REVISTA MATEMÁTICA COMPLUTENSE
Volumen 12, número 2: 1999
http://dx.doi.org/10.5209/rev_REMA.1999.v12.n2.17111

Extension and splitting theorems for Fréchet spaces of type 2.

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Abstract

We prove the following common generalization of Maurey's extension theorem and Vogt's (DN)-(Ω) splitting theorem for Fréchet spaces: if T is an operator from a subspace E of a Fréchet space G of type 2 to a Fréchet space F of dual type 2, then T extends to a map from G into F'' whenever G/E satisfies (DN) and F satisfies (Ω).

In the paper [M] Maurey proved a very strong extension theorem for Banach spaces: each operator from a subspace E of a Banach space G of type 2 into a Banach space of cotype 2 extends to the whole space G. In particular, this implies a result of Kadec and Pełczyński [KP, Cor. 1] that every Hilbert subspace of $L_p(\mu)$ for $2 \leq p < \infty$ is always complemented. It is well-known that in the Fréchet case even a subspace of a hilbertizable space (i.e., a projective limit of Hilbert spaces) could be uncomplemented. Thus there is no straightforward generalization of Maurey's result. On the other hand, for hilbertizable spaces the splitting result of Vogt holds ([V3] or [MV2, 30.1]): a closed subspace F of a hilbertizable Fréchet space G such that F has property (Ω) and G/F has property (DN), is complemented. We will prove (Cor. 4.4) that a hilbertizable Fréchet subspace F of a Fréchet space G of type 2 (i.e., a projective limit of type 2 Banach spaces) is complemented

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1991 Mathematics Subject Classification: Primary 46A04, 46A22. Secondary 46B70.
The research of the third named author was partially supported by Komitet Badań Naukowych (State Committee for Scientific Research), Poland, grant no. 2PO3A 050 09.
Servicio Publicaciones Univ. Complutense. Madrid, 1999.

whenever F and G/F satisfy Vogt's conditions. Similarly, we prove an extension version of this result (Cor. 4.5). In view of the preceding this result seems to be a natural generalization of Maurey's extension theorem for Fréchet spaces. The proof is deeply influenced by [V3]. For recent splitting results see [F] and [FW].

The techniques used are based on interpolation theory and the local theory of Banach spaces, in particular, a deep result of Kouba [K] (Th. 2.2 below), as well as the very subtle splitting criterion of Vogt [V1] (Th. 3.2 below). In order to make the paper selfcontained and accessible for non-specialist in one of the involved fields we start with a short presentation of notions and results from interpolation theory, local theory of Banach spaces and the splitting theory for Fréchet spaces.

Another different generalization of Maurey's extension theorem for operators from Fréchet spaces into Banach spaces has been obtained recently by Peris and the first named author [DP].

In general we follow the notation and terminology of [J], of [DF] (for tensor products) and of [BL] (for interpolation theory). Operator means a linear continuous map. An arbitrary Fréchet space E is a reduced projective limit of a sequence of Banach spaces (E_n). We always denote by $i_n^k : E_k \to E_n$ the linking maps and by $i_k : E \to E_k$ the standard projections. Reduced means that $i_k(E)$ is dense in E_k. By $\|\cdot\|_n$, we denote the norm in E_n as well as the induced seminorm on E, and by E', E'' the strong dual and bidual, respectively.

1 Preliminaries from interpolation theory

A pair $\bar{X} = (X_0, X_1)$ of Banach spaces is called a *Banach couple* if X_0 and X_1 are both continuously embedded in some Hausdorff topological vector space $\mathcal{X}$. The couple is called *ordered* if $X_1 \hookrightarrow X_0$, where $\hookrightarrow$ means a continuous embedding. For a Banach couple $\bar{X}$ we form the intersection $X_0 \cap X_1$ and the sum $X_0 + X_1$. They are both Banach spaces equipped with the norms

$$\| x \| := J(1, x; \bar{X}) \qquad \text{and} \qquad \| x \| := K(1, x; \bar{X}),$$

respectively, where for $t > 0$ [BL, 2.6]:

$$J(t, x; \bar{X}) := \max\{\| x \|_{X_0}, t \| x \|_{X_1}\}$$

$K(t,x;\bar{X}) := \inf\{\| x_0 \|_{X_0} + t \| x_1 \|_{X_1} : x = x_0 + x_1, x_0 \in X_0, x_1 \in X_1\}$.

A Banach space X is called an *intermediate space with respect to* $\bar{X}$ if $X_0 \cap X_1 \hookrightarrow X \hookrightarrow X_0 + X_1$.

Let $0 < \theta < 1$ and let X be an intermediate space with respect to $\bar{X}$. Then X is said to be of *J-type* θ, shortly $X \in C_J(\theta; \bar{X})$ (respectively, *K-type* θ, shortly $X \in C_K(\theta; \bar{X})$) [BL, 3.5.1] if there exists $C > 0$ such that for all positive t:

$$\| x \|_X \leq C t^{-\theta} J(t,x;\bar{X}) \qquad \text{for all } x \in X_0 \cap X_1$$

(respectively, $K(t,x;\bar{X}) \leq C t^{\theta} \| x \|_X$ for all $x \in X$). Moreover, we need the following two important "interpolation methods". If $\bar{X} = (X_0, X_1)$ is a Banach couple and $0 < \theta < 1$, then the Lions-Peetre space $\bar{X}_{\theta,1}$ [BL, 3.1] is given by:

$$\bar{X}_{\theta,1} := \{x \in X_0 + X_1 : \| x \|_{\theta,1} := \left(\int_0^\infty t^{-\theta} K(t,x;\bar{X}) \frac{dt}{t} \right) < \infty\},$$

whereas the complex interpolation space of Calderón [BL, 4.1] is defined as follows: assuming that X_0 and X_1 are complex spaces, denote by $\mathcal{F}(\bar{X})$ the space of all continuous functions f on the closed strip $\{z \in \mathbb{C} : 0 \leq \text{Re } z \leq 1\}$ with values in $X_0 + X_1$, which are analytic on the interior, and such that the functions $f(j+it)$ are continuous into X_j $(j = 0,1)$ and tend to zero whenever $|t| \to \infty$. Then

$$\| f \|_{\mathcal{F}(\bar{X})} := \max_{j=0,1} \sup_{t \in \mathbf{R}} \| f(j+it) \|_{X_j}$$

defines a norm on $\mathcal{F}(\bar{X})$, and for $0 < \theta < 1$ the interpolation space $[\bar{X}]_\theta$ of Calderón is defined by $[\bar{X}]_\theta := \{f(\theta) : f \in \mathcal{F}(\bar{X})\}$ and equipped with the corresponding quotient norm.

The following proposition is well-known:

Proposition 1.1. *Let $\bar{X} = (X_0, X_1)$ be a Banach couple.*

(a) [BL, 4.1.2] The spaces $[\bar{X}]_\theta$ are of K-type θ.

(b) [BL, p. 49] An intermediate space X is of J-type θ if and only if $\bar{X}_{\theta,1} \hookrightarrow X$ if and only if there is $C > 0$ such that

$$\| x \|_X \leq C \| x \|_{X_0}^{1-\theta} \| x \|_{X_1}^{\theta} \qquad \text{for all } x \in X_0 \cap X_1.$$

Corollary 1.2. *If $\bar{X}$ is ordered and X is of K-type ν, $\theta < \nu$, then $X \hookrightarrow \bar{X}_{\theta,1}$. In particular, if Y is a Banach space of J-type θ, then $X \hookrightarrow Y$.*

Proof. It suffices to show that any $x \in X$ belongs to $\bar{X}_{\theta,1}$. Since $K(t, x; \bar{X}) \leq \| x \|_{X_0}$, it follows that

$$\int_1^\infty t^{-\theta} K(t, x; \bar{X}) \frac{dt}{t} < \infty.$$

On the other hand, since X is of K-type ν we get

$$\int_0^1 t^{-\theta} K(t, x; \bar{X}) \frac{dt}{t} \leq C \int_0^1 t^{\nu-\theta} \| x \|_X \frac{dt}{t} < \infty,$$

which yields $\| x \|_{\theta,1} < \infty$.

We will also need the following lemma:

Lemma 1.3. *Let (X, Y) be a Banach couple. If Z is an intermediate space of J-type θ, then for any $f \in Z'$ and any $\varepsilon > 0$ there are $g \in X'$, $h \in Y'$ such that*

$$f|_{X \cap Y} = g|_{X \cap Y} + h|_{X \cap Y} \quad \text{and} \quad \| g \|_{X'} \leq \varepsilon. \tag{1.1}$$

Proof. For a fixed $t > 0$ we equip $X \oplus Y$ with the norm $|\cdot|_t$ defined by

$$|(x, y)|_t := \max(\| x \|_X, t \| y \|_Y)$$

and obtain an isometry

$$I : (X \cap Y, J(t, \cdot; (X, Y))) \to (X \oplus Y, |\cdot|_t), \qquad I(x) := (x, x).$$

Moreover, by assumption, the embedding $(X \cap Y, J(t, \cdot; (X, Y))) \overset{j}{\hookrightarrow} Z$ is of norm $\leq Ct^{-\theta}$ (C independent of t).

Now, take $f \in Z'$ and consider $j'(f) \in (X \cap Y, J(t, \cdot; (X, Y)))'$. Obviously, there is a pair $(g, h) \in (X \oplus Y, |\cdot|_t)'$ such that

$$I'(g, h) = j'(f), \tag{1.2}$$

and

$$|(g, h)|'_t = \| g \|_{X'} + \frac{1}{t} \| h \|_{Y'} \leq 2 \| j' \| \| f \|_{Z'} \leq 2Ct^{-\theta} \| f \|_{Z'} .$$

Taking t big enough we get $\| g \|_{X'} \leq \varepsilon$. The equality (1.2) means exactly (1.1).

2 Preliminaries from the local theory of Banach spaces

Let us recall (see [Pi2, Ch. 3] or [DF, p. 86]) that a Banach space X is of *type* 2 (*cotype* 2, respectively), whenever there is a constant $C > 0$ such that for any finite sequence $(x_1, \ldots, x_n) \subseteq X$ the following inequality holds:

$$2^{-n} \sum_{\varepsilon} \| \sum_{i=1}^{n} \varepsilon_i x_i \| \leq C \left(\sum_{j=1}^{n} \| x_j \|^2 \right)^{1/2}$$

$$(2^{-n} \sum_{\varepsilon} \| \sum_{i=1}^{n} \varepsilon_i x_i \| \geq C \left(\sum_{j=1}^{n} \| x_j \|^2 \right)^{1/2}, \qquad \text{respectively}),$$

where the first sum $\sum_{\varepsilon}$ is taken over all sequences $\varepsilon = (\varepsilon_i)$ of signs ± 1.

The only spaces which are both of type and cotype 2 are the Hilbert spaces ([Pi2, Th. 3.3] or [DF, 30.5]), while $L_p(\mu)$ is of type 2 for $2 \leq p < \infty$ and of cotype 2 for $1 \leq p \leq 2$ (see [DF, Prop. 8.6]). It is also clear that type is inherited by subspaces and quotients while cotype only by subspaces.

Proposition 2.1. ([DF, 31.2], [Pi2, Prop. 3.2]) *If a Banach space X is of type 2, then X' is of cotype 2.*

Remark. This yields that X is of cotype 2, whenever X' is of type 2. The converse holds iff X is so-called K-convex iff X has some type strictly larger than 1 [Pi1].

Now, we call a Fréchet space *hilbertizable* (*of type* 2, *of cotype* 2, *of dual type* 2) whenever it is a projective limit of Hilbert spaces (Banach spaces of type 2, of cotype 2, of spaces with duals of type 2, respectively). Again it is clear that for Fréchet spaces type 2 is inherited by subspaces and quotients while cotype by subspaces only. Thus, without loss of generality, we can consider reduced spectra only. In general, a Fréchet space of dual type 2 is of cotype 2 while a projective limit of K-convex Banach spaces with cotype 2 is a Fréchet space of dual type 2.

The following deep result of Kouba [K, Th. 4.4] for completed projective tensor products $\hat{\otimes}$ will be essential:

Theorem 2.2. *Let (X_0, X_1) and (Y_0, Y_1) be two Banach couples and let X_0, X_1, Y_0, Y_1 be of type 2. Then $(X_0 \hat{\otimes} Y_0, X_1 \hat{\otimes} Y_1)$ is a Banach couple* *and for* $0 < \theta < 1$:

$$[X_0\hat{\otimes}Y_0, X_1\hat{\otimes}Y_1]_\theta = [X_0, X_1]_\theta\hat{\otimes}[Y_0, Y_1]_\theta.$$

3 Preliminaries from the theory of short exact sequences

Let us assume that all the considered spaces (E, F, G etc.) are Fréchet. The following diagram

$$0 \to F \xrightarrow{j} G \xrightarrow{q} E \to 0 \tag{3.1}$$

is called *short exact sequence*, whenever $\ker q = \operatorname{im} j$ and j, q are a topological embedding and a quotient map, respectively. If j has a left inverse (or, equivalently, q has a right inverse), then (3.1) *splits*. We say that (3.1) *splits locally* if for any continuous seminorm $\|\cdot\|$ on F there is another continuous seminorm $\|\cdot\|_1$ on G and linear map $r : G \to F$ such that r is a left inverse for j and continuous as a map $r : (G, \|\cdot\|_1) \to (F, \|\cdot\|)$. Equivalently, local splitting can be formulated in terms of a linear map $E \to G$.

The following certainly known fact shows that splitting results usually lead to extension theorems and vice versa.

Proposition 3.1. *Let E, F, G, H be Fréchet spaces, $F = \operatorname{proj} F_n$, $G = \operatorname{proj} G_n$, $H = \operatorname{proj} H_n$ and $T : F \to H$ be an operator.*

(a) For each short exact sequence (3.1) there is a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H & \xrightarrow{j_0} & G_0 & \xrightarrow{q_0} & E & \longrightarrow & 0 \\ & & \uparrow T & & \uparrow T_0 & & \uparrow \mathrm{id} & & \\ 0 & \longrightarrow & F & \xrightarrow{j} & G & \xrightarrow{q} & E & \longrightarrow & 0. \end{array} \tag{3.2}$$

(b) The upper row in (3.2) splits iff T extends onto G.

(c) The upper row in (3.2) splits locally iff for every n there are k and l such that the map $T_n^k : F_k \to H_n$ induced by T factorizes through some map $j_l^k : F_k \to G_l$ induced by j.

Proof. (a): Take the closed subspace $A := \{(Ty, -jy) : y \in F\} \subseteq H \oplus G$, the quotient $G_0 := H \oplus G/A$ and define T_0, j_0, q_0 canonically.

(b): If S is an extension of T, then $T_0 - j_0S$ induces a lifting of $\mathrm{id} : E \to E$ into G_0 and the upper row splits. Conversely, if $R : E \to G_0$ is a right inverse for q_0, then $T_0 - Rq$ gives the required extension.

(c): Analogous to (b).

The fact that each sequence (3.1) splits, where E, F are fixed, is denoted traditionally by $\mathrm{Ext}^1(E, F) = 0$. Vogt's theory of the functor Ext^1 gives a very precise criterion which allows to conclude splitting from local splitting (comp. [V1,Prop. 2.1] and its proof, or [MV2]):

Theorem 3.2. *Let F be a projective limit and E a reduced projective limit of sequences of Banach spaces (F_n) and (E_n), respectively. Then every locally splitting sequence (3.1) splits whenever the following sufficient condition holds:*

$$\exists\, n\ \forall\, k\ \exists\, l\ \forall\, m, p, \varepsilon > 0\ \exists\, r\ \forall\, \phi \in L(E_p, F_l)\ \exists\, \psi \in L(E_n, F_k),$$

$$\chi \in L(E_r, F_m) : \|\, \psi \,\| \leq \varepsilon \quad \text{and} \quad i_k^l \phi i_p^r = \psi i_n^r + i_k^m \chi.$$

Note that this formulation is for non-necessarily reduced projective spectra (F_n) and an individual sequence (3.1) its proof follows verbatim the proof in [V1].

In the splitting theory the following conditions play a fundamental role:

A Fréchet space E has property (DN) iff:

$$\exists\, n\ \forall\, p, 0 < \tau < 1\ \exists\, r, C : \|\, x \,\|_p \leq C \,\|\, x \,\|_n^{1-\tau} \|\, x \,\|_r^{\tau} \qquad \text{for } x \in E,$$

and property (Ω) iff:

$$\forall\, k\ \exists\, l\ \forall\, m\ \exists\, 0 < \nu < 1, C : \|\, x \,\|_l^* \leq C \,\|\, x \,\|_k^{*1-\nu} \|\, x \,\|_m^{*\nu} \qquad \text{for } x \in E_m',$$

where $\|\, x \,\|_k^* := \sup_{\|y\|_k \leq 1} |x(y)|$.

It is proved [V2, Lemma 5.7] (comp. [MV2, p. 362]) that property (DN) of E implies that E is countably normed, i.e., the sequence of seminorms $(\|\cdot\|_n)$ on E could be chosen in such a way that all $\|\cdot\|_n$ are norms and $i_n^{n+1} : E_{n+1} \to E_n$ are injective, $n \in \mathbb{N}$.

4 Splitting for spaces of suitable type and cotype

We start with the following local splitting theorem:

Proposition 4.1. *Let H and G be Fréchet spaces of cotype 2 and of type 2, respectively. Then in the following commutative diagram with exact rows:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H & \longrightarrow & G_0 & \longrightarrow & E & \longrightarrow & 0 \\ & & \uparrow T & & \uparrow & & \uparrow \mathrm{id} & & \\ 0 & \longrightarrow & F & \xrightarrow{j} & G & \xrightarrow{q} & E & \longrightarrow & 0 \end{array}$$

the upper row splits locally.

Proof. Let H be a reduced projective limit of spaces (H_n) of cotype 2. Without loss of generality we may assume that $T_n^n : F_n \to H_n$ (the map induced by T) factorizes through a subspace of some G_k, where G is a reduced projective limit of Banach spaces (G_n) of type 2. By Maurey's extension theorem (see the introduction), T_n^n factorizes through some G_k which, by Prop. 3.1 (c), implies local splitting of the upper row.

Now, in view of Theorem 3.2, we need the following more subtle version of Vogt's "Decomposition Lemma" (see [V3, Prop. 2.4] or [MV2, 30.4]):

Theorem 4.2. *Let (E_0, E_2) and (F_2, F_0) be ordered Banach couples consisting of spaces of type 2. Assume that E_1, F_1 are intermediate with respect to (E_0, E_2) and (F_2, F_0), respectively, such that*

$$E_1 \in C_J(\tau, (E_0, E_2)) \quad \textit{and} \quad F_1 \in C_J(1-\nu, (F_2, F_0))$$

with $0 \leq \tau < \nu \leq 1$. Then for any $T \in L(E_1, F_1')$ and $\varepsilon > 0$ there exist $T_0 \in L(E_0, F_0')$, $T_2 \in L(E_2, F_2')$ such that

$$\| T_0 \| \leq \varepsilon \qquad \textit{and} \qquad i_0^1 T i_1^2 = T_0 i_0^2 + i_0^2 T_2,$$

where the i_l^k denote the linking maps in the respective triples of Banach spaces.

Proof. Without loss of generality, by taking complexifications if necessary, we may assume that all spaces are complex.

Let $0 \leq \tau < \theta < \nu \leq 1$. By Prop. 1.1 and Cor. 1.2, we obtain the following continuous inclusions:

$$[E_0, E_2]_\theta \overset{i_1^\theta}{\hookrightarrow} E_1 \qquad \text{and} \qquad [F_2, F_0]_{1-\theta} \overset{j_1^\theta}{\hookrightarrow} F_1,$$

and hence, by duality, the following commutative diagram:

$$\begin{tikzcd}
E_0 \arrow[r, "\mathrm{id}"] & E_0 & F_0' \arrow[r, "\mathrm{id}"] & F_0' \\
{[E_0, E_2]_\theta} \arrow[u, "i_0^\theta"] \arrow[r, "i_1^\theta"] & E_1 \arrow[u, "i_0^1"] \arrow[r, "T"] & F_1' \arrow[u, "i_0^1"] \arrow[r, "(j_1^\theta)'"] & ([F_0, F_2]_\theta)' \arrow[u, "i_0^\theta"] \\
E_2 \arrow[u, "i_\theta^2"] \arrow[r, "\mathrm{id}"] & E_2 \arrow[u, "i_1^2"] & F_2' \arrow[u, "i_1^2"] \arrow[r, "\mathrm{id}"] & F_2' \arrow[u, "i_\theta^2"]
\end{tikzcd}$$

Define $S = (j_1^\theta)' T i_1^\theta \in L([E_0, E_2]_\theta, ([F_0, F_2]_\theta)')$ and $X := E_0 \hat{\otimes} F_0$, $Y := E_2 \hat{\otimes} F_2$. Then, by Kouba's Th. 2.2,

$$[X, Y]_\theta = [E_0, E_2]_\theta \hat{\otimes} [F_0, F_2]_\theta, \tag{4.1}$$

and, by the well-known duality of projective tensor products [DF, Prop. 3.2],

$$([X, Y]_\theta)' = ([E_0, E_2]_\theta \hat{\otimes} [F_0, F_2]_\theta)' = L([E_0, E_2]_\theta, ([F_0, F_2]_\theta)').$$

Hence, Lemma 1.3 applied to $S = f \in ([X, Y]_\theta)'$ yields that there are $T_0 \in L(E_0, F_0')$, $T_2 \in L(E_2, F_2')$ such that $\| T_0 \| \leq \varepsilon$ and

$$S|_{E_2 \otimes F_0} = T_0|_{E_2 \otimes F_0} + T_2|_{E_2 \otimes F_0}.$$

This means

$$i_0^\theta S i_\theta^2 = T_0 i_0^2 + i_0^2 T_2,$$

which completes the proof.

Remark. The assumption type 2 is only used to derive formula (4.1).

Theorem 4.3. *Let $F \in (\Omega)$ be of dual type 2 and let $E \in (DN)$ be of type 2. Then the following locally splitting short exact sequence splits*

$$0 \to F'' \to G \to E \to 0.$$

Proof. We check the condition from Th. 3.2 — without loss of generality, assume that F is a reduced projective limit of (F_n) with F_n' of type 2. Choose n as in (DN) for E and fix k, then l as in (Ω) for F and for any m find $0 < \nu < 1$ and C also as in (Ω) for F. Finally, choose p and $\tau < \nu$ and a corresponding r as in (DN). Since the (DN) condition implies that E is countably normed we may assume that the mappings

$$i_n^p : E_p \to E_n \qquad \text{and} \qquad i_p^r : E_r \to E_n$$

are injective. Now, apply Theorem 4.2 to the triples

$$E_r \to E_p \to E_n \qquad \text{and} \qquad F_k' \to F_l' \to F_m',$$

where the linking maps are injective also in the second triple because the spectrum (F_n) is reduced. Fréchet spaces with condition (Ω) are quasinormable [MV1] and, thus, distinguished [MV2, 26.18]. Hence F'' is a projective limit of the bidual spectrum (F_n''). Since (DN) and (Ω) mean exactly that $E_p \in C_J(\tau, (E_n, E_r))$ and $F_l' \in C_J(1-\nu, (F_m', F_k'))$ (see 1.1 (b)), we obtain the desired condition from 3.2 for F'' and E.

Remark. Note again that the type 2 assumptions on E and F are only required in order to make sure that the couples (E_n, E_r) and (F_m', F_k') satisfy Kouba's formula (4.1).

The following corollary weakens the assumption of the main result of [V3] and is an analogue of the result of Kadec and Pełczyński mentioned in the introduction:

Corollary 4.4. *If F is a hilbertizable Fréchet space satisfying (Ω) and E is a Fréchet space of type 2 satisfying (DN), then the following short exact sequence*

$$0 \to F \to G \to E \to 0$$

splits if and only if G is of type 2.

Remark. Note that even if F, E are Hilbert spaces, G need not be of type 2 [ELP]. See also [KP].

Proof. The necessity is obvious, the sufficiency follows from 4.3 and 4.1.

Finally, we state a Fréchet version of Maurey's extension theorem:

Corollary 4.5. *Let F and G be Fréchet spaces, F of dual type 2 and G of type 2. Then every operator $T : E \to F$ defined on a subspace E of G extends to a map $T_1 : G \to F''$ whenever F has (Ω) and G/E has (DN).*

Remark. Since in the Banach case each map from a type 2 space into a cotype 2 space factorizes through a Hilbert space ([M] or [Pi2, Cor. 3.6]), the above result implies Maurey's extension theorem in the Banach setting.

Proof. Let $i : F \to F''$ be the canonical embedding. By Prop. 3.1 (a), we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F'' & \longrightarrow & G_0 & \longrightarrow & G/E & \longrightarrow & 0 \\
 & & \uparrow iT & & \uparrow & & \uparrow \mathrm{id} & & \\
0 & \longrightarrow & E & \xrightarrow{j} & G & \xrightarrow{q} & G/E & \longrightarrow & 0
\end{array}$$

The duals F_n' are of type 2, thus the F_n'' are of cotype 2 (Prop. 2.1). Hence, by 4.1, the upper row locally splits. Now, 4.3 implies that it splits because G/E is of type 2 as a quotient of the type 2 space G. Again, by 3.1 (b), T extends to a map $T_1 : G \to F''$.

Remarks. There are plenty of (reflexive) examples of spaces satisfying the assumptions concerning F. The most natural seem to be suitable projective limits of $L_p(\mu)$-spaces for $1 < p \leq 2$, in particular, some weighted projective limits of L_p. It is worth observing that, by Hölder's inequality, such a space with a sequence of weights (w_i) satisfies (Ω) iff

$$\forall k\, \exists l\, \forall m\, \exists n, C\ : C w_l^{n+1} \geq w_m w_k^n \qquad \mu - \text{almost everywhere.}$$

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Recibido: 2 de Diciembre de 1998
Revisado: 7 de Abril de 1999