_s|\mathrm{Im}\, z_s| + (a_s - b_s)|\mathrm{Im}\, \mu_s| - k\omega(z_s) + m\omega(\mu_s) + C$$

for all $z_s, \mu_s \in \mathbb{C}$.

We can take now as v_λ the functions

$$v_\lambda(t) := P_\lambda(t) - a_s|\mathrm{Im}\, \lambda| + m_4\omega(\lambda) + C_4.$$

For any weight function σ, any $c \in \mathbf{R}^p$ with $c \geq 0$ and any function $h \in B_\sigma(c,p)$ we denote by M_h the multiplication operator with the function h, i.e. $M_h(f) := hf$. For all $r > 0$ the operator M_h is continuous and linear from $A_\sigma(r,p)$ into $A_\sigma(r+c,p)$. Let $h(D)$ the adjoint to M_h operator from $D_\sigma(\Pi_{r+c})'_\beta$ into $D_\sigma(\Pi_r)'_\beta$. Note that for each $z \in \mathbb{C}^p$ the equality $h(D)(e_z) = h(z)e_z$ holds.

If $c = 0$ we say that $h(D)$ is an ultradifferential operator of class σ.

2.6 Theorem. *(I) Let $\omega(t) = \log^+ t$. The representation operator $R : K(b) \to E(b,p)$ has a right inverse if and only if $b < a$.*

(II) Let ω be a strong weight function. The representation operator $R : K(b) \to E(b,p)$ has a right inverse if and only if $a > b$ or there is $1 \leq j \leq p$ with $a_j = b_j$ and ω is a (DN)-weight function.

(III) If a right inverse for $R : K(b) \to E(b,p)$ exists, there exists the function Q as in 2.2 (iii) and a map

$$S(u) := \left((-i)^p \frac{\Omega_Q(\lambda_{(n)}, \lambda_{(n)}, u)}{L'(\lambda_{(n)})}\right)_{n\in\mathbb{N}^p}, \quad u \in E(b,p),$$

is a right inverse for the representation operator $R : K(b) \to E(b,p)$.

(IV) Let $p = 1$. If $S : E(b,1) \to K(b)$ is a right inverse for $R : K(b) \to E(b,1)$, then there is an unique function Q as in 2.2 (iii) such that

$$S = \left(-i\frac{\Omega_Q(\lambda_{(n)}, \lambda_{(n)}, \cdot)}{L'(\lambda_{(n)})}\right)_{n\in\mathbb{N}}.$$

Proof. NECESSITY IN (I): We assume that there are $1 \leq j \leq p$ with $a_j = b_j$ and a right inverse S for $R : K(b) \to E(b,p)$. By 2.5 Lemma there are the subharmonic in $\mathbb{C}$ functions v_λ, $\lambda \in \mathbb{C}$, such that $v_\lambda(\lambda) \geq 0$ and for any $k \in \mathbb{N}$ there are $m \in \mathbb{N}$ and a constant C with

$$v_\lambda(z) \leq m\log^+|z| - k\log^+|\lambda| + C \text{ for all } \lambda, z \in \mathbb{C}.$$

The upper semicontinuous regularization $\widetilde{v}_t(\mu)$ of $\sup\{v_t(e^{i\theta}\mu) \mid \theta \in \mathbb{R}\}$ is radial subharmonic function in $\mathbb{C}$ such that $\widetilde{v}_t(t) \geq 0$ and for any $k \in \mathbb{N}$ there are $m \in \mathbb{N}$ and a constant C with

$$\tilde{v}_t(x) \leq m\log^+ x - k\log^+ t + C \text{ for all } t, x \geq 0.$$

Since the functions $y \mapsto \tilde{v}_t(e^y)$ are convex on $\mathbb{R}$, this contradicts 2.4 Lemma.

NECESSITY IN (II): It follows from 2.4 Lemma and 1.8 Lemma.

(III) (AND SUFFICIENCY IN (I) AND (II)) : If a right inverse for R exists, by the necessity in (I) (resp. (II)) $a > b$ if $\omega(t) := \log^+ t$ (resp. $a > b$ or there is $1 \leq j \leq p$ with $a_j = b_j$ and ω is a (DN)-function). By 2.3 Lemma there exists a function Q as in 2.2 (iii). By 2.2 (iii) S is a continuous linear map of $E(b,p)$ into $K(b)$. We prove that $R \circ S = \mathrm{id}_{E(b,p)}$.

By 2.2 (i) and 1.15 (ii) we have for any $z \in \mathbb{C}^p$

$$L(z)(R \circ S)(e_z) = \sum_{n \in \mathbb{N}^p} L(z) \frac{Q(z, \lambda_{(n)})}{L'(\lambda_{(n)})(z - \lambda_{(n)})^1} e_{\lambda_{(n)}} =$$

$$Q(z, D)\left(\sum_{n \in \mathbb{N}^p} \frac{L(z)}{L'(\lambda_{(n)})(z - \lambda_{(n)})^1} e_{\lambda_{(n)}}\right) = Q(z, z)e_z = L(z)e_z.$$

(From the estimates from above for $|Q|$ it follows that for each $z \in \mathbb{C}^p$ the operator $Q(z, D)$ is continuous from $E(a,p)$ into $E(b,p)$.) Consequently $(R \circ S)(e_z) = e_z$ for all $z \in \mathbb{C}^p$. Since the set $\{e_z \mid z \in \mathbb{C}^p\}$ is total in $E(b,p)$, this implies that $(R \circ S)(u) = u$ for any $u \in E(b,p)$.

(IV): Let $S : E(b,1) \to K(b)$ be a right inverse for R. Then its adjoint operator $S' : \Lambda(b) \to A_\omega(b,1)$ is a continuous linear left inverse for $R' : A_\omega(b,1) \to \Lambda(b)$. As in the proof of 2.5 Lemma we put $e_n^* :=$

$(\delta_{ln})_{l\in\mathbb{N}}$, $f_n := S'(e_n^*)$, $n \in \mathbb{N}$, and choose a function $h_z \in A(\mathbb{C})$ such that

$$h_z(\mu)e_\mu := \sum_{n=1}^{\infty} \frac{L(\mu)}{\mu - \lambda_{(n)}} f_n(z)(z - \lambda_{(n)})e_{\lambda_{(n)}} \text{ for all } z, \mu \in \mathbb{C}. \qquad (20)$$

Let $Q(z,\mu) := h_z(\mu)$, $z, \mu \in \mathbb{C}$. From (20) it follows that $Q \in A(\mathbb{C}^2)$. By the proof of 2.5 Lemma we get $Q(z,z) = L(z)$ for all $z \in \mathbb{C}$. From (20), 1.10 we have the estimates from above for $|Q|$ as in 2.2 (iii).

By (20) the following holds

$$f_n(z) = \frac{Q(z, \lambda_{(n)})}{L'(\lambda_{(n)})(z - \lambda_{(n)})} \text{ for all } z \in \mathbb{C} \text{ and } n \in \mathbb{N}. \qquad (21)$$

To show that $S = (-i\Omega_Q(\lambda_{(n)}, \lambda_{(n)}, \cdot)/L'(\lambda_{(n)}))_{n\in\mathbb{N}}$ we note at first that by (21) for all $z \in \mathbb{C}$ and $c \in \Lambda(b)$

$$S'(c)(z) = \sum_{n\in\mathbb{N}} c_n \frac{Q(z, \lambda_{(n)})}{(z - \lambda_{(n)})L'(\lambda_{(n)})} = < S'(c), e_z > = < c, S(e_z) > .$$

Hence $S(e_z) = \left(\frac{Q(z, \lambda_{(n)})}{(z - \lambda_{(n)})L'(\lambda_{(n)})} \right)_{n\in\mathbb{N}}$. By (13) we have

$$S(e_z) = \left(-i \frac{\Omega_Q(\lambda_{(n)}, \lambda_{(n)}, e_z)}{L'(\lambda_{(n)})} \right)_{n\in\mathbb{N}} \quad \text{for all } z \in \mathbb{C}.$$

By (III) the linear operator $u \mapsto \left(-i \frac{\Omega_Q(\lambda_{(n)}, \lambda_{(n)}, u)}{L'(\lambda_{(n)})} \right)_{n\in\mathbb{N}}$ is continuous from $E(b,1)$ into $K(b)$. Hence for all $u \in E(b,1)$ $S(u) = \left(-i \frac{\Omega_Q(\lambda_{(n)}, \lambda_{(n)}, u)}{L'(\lambda_{(n)})} \right)_{n\in\mathbb{N}}$.

To prove the uniqueness of Q, we assume that the functions Q_1, Q_2 satisfy of the conditions of 2.2 (iii) and for all $u \in E(b,1)$ and for all $n \in \mathbb{N}$ $\Omega_{Q_1}(\lambda_{(n)}, \lambda_{(n)}, u)/L'(\lambda_{(n)}) = \Omega_{Q_1}(\lambda_{(n)}, \lambda_{(n)}, u)/L'(\lambda_{(n)})$. Then by 2.2 (i) Lemma $Q_1(z, \lambda_{(n)}) = Q_2(z, \lambda_{(n)})$ for all $z \in \mathbb{C}$ and $n \in \mathbb{N}$.

We fix $z \in \mathbb{C}$. The estimates from above for $|Q_1|, |Q_2|$ imply that $Q_1(z, \cdot), Q_2(z, \cdot) \in B_\omega(a-b, 1)$. Since by 1.15 Theorem, (vi) $(\lambda_{(n)})_{n\in\mathbb{N}}$ is the uniqueness set for $A_\omega(a,1)$, by 1.14 Lemma $(\lambda_{(n)})_{n\in\mathbb{N}}$ is the uniqueness set for $B_\omega(a-b,1)$) too. Hence $Q_1(z,\mu) = Q_2(z,\mu)$ for all $\mu \in \mathbb{C}$ and $Q_1 \equiv Q_2$ in $\mathbb{C}^2$.

We give in conclusion an adding to 8 Theorem in Braun [1].

2.7 Corollary. *For any $b > 0$ and any weight function σ with $\omega(t) = o(\sigma(t))$ as $t \to +\infty$ there are an ultradifferential operator of class σ $h(D) : \mathcal{E}_\omega(\Pi_b) \to D_\sigma(\Pi_b)'_\beta$ such that there is a continuous and linear operator $T : E(b,p) \to \mathcal{E}_\omega(\Pi_b)$ with $h(D)\circ T = \mathrm{id}_{E(b,p)}$.*

Proof. We exploit an idea of Korobeinik of the construction of a right inverse for a convolution operator with the help of a right inverse for a representation operator. By 1.19 Corollary and 2.6 Theorem there is an ARS $(e_{\lambda_{(n)}})_{n\in\mathbb{N}^p}$ in $E(b,p)$ with $\lambda_{(n)} \in \mathbb{R}^p$ for all $n \in \mathbb{N}^p$ and such that the representation operator $R : K(b) \to E(b,p)$ have a right inverse S. By Braun [1] (Theorem 7) there is a function $h \in B_\sigma(0,p)$ such that $|h(x)| \geq \exp(\sigma(x))$ for all $x \in \mathbb{R}^p$. The ultradifferential operator $h(D)$ maps $D_\sigma(\Pi_b)'_\beta$ into $D_\sigma(\Pi_b)'_\beta$ and, consequently, $\mathcal{E}_\omega(\Pi_b) \subset D_\sigma(\Pi_b)'_\beta$ into $D_\sigma(\Pi_b)'_\beta$ continuously and linearly. We put

$$T(u) := \sum_{n\in\mathbb{N}^p} \frac{(S(u))_n}{h(\lambda_{(n)})} e_{\lambda_{(n)}}, \quad u \in E(b,p).$$

By the estimates from above for the norms of $e_{\lambda_{(n)}}$ in $\mathcal{E}_\omega(\Pi_b)$ (see Braun, Meise, Taylor [3], 7.1) this series converges absolutely in $\mathcal{E}_\omega(\Pi_b)$ for all $u \in E(b,p)$. Hence by Banach-Steinhaus theorem the operator T is continuous and linear from $E(b,p)$ into $\mathcal{E}_\omega(\Pi_b)$. Moreover we have

$$h(D)\circ T(u) = \sum_{n\in\mathbb{N}^p} (S(u))_n e_{\lambda_{(n)}} = u \text{ for all } u \in E(b,p).$$

We note that if ω is a weight function, by Braun, Meise, Taylor [3] (1.6 Lemma) there exists a weight function σ with $\omega(t) = o(\sigma(t))$ as $t \to +\infty$. If $\omega(t) := \log^+ t$, for any weight function σ by 1.1 (γ) we have $\omega(t) = o(\sigma(t))$ as $t \to +\infty$.

References

[1] Braun, R., *An extension of Komatsu's second structure theorem for ultradistributions,* J. Fac. Sci. Univ. Tokyo. Sect. IA, Math. 40 (1993), 411–417.

[2] Braun, R., Meise, R., *Generalized Fourier expansions for zero-solutions of surjective convolution operators on $D_{\{\omega\}}(\mathbb{R})'$,* Arch. Math. 55 (1990), 55–63.

[3] Braun, R., Meise, R., Taylor, B. A., *Ultradifferentiable functions and Fourier analysis,* Results Math. 17 (1990), 206–237.

[4] Bremermann, H., *Distributions, complex variables, and Fourier transform,* Addison-Wesley, 1965.

[5] Edvards, R., *Fourier Series. A Modern Introduction. II,* Springer-Verlag, New York Heidelberg Berlin, 1982.

[6] Franken, U., Meise, R., *Generalized Fourier expansions for zero-solutions of surjective convolution operators on $D'(\mathbb{R})$ and $D'_\omega(\mathbb{R})$,* Note Mat. 10, Suppl.1 (1990), 251–272.

[7] Grothendieck, A., *Produits tensoriel topologiques et espaces nucléaires,* Mem. Amer. Math. Soc. 16, Providence 1955.

[8] Hörmander, L., *The Analysis of Linear Partial Differential Operators I,* Springer, 1983.

[9] Hörmander, L., *An Introduction to Complex Analysis in Several Variables,* Princeton Univ. Press, 1967.

[10] Korobeinik, Yu. F., *Representing systems,* Russian Math. Surveys 36 (1981), 75–137.

[11] Langenbruch, M., *Splitting of the $\overline{\partial}$-complex in weighted spaces of square integrable functions,* Rev. Mat. Univ. Complut. Madrid 5 (1992), 201–223.

[12] Langenbruch, M., *The splitting conditions for the weighted $\overline{\partial}$-complex,* Results Math. 22 (1992), 560–597.

[13] Leont'ev, A. F., *Exponential Series*, Moskow, Nauka, 1976 (in Russian).

[14] Meise, R., *Sequence space representations for zero-solutions of convolution equations on ultradifferentiable functions of Roumieu type*, Studia Math. 125 (1989), 211–230.

[15] Meise, R., Momm, S., Taylor, B. A., *Splitting of slowly decreasing ideals in weighted algebras of entire functions*, in: Complex analysis II, C. A. Berenstein (ed.), Lecture Notes in Math. 1276, Springer, 1987, 229–252.

[16] Meise, R., Taylor, B.A., *Splitting of closed ideals in (DFN)-algebras of entire functions and the property (DN)*, Trans. Amer. Math. Soc. 302 (1987), 341–370.

[17] Meise, R., Taylor, B. A., *Whitney's extension theorem for ultradifferentiable functions of Beurling type*, Ark. Mat. 26 (1988), 265–287.

[18] Melikhov, S. N., *Absolutely convergent series in the canonical inductive limits*, Math. Notes 39 (1986), 475–480.

[19] Melikhov, S. N., *On expansion of analytic functions in exponential series*, Math. USSR Izv. 33 (1989), 317–329.

[20] Melikhov, S. N., *Nontrivial expansions of zero and representative subspaces*, Soviet Math. (Iz. VUZ), 34 (1990). No. 8, 60–71.

[21] Melikhov, S. N., *Generalized Exponential Series for Distributions*, Russian Acad. Sci. Dokl. Math. 53 (1996), No. 2, 266–269.

[22] Meyer, T., *Die Fourier-Laplace-Transformation quasi-analytischer Funktionale und ihre Anwendung auf Faltungsoperatoren*, Diplomarbeit, Düsseldorf, 1989.

[23] Napalkov, V. V., *On discrete sufficient sets in some spaces of entire functions*, Soviet Math. Dokl. 21 (1980), No. 1, 218–221.

[24] Pietsch, A., *Nuclear Locally Convex Spaces*, Springer-Verlag, 1972.

[25] Vladimirov, V. S., *Distributions en Physique Mathématique*, Editions Mir, Moskou, 1979.

[26] Yulmukhametov, R. S., *On an approximation of the subharmonic functions*, Anal. Math. 11 (1985), 257–282 (in Russian).

Dept. of Mechanics and Mathematics
Rostov State University
Zorge St. 5
344104 Rostov on Don, Russia
e-mail: melih@ms.math.rsu.ru

Recibido: 23 de Septiembre de 1998
Revisado: 24 de Febrero de 1999