REVISTA MATEMÁTICA COMPLUTENSE
Volumen 12, número 2: 1999
http://dx.doi.org/10.5209/rev_REMA.1999.v12.n2.17103

A p-adic behaviour of dynamical systems.

Stany De SMEDT, Andrew KHRENNIKOV

Abstract

We study dynamical systems in the non-Archimedean number fields (i.e. fields with non-Archimedean valuation). The main results are obtained for the fields of p-adic numbers and complex p-adic numbers. Already the simplest p-adic dynamical systems have a very rich structure. There exist attractors, Siegel disks and cycles. There also appear new structures such as "fuzzy cycles". A prime number p plays the role of parameter of a dynamical system. The behaviour of the iterations depends on this parameter very much. In fact, by changing p we can change crucially the behaviour : attractors may become centers of Siegel disks and vice versa, cycles of different length may appear and disappear...

1 Introduction

During the last 100 years p-adic numbers were considered only in pure mathematics. But last years numerous applications of these numbers to theoretical physics were proposed by I.Volovich, P. Freund, E. Witten, G. Parisi, I. Aref'eva, E. Marinari, B. Dragovic (string theory, see [1], [5], [11], [16], [17]), V. Vladimirov, E. Zelenov (quantum mechanics) [15], A. Khrennikov (p-adic valued physical observables) [6] and many others, see the books [7], [15]. A number of models of p-adic physics might not be described by the ordinary theory of probability based on Kolmogorov's axiomatic [10]. A new class of probability models, p-adic valued probability models, was investigated in [7]. Here p-adic probabilities are defined as limits of relative frequencies $\nu_N = n/N$ but

1991 Mathematics Subject Classification: 46S10, 58F12.
Servicio Publicaciones Univ. Complutense. Madrid, 1999.

with respect to a p-adic metric on the field of rational numbers $\mathbf{Q}$ (frequencies are always rational numbers) . There exist random sequences ω, see [7], where limits of ν_N do not exist in $\mathbf{R}$, but they exist in one of the p-adic number fields $\mathbf{Q}_p$. This situation can be considered as the classification of chaos. A sequence, which is totally chaotic from the real point of view, may have a definite p-adic structure.

In the present paper we study dynamical systems (with analytic functions) in non-Archimedean number fields. The most interesting examples of such fields are the fields $\mathbf{Q}_p$ of p-adic numbers and $\mathbf{C}_p$ of complex p-adic numbers. We start with general results for an arbitrary non-Archimedean field K and then apply these results to study the behaviour of the dynamical system $y = x^n$ in $\mathbf{Q}_p$ and $\mathbf{C}_p$. Already this simplest dynamical system has a very rich structure. We obtain some general results about this dynamical system. However, many properties of this dynamical system change crucially by changing p. Here we could not find some general laws[1], but we illustrate these properties by numerous examples. We further continue these investigations with the aid of a computer with the aid of the complex of programs for p-adic dynamical systems.

As we hope that this paper would be interesting for scientists working in applications (mathematical physics, dynamical systems, chaos), we present all primary notions about non-Archimedean fields and p-adic numbers (see, for example, [4], [13]) in the next section.

2 Non-Archimedean fields, fields of p-adic numbers

Let F be a field. Recall that a *valuation* is a mapping $|\cdot|_F : F \to \mathbf{R}_+$ satisfying the following conditions :

$$(1)\ |x|_F = 0 \Longleftrightarrow x = 0$$
$$(2)\ |xy|_F = |x|_F |y|_F$$
$$(3)\ |x + y|_F \leq |x|_F + |y|_F$$

[1] Probably, it is impossible to find such laws, because the set of prime numbers has a very complicated structure.

The last inequality is the well known triangle axiom. A valuation is said to be non-Archimedean if the strong triangle axiom holds, i.e.

$$|x+y|_F \leq \max(|x|_F, |y|_F). \tag{1}$$

A field F with a non-Archimedean valuation is said to be a non-Archimedean field.

Everywhere in what follows K denotes a complete non-Archimedean field with a nontrivial valuation $|\cdot|_K$. [2] The case $Char\ K = 0$ is considered.

We shall often use the following property of the non-Archimedean valuation $|\cdot|_K$:

$$|a+b|_K = \max(|a|_K, |b|_K) \text{ if } |a|_K \neq |b|_K. \tag{2}$$

Set $\mathcal{U}_r(a) = \{x \in K : |x-a|_K \leq r\}, a \in K, r > 0$. This is by definition a closed ball in K with center in a of radius r. The term "closed ball" is only a terminology. In fact, these balls are at the same time closed and open ("clopen"). We introduce also open balls $\mathcal{U}_r^-(a) = \{x \in K : |x-a|_K < r\}$ which are also clopen sets. Set $S_r(a) = \{x \in K : |x-a|_K = r\}$. This is by definition the sphere in K with center in a of radius r. It is also a clopen set.

The following simple fact will be very useful for us :

Lemma 2.1. *Let $a \in S_1(0)$. Then $S_1(0) \setminus \mathcal{U}_1^-(a) \subset S_1(a)$.*

Set $|K| = \{r = |x|_K : x \in K\}$. A function $f : \mathcal{U}_r(a) \to K, r \in |K|$ is said to be analytic if it is expanded into a power series $f(x) = \sum_{n=0}^{\infty} f_n(x-a)^n, f_n \in K$, which converges uniformly on the ball $\mathcal{U}_r(a)$. Set $\|f\|_r = \max_n |f_n|_K r^n$. This is the norm on the space of analytic functions $f : \mathcal{U}_r(a) \to K$ (see, for example, [4] for the theory of analytic functions).

The field of real numbers $\mathbf{R}$ is constructed as the completion of the field of rational numbers $\mathbf{Q}$ with respect to the metric $\rho(x,y) = |x-y|$, where $|\cdot|$ is the usual valuation given by the absolute value. The fields of p-adic numbers $\mathbf{Q}_p$ are constructed in a corresponding way, by using other valuations. For any prime number the p-adic valuation $|\cdot|_p$ is defined in the following way. First we define it for natural numbers.

[2] A valuation is called trivial if it is equal to 1 for any nonzero element.

Every natural number n can be represented as the product of prime numbers : $n = 2^{r_2}3^{r_3}\dots p^{r_p}\dots$. Then we define $|n|_p = p^{-r_p}$, and set $|0|_p = 0$ and $|-n|_p = |n|_p$. We extend the definition of the p-adic valuation $|\cdot|_p$ to all rational numbers by setting for $m \neq 0$: $|n/m|_p = |n|_p/|m|_p$. The completion of $\mathbf{Q}$ with respect to the metric $\rho_p(x,y) = |x-y|_p$ is the locally compact field of p-adic numbers $\mathbf{Q}_p$. It is well known (Ostrovsky's theorem), see [4], [13], that $|\cdot|$ and $|\cdot|_p$ are the only possible valuations on $\mathbf{Q}$. The p-adic valuation satisfies the strong triangle inequality. Denote the ring of p-adic integers by $\mathbf{Z}_p(=\mathcal{U}_1(0))$. For any $x \in \mathbf{Q}_p$ we have a unique canonical expansion (converging in the $|\cdot|_p$-norm) of the form

$$x = a_{-n}p^{-n} + \dots + a_0 + \dots + a_k p^k + \dots = \dots\alpha_k\dots\alpha_1\alpha_0, \alpha_{-1}\dots\alpha_{-n}, \quad (3)$$

where $a_j = 0, 1, \dots, p-1$, are the "digits" of the p-adic expansion. We shall use the following property of the binomial coefficients $C_n^k = \frac{n!}{k!(n-k)!}, k \leq n$: $|C_n^k|_p \leq 1$. The proof can be achieved by observing that the binomial coefficient $\frac{a!}{b!(a-b)!}$ is integer and, therefore, its p-adic norm is ≤ 1.

Denote the ring of residue classes with respect to *mod n* by F_n, $F_n = \{0, 1, \dots, n-1\}$; F_n^* is its multiplicative semigroup (if $n = p$ is a prime, then F_p is a field). Using (3) it is easy to show that $\mathbf{Z}_p/p\mathbf{Z}_p = F_p$. As usual we introduce the factor map $\mathbf{Z}_p \to F_p, x \to \bar{x}$.

The greatest common divisor of natural numbers n and m is denoted by (n,m). If $(n,m) = 1$, then n and m are called relatively prime.

Denote the algebraic closure of $\mathbf{Q}_p$ by $\mathbf{Q}_p^a$. It is an infinite dimensional space over $\mathbf{Q}_p$. We note that an extension of any finite order of $\mathbf{Q}_p$ is not algebraically closed. The p-adic valuation $|\cdot|_p$ can be extended canonically to the valuation on $\mathbf{Q}_p^a$ (see [4], [13]). However, the field $\mathbf{Q}_p^a$ is not complete with respect to this valuation. By Krasner's theorem [4] the completion $\mathbf{C}_p$ of $\mathbf{Q}_p^a$ is algebraically closed. We use the same symbol $|\cdot|_p$ for the valuation on $\mathbf{C}_p$ which is constructed as the unique prolongation of the valuation on $\mathbf{Q}_p^a$. The roots of unity in $\mathbf{C}_p$ will play an important role in our considerations. Denote the group of m-th roots of unity, $m = 1, 2, \dots$, by $\Gamma^{(m)}$. Set

$$\Gamma = \cup_{m=1}^{\infty}\Gamma^{(m)}, \Gamma_m = \cup_{j=1}^{\infty}\Gamma^{(m^j)}, \Gamma_u = \cup_{(m,p)=1}\Gamma_m$$

By elementary group theory we have $\Gamma = \Gamma_u \cdot \Gamma_p, \Gamma_u \cap \Gamma_p = \{1\}$. Denote the kth roots of unity by $\theta_{j,k}, j = 1, \dots, k, \theta_{1,k} = 1$.

Lemma 2.2. *Let* $y^n = a, a = \theta_{j,n-1}$ *for some* $j = 1, 2, ..., n-1$, *and* $y \neq a$. *If* $(n,p) = 1$, *then* $y \in S_1(a)$.

¿From this lemma we get the well known consequence :

Corollary 2.1. $\Gamma_u \subset S_1(1)$.
The following two results can be found, for example, in [4], [13].

Lemma 2.3. $|C^j_{p^k}|_p \leq 1/p$ *for all* $j = 1, ..., p^k - 1$.

Lemma 2.4. $\Gamma_p \subset \mathcal{U}_1^-(1)$.

To find fixed points and cycles of functions $f(x) = x^n$ in $\mathbf{Q}_p$, we have to know whether the roots of unity belong to $\mathbf{Q}_p$. Denote by $\xi_l, l = 1, 2, ...$, a primitive lth root of 1 in $\mathbf{C}_p$. We are interested whether $\xi_l \in \mathbf{Q}_p$.

Proposition 2.1. (Primitive roots) *If* $p \neq 2$, *then* $\xi_l \in \mathbf{Q}_p$ *if and only if* $l \mid p-1$. *The field* $\mathbf{Q}_2$ *contains only* $\xi_1 = 1$ *and* $\xi_2 = -1$.

In fact, this proposition is a consequence of the same result for F_p (which is well known in the elementary number theory) and the Hensel lemma (see appendix).

Corollary 2.2. *The equation* $x^k = 1$ *has* $g = (k, p-1)$ *different roots in* $\mathbf{Q}_p$.

The same result is valid for F_p.

3 Dynamical systems in non-Archimedean fields

We study the dynamical system :

$$\mathcal{U} \to \mathcal{U}, x \to f(x) \tag{4}$$

where $\mathcal{U} = \mathcal{U}_R(a)$ or K and $f : \mathcal{U} \to \mathcal{U}$ is an analytic function. First we shall prove a general theorem about the behaviour of iterations $x_n = f^n(x_0), x_0 \in \mathcal{U}$. As usual $f^n(x) = f \circ ... \circ f(x)$. Then we shall use this result to study the behaviour of the concrete dynamical systems, $f(x) = x^n, n = 2, 3, ...$, in the fields of complex p-adic numbers $\mathbf{C}_p$.

We shall use the standard terminology of the theory of dynamical systems : fixed and periodic points, cycles, stable and unstable points, attractors, basin of attraction (we denote the basin of attraction for the attractor a by $A(a)$), repeller (see, for example, [3], [12]).

We have to be more careful to define a non-Archimedean analogue of a Siegel disk. Let $a \in \mathcal{U}$ be a fixed point of a function $f(x)$. The ball $\mathcal{U}_r^-(a)$ (contained in $\mathcal{U}$) is said to be a Siegel disk if each sphere $S_\rho(a), \rho < r$, is an invariant sphere of $f(x)$, i.e. if one takes an initial point on one of the spheres $S_\rho(a), \rho < r$, all iterated points will also be on it. The union of all Siegel disks with center in a is said to be a maximal Siegel disk. Denote the maximal Siegel disk by $SI(a)$.

In the same way we define a Siegel disk with center in a periodic point $a \in \mathcal{U}$ with the corresponding cycle $\gamma = \{a, f(a), ..., f^{n-1}(a)\}$ of the period n. Here the spheres $S_\rho(a), \rho < r$, are invariant spheres of the map $f^n(x)$.

As usual in the theory of dynamical systems, we can find attractors, repellers and Siegel disks using properties of the derivative of $f(x)$.

Let a be a periodic point with period n of the C^1-function $g : \mathcal{U} \to \mathcal{U}$. Set $\lambda = \frac{d}{dx} g^n(a)$. This point is called : 1) attractive if $0 \leq |\lambda|_K < 1$; 2) indifferent if $|\lambda|_K = 1$; 3) repelling if $|\lambda|_p > 1$.

Lemma 3.1. *Let $f : \mathcal{U} \to \mathcal{U}$ be an analytic function and let $a \in \mathcal{U}$ and $f'(a) \neq 0$. Then there exists $r > 0$ such that*

$$s = \max_{2 \leq n < \infty} \left| \frac{1}{n!} \frac{d^n f}{dx^n}(a) \right|_K r^{n-1} < |f'(a)|_K. \tag{5}$$

If $r > 0$ satisfies this inequality and $\mathcal{U}_r(a) \subset \mathcal{U}$, then

$$|f(x) - f(y)|_K = |f'(a)|_K |x - y|_K \tag{6}$$

for all $x, y \in \mathcal{U}_r(a)$.

Proof. We consider the case $\mathcal{U} = \mathcal{U}_R(a)$. We have : $f(x) - f(y) = [f'(a) + T(x, y, a)](x - y)$ with

$$T(x, y, a) = \sum_{n=2}^{\infty} \frac{1}{n!} \frac{d^n f}{dx^n}(a)[(x-a)^{n-1} + (y-a)(x-a)^{n-2} + \cdots + (y-a)^{n-1}]$$

Denote the expression in the square brackets by $B_n(x, y, a)$. Let $x, y \in \mathcal{U}_r(a), r \leq R$. Using the strong triangle inequality, we get : $|B_n(x, y, a)|_K \leq r^{n-1}$. Set

$$\sigma(\rho) = \max_{2 \leq n < \infty} \left| \frac{1}{n!} \frac{d^n f}{dx^n}(a) \right|_K \rho^{n-2}, \rho > 0.$$

By analyticity of f on $\mathcal{U}_R(a)$ we have $\sigma(R) \leq \|f\|_R/R^2 < \infty$. As $\sigma(r) \leq \sigma(R)$ for any $r \leq R$, we get :

$$\sup_{x,y\in\mathcal{U}_r(a)} |T(x,y,a)|_K \leq r\sigma(R) \to 0, r \to 0. \tag{7}$$

Hence, if $f'(a) \neq 0$, then there exists $r > 0$ satisfying (5). By (2) we obtain (6) for such r.

Theorem 3.1. *Let a be a fixed point of the analytic function $f : \mathcal{U} \to \mathcal{U}$. Then :*

1) If a is an attractive point of f, then it is an attractor of the dynamical system (4). If $r > 0$ satisfies the inequality :

$$q = \max_{1\leq n<\infty} \left|\frac{1}{n!}\frac{d^n f}{dx^n}(a)\right|_K r^{n-1} < 1 \tag{8}$$

and $\mathcal{U}_r(a) \subset \mathcal{U}$, then $\mathcal{U}_r(a) \subset A(a)$.

2. If a is an indifferent point of f, then it is the center of a Siegel disk. If $r > 0$ satisfies the inequality (5) and $\mathcal{U}_r(a) \subset \mathcal{U}$, then $\mathcal{U}_r(a) \subset SI(a)$.

3. If a is a repelling point of f, then a is a repeller of the dynamical system (4).

Proof. If $f'(a) \neq 0$ and $r > 0$ satisfies (5) (with $\mathcal{U}_r(a) \subset \mathcal{U}$), then it suffices to use the previous lemma.

If a is an arbitrary attractive point, then again by (7) there exists $r > 0$ satisfying (8). Thus we have $|f(x) - f(y)|_K < q|x-y|_K, q < 1$, for all $x, y \in \mathcal{U}_r(a)$. Consequently, a is an attractor of (4) and $\mathcal{U}_r(a) \subset A(a)$.

We note that (in the case of an attractive point) the condition (8) is less restrictive than the condition (5).

To study dynamical systems for nonanalytic functions, we can use the following theorem of non-Archimedean analysis [13] :

Theorem 3.2. (Local injectivity of C^1-functions) *Let $f : \mathcal{U}_r(a) \to K$ be C^1 at the point a. If $f'(a) \neq 0$ there is a ball $\mathcal{U}_s(a), s \leq r$, such that (6) holds for all $x, y \in \mathcal{U}_s(a)$.*

However, Theorem 3.1 is more useful for our considerations, because Theorem 3.2 is a so-called "existence theorem". This theorem does not say anything about the value of s. Thus we cannot estimate the volume of $A(a)$ or $SI(a)$. Theorem 3.1 gives us such possibility. We need only to test one of the conditions (8) or (5). Moreover, the case $f'(a) = 0$ is "a pathological case" for nonanalytic functions of a non-Archimedean argument. For example, there exist functions g which are not locally constant but for which $g' \equiv 0$. In our analytic framework we have not such problems.

The *Julia set* J_f for the dynamical system (4) is defined as the closure of the set of all repelling periodic points of f. The set $F_f = \mathcal{U} \setminus J_f$ is called the *Fatou set*. These sets play an important role in the theory of real dynamical systems. In the non-Archimedean case the structures of these sets are more or less trivial.

We shall also use an analogue of Theorem 3.1 for periodic points. There we must apply our theorem to the function $f^n(x)$.

4 Dynamical systems in the field of complex p-adic numbers

As an application of Theorem 3.1, we study dynamical systems with $p_n(x) = x^n, n = 2, 3, \ldots$ in the fields of the complex p-adic numbers $\mathbf{C}_p$. It is evident that the points $a = 0$ and $a = \infty$ are attractors with basins of attraction $A(0) = \mathcal{U}_1^-(0)$ and $A(\infty) = \mathbf{C}_p \setminus \mathcal{U}_1(0)$ respectively. Thus the main scenarios is developed on the sphere $S_1(0)$. Fixed points of $p_n(x)$ belonging to this sphere are the roots $\theta_{j,n-1}, j = 1, \ldots, n-1$, of unity of the degree $(n-1)$. There are two essentially different cases : 1) n is not divisible by p; 2) n is divisible by p.

Theorem 4.1. *The dynamical system $p_n(x)$ has $(n-1)$ fixed points $a_j = \theta_{j,n-1}, j = 1, \ldots, n-1$, on the sphere $S_1(0)$.*

1. Let $(n, p) = 1$. Then all these points are centers of Siegel disks and $SI(a_j) = \mathcal{U}_1^-(a_j)$. If $(n-1, p) \neq 1$, then $SI(a_j) = SI(1) = \mathcal{U}_1^-(1)$ for all $j = 1, \ldots, n-1$. If $(n-1, p) = 1$, then $a_j \in S_1(1), j = 2, \ldots, n-1$, and $SI(a_j) \cap SI(a_i) = \emptyset, i \neq j$. For any $k = 2, 3, \ldots$ all k-cycles are also centers of Siegel disks of unit radius.

2. If $(n, p) \neq 1$, then these points are attractors and $\mathcal{U}_1^-(a_j) \subset A(a_j)$. For any $k = 2, 3, \ldots$ all k-cycles are also attractors and open unit balls are contained in basins of attraction.

Proof. 1.Consider the first case. There we have $|p_n'(a_j)|_p$ -

Theorem 3.1 all points a_j are centers of Siegel disks. We are interested in the radius of the maximal Siegel disk. We use the condition (5). As $|\frac{1}{l!}\frac{d^l p_n}{dx^l}(b)|_p = |C_n^l|_p$ for any $b \in S_1(0)$, the condition (5) has the form

$$r \max_{2 \le k < \infty} r^{k-2}|C_n^k|_p < 1.$$

If $r < 1$, then this condition is satisfied. Thus $\mathcal{U}_1^-(a_j) \subset SI(a_j)$. We need only to show that the spheres $S_1(a_j)$ are not invariant sets for p_n. There Lemma 2.2 is used. We choose $x_0 = y$ where $y^n = a_j$ and $y \in S_1(a_j)$. Then $p_n(y) = a_j$.

If $(n-1, p) \neq 1$, then by Lemma 2.4 all $a_j \in \mathcal{U}_1^-(1)$. Hence $\mathcal{U}_1^-(a_j) = \mathcal{U}_1^-(1)$ and $SI(a_j) = SI(1)$. Thus the dynamical system $p_n(x)$ describes the following motion in the ball $\mathcal{U}_1^-(1)$. *There exist $(n-1)$ points $a_1, ..., a_{n-1}$ such that for any initial point $x_0 \in \mathcal{U}_1^-(1)$ the distances between iterations x_n of x_0 and these points are constants of the motion.*

If $(n-1, p) = 1$, then by Corollary 2.1 $a_j \in S_1(1)$ for $j = 2, ..., n-1$. Thus there are $(n-1)$ different Siegel disks which have empty intersections.

To study k-cycles, we use the fact that $(n^k, p) = 1$ if and only if $(n, p) = 1$. Hence each fixed point of the map $p_n^k(x)$ is the center of a Siegel disk.

2. Now we consider the second case. Let $n = p^k m, k \geq 1$, with $(m, p) = 1$. Then we have $|p_n'(a_j)|_p = 1/p^k$. Thus all points a_j are attractors. Further we are interested in basins of attraction. We use the condition (8) which has the form :

$$q = \max(1/p^k, r|C_n^2|_p, ..., r^{n-1}) < 1$$

If $r < 1$, then this inequality is satisfied. Thus $\mathcal{U}_1^-(a_j) \subset A(a_j)$.

To study k-cycles, we use the fact that $(n^k, p) \neq 1$ if and only if $(n, p) \neq 1$. Hence each fixed point of the map $p_n^k(x)$ is an attractor.

Corollary 4.1. *Let $(n, p) = 1$. There exist an infinite number of k such that $(n^k - 1, p) \neq 1$. Any k-cycle $\gamma = (a_1, ..., a_k)$ for such k is located in the ball $\mathcal{U}_1^-(1)$; it has the behaviour of a Siegel disk with $SI(\gamma) = \cup_{j=1}^k \mathcal{U}_1^-(a_j) = \mathcal{U}_1^-(1)$. During the process of the motion the distances $c_j = \rho_p(x_0, a_j), j = 1, ..., k$, where $x_0 \in \mathcal{U}_1^-(1)$ is an arbitrary initial* *point, are changed according to the cyclic law :* $(c_1, c_2, ..., c_{n-1}, c_n) \to (c_n, c_1, ..., c_{n-2}, c_{n-1}) \to ...$

Proof. By the Fermat theorem $x^{p-1} = 1$ for any $x \in F_p^*$. Thus, as $(n,p) = 1$, we have $n^{j(p-1)} = 1 \bmod p$.

Now we prove the cyclic law for the distances. It is a simple consequence of Lemma 3.1 :

$$|x_1 - a_j|_p = |f(x_0) - f(a_{j-1})|_p = |f'(a_{j-1})|_p |x_0 - a_{j-1}|_p = c_{j-1}.$$

Thus in the case $(n,p) = 1$ the motion of a point in the ball $\mathcal{U}_1^-(1)$ is very complicated. It moves cyclic (with different periods) around an infinite number of centers.

Examples. Let $n = 2$ in all following examples.
1. Let $p = 3$. Then $(n^{2k} - 1, p) \neq 1$ and $(n^{2k+1} - 1, p) = 1$. Thus all even cycles (and only they) are located in the ball $\mathcal{U}_1^-(1)$.
2. Let $p = 5$. Then $(n^{4k} - 1, p) \neq 1$ and $(n^{4k+j} - 1, p) = 1$ for $j = 1, 2, 3$. Thus all $4k$-cycles (and only they) are located in the ball $\mathcal{U}_1^-(1)$.
3. Let $p = 7$. Then $(n^{3k} - 1, p) \neq 1$ and $(n^{3k+j} - 1, p) = 1$ for $j = 1, 2$. Thus all $3k$-cycles (and only they) are located in the ball $\mathcal{U}_1^-(1)$.

Now we find the basins of attraction $A(a_j), j = 1, ..., n-1, (n,p) \neq 1$, exactly. We begin from the attractor $a_1 = 1$.

Let $n = mp^k, (m,p) = 1$ and $k \geq 1$.

Lemma 4.1. *The basin of attraction* $A(1) = \cup_\xi \mathcal{U}_1^-(\xi)$ *where* $\xi \in \Gamma_m$; *these balls have empty intersections for different points* ξ.

Proof. 1. Let $\xi \in \Gamma_{m^j}$ and $y = \xi + \gamma, |\gamma|_p < 1$. Then

$$|y^{n^j} - 1|_p = \left| \sum_{i=0}^{n^j - 1} C_{n^j}^i \xi^i \gamma^{n^j - i} \right|_p \leq |\gamma|_p < 1$$

i.e. the j-th iteration of y belongs to the ball $\mathcal{U}_1^-(1) \subset A(1)$. Hence $\mathcal{U}_1^-(\xi) \subset A(1)$. These balls have empty intersections for different ξ, because $|a - b| = 1$ for any $a, b \in \Gamma_u, a \neq b$.

2. Now let $s = m^\alpha t, \alpha \geq 0, (s,p) = 1, (t,m) = 1, t \neq 1$, and $\xi \in \Gamma^{(s)} \setminus \Gamma^{(m^\alpha)}$. Then $\xi = uv, u \in \Gamma^{(m^\alpha)}, v \in \Gamma^{(t)}$ and $v \neq 1$. Let $y = \xi + \gamma, |\gamma|_p < 1$. Then $y^{n^k} = \xi^{n^k} + \beta$ where $|\beta|_p < 1$, i.e. $y^{n^k} \in \mathcal{U}_1^-(\xi^{n^k})$. It suffices to show that $\xi^{n^k} \in S_1(1)$ for all sufficiently large k.

First we note that if $k \geq \alpha$, then $u^{n^k} = 1$. Further let $n^k = jt + q$ with the remainder $q = 1, ..., t-1$. If $k \geq \alpha$, then we have $\xi^{n^k} = v^q$. Let $v = \xi_t^{\lambda}, \lambda = 1, ..., t-1$ (as usual ξ_t is a t-th primitive root). Then $v^q = \xi_t^{\lambda q}$. Finally we have to show that $\lambda q \neq 0 \bmod t$. Suppose that $\lambda q = 0 \bmod t$. It implies that $(q,t) = \omega \neq 1$. Thus $m^k p^{lk} = j'\omega$. It contradicts the condition $(m,t) = (p,t) = 1$.

Thus it has been shown that $\mathcal{U}_1^-(\xi) \subset \mathbf{C}_p \setminus A(1)$ for any ξ such that it is a s-th root of unity, $(s,p) = 1$, and it is not a m^j-th root of unity, $j \geq 1$. Finally we use the following fact which can be obtained by Lemma 2, p.103, [13] :

$$S_1(1) = \cup \mathcal{U}_1^-(\xi) \text{ where } \xi \in \Gamma_u, \xi \neq 1.$$

Corollary 4.2. *Let $n = p^l, l \geq 1$. Then $S_1(1)$ is the invariant sphere of the dynamical system $p_n(x)$.*

Examples. 1.Let $n = p^l, l \geq 1$. Then $A(1) = \mathcal{U}_1^-(1)$.

2. Let $p \neq 2$ and $n = 2p^l, l \geq 1$. Then $A(1) = \cup \mathcal{U}_1^-(\xi)$ where $\xi \in \Gamma_2$.

Theorem 4.2. *The basin of attraction $A(a_k) = \cup_{\xi} \mathcal{U}_1^-(\xi a_k)$ where $\xi \in \Gamma_m$. These balls have empty intersections for different points ξ.*

Proof. This theorem is a simple consequence of Lemma 4.1. We introduce the map $T_k : \mathcal{U}_1(1) \to \mathcal{U}_1(a_k), T_k x = a_k x$. It is an isometric map. If $x_0 \in \mathcal{U}_1(1)$ and $x_0' = T_k(x_0)$, then we have for the iterations of the initial point $x_0' \in \mathcal{U}_1(a_k)$: $x_N' = p_n^N(x_0') = T_k(x_N)$ where $x_N = p_n^N(x_0)$. Thus the behaviour of the dynamical system $p_n(x)$ in the ball $\mathcal{U}_1(a_k)$ can be obtained as the T_k-image of the behaviour in the ball $\mathcal{U}_1(1)$.

The dynamical system $p_n(x)$ has no repelling points in $\mathbf{C}_p$ for any p. Thus the Julia set $J_{p_n} = \emptyset$ and the Fatou set $F_{p_n} = \mathbf{C}_p$. Therefore in the p-adic case the Julia set does not play the role of a set where "the scenarios of chaos is developed." It seems to be interesting to study the behaviour of dynamical systems on the *intermediate set* which is defined as $INT_f = \mathcal{U} \setminus [(\cup_a A(a)) \cup (\cup_b SI(b))]$ where $\{a\}$ are all attractors and $\{b\}$ are all centers of Siegel disks of a dynamical system. In the case $(n,p) = 1$ this is the set $INT_{p_n} = S_1(0) \setminus \cup_{j=1}^{n-1} SI(a_j)$; in the case $(n,p) \neq 1$ this is the set $INT_{p_n} = S_1(0) \setminus \cup_{j=1}^{n-1} A(a_j)$. These sets contain cycles of all lengths.

5 Dynamical systems in the fields of p-adic numbers

Here we study the behaviour of the dynamical system $p_n(x) = x^n, n = 2, 3, ...,$ in $\mathbf{Q}_p$. In fact, this behaviour can be obtained on the basis of the corresponding behaviour in $\mathbf{C}_p$. We need only to apply the results of the second section about the roots of unity in $\mathbf{Q}_p$

Proposition 5.1. *The dynamical system $p_n(x)$ has $m = (n-1, p-1)$ fixed points $a_j = \theta_{j,m}, j = 1, ..., m$, on the sphere $S_1(0)$ of $\mathbf{Q}_p$. The character of these points is described by Theorem 4.1. Fixed points $a_j \neq 1$ belong to the sphere $S_1(1)$.*

Proof. The first statement is a consequence of Corollary 2.2. We must only show that all fixed points $a_j \neq 1$ belong to $S_1(1)$. As $m = (n-1, p-1)$, then $(m, p) = 1$. Finally we use Corollary 2.1.

We remark that the number of attractors or Siegel disks for the dynamical system $p_n(x)$ on the sphere $S_1(0)$ is less than or equal to $(p-1)$.

To study k-cycles in $\mathbf{Q}_p$, we use the following numbers : $m_k = (l_k, p-1), k = 1, 2, ...,$ with $l_k = n^k - 1$ (i.e. $m \equiv m_1$).

Proposition 5.2. *The dynamical system $p_n(x)$ has k-cycles ($k \geq 2$) in $\mathbf{Q}_p$ if and only if m_k does not divide any $m_j, j = 1, ..., k-1$. All these cycles are located on $S_1(1)$.*

Proof. 1. Suppose that m_k does not divide $m_j, 1 \leq j \leq k-1$. We choose $a_1 = \xi_{m_k}$. It is, a primitive m_k-th root. Then we have $a_1^{n^k} = a_1$. Suppose that $a_1^{n^s} = a_1$ for $s < k$. Then $a_1^{m_s} = 1$. Hence m_k divides m_s. It is a contradiction.

2. Suppose that m_k divides m_j for some $j < k$. Let $a_1^{l_k} = 1$. Then $a_1^{m_k} = 1$ and, consequently, $a_1^{m_j} = 1$. Thus $a_1^{l_s} = 1$. Thus the cycle $\gamma = \gamma(a_1)$ has length $\leq s < k$.

In particular, if $(n, p) = 1$ (i.e. all fixed points and k-cycles are centers of Siegel disks), there is no such complicated motion around a group of centers in $\mathcal{U}_1^-(1) (= \mathcal{U}_{1/p}(1))$.

Corollary 5.1. *The dynamical system $p_n(x)$ has only a finite number of cycles in $\mathbf{Q}_p$ for any prime number p.*

Example. Let $n = p^l, l \geq 1$. Then $m_1 = p - 1$ and there are $p - 1$ attractors $a_j = \theta_{j,p-1}, j = 1, ..., p - 1$, with the basins of attraction $A(a_j) = \mathcal{U}_{1/p}(a_j)$ and there is no k-cycle, $k \geq 2$. As we can choose $a_j = j \bmod p$, then $\mathcal{U}_{1/p}(a_j) = \mathcal{U}_{1/p}(j)$. As $S_1(0) = \cup_{j=1}^{p-1}\mathcal{U}_{1/p}(j)$, the intermediate set is empty. In particular, if $p = 2$ then all points of the sphere $S_1(0)$ are attracted by a_1.

To study the general case $n = qp^l, l \geq 1, (q,p) = 1$, we use the following elementary fact.

Lemma 5.1. *Let $n = qp^l, l \geq 1, (q,p) = 1$. Then $m_k = (l_k, p - 1) = (q^k - 1, p - 1), k = 1, 2, ...$*

Proof. Set $a = (q^k - 1, p - 1), a' = (l_k, p - 1)$. We have : $q^k = ab + 1, p = ac + 1$, where $b, c \in \mathbf{N}$. Hence $l_k = n^k - 1 = (ab+1)(ac+1)^{kl} - 1$ and $a|l_k$ and, consequently, $a|a'$. On the other hand, we have $n^k = a'd + 1, p = a'x + 1$, where $d, x \in \mathbf{N}$. Hence $q^k(a'x+1)^{kl} = (q^k - 1)(a'x+1)^{kl} + (a'x + 1)^{kl} = a'd + 1$ and, consequently, $a'|(q^k - 1)(a'x + 1)^{kl}$, i.e. $a'|(q^k - 1)$. Thus $a'|a$.

Examples. 1). Let $n = 2p, p \neq 2$. There is only one attractor $a_1 = 1$ on $S_1(0)$ for all p. To find k-cycles, $k \geq 2$, we have to consider the numbers $m_k, k = 2, ...$ However, by Lemma 5.1 $m_k = (2^k - 1, p - 1)$. Thus the number of k-cycles for the dynamical system $p_{2p}(x)$ coincides with the corresponding number for the dynamical system $p_2(x)$. An extended analysis of the dynamical system $p_2(x)$ will be presented after Proposition 5.4. Of course, it should be noted that the behaviours of k-cycles for $p_{2p}(x)$ and $p_2(x), p \neq 2$, are very different. In the first case these are attractors; in the second case these are centers of Siegel disks.

2). Let $n = 3p, p \neq 2$. There are two attractors $a_1 = 1$ and $a_2 = -1$ on $S_1(0)$ for all p.

3). Let $n = 4p$. Then there is a more complicated picture : 1 attractor for $p = 2, 3, 5, 11, 17, 23, ...$ and 3 attractors for $p = 7, 13, 19, 29, 31, ...$

4). Let $n = 5p$. Then we have : 1 attractor for $p = 2$; 2 attractors for $p = 3, 7, 11, 23, 31, ...$ and 4 attractors for $p = 5, 13, 17, ...$

Now we study basins of attraction (in the case $n = qp^l, l \geq 1, (q,p) = 1$). As a consequence of our investigations for the dynamical system in $\mathbf{C}_p$, we get that $A(1) = \cup_\xi \mathcal{U}_{1/p}(\xi)$ where $\xi \in \Gamma_q \cap \mathbf{Q}_p$. We have $\Gamma_q \cap \mathbf{Q}_p \neq \{1\}$ if and only if $(q, p-1) \neq 1$.

Examples. 1). Let $p = 5$ and $n = 10$, i.e. $q = 2$. As $(q^2, p - 1) = 4$, then $\Gamma_2 \cap \mathbf{Q}_5 = \Gamma^{(4)}$ and $A(1) = \cup_{j=1}^{4} \mathcal{U}_{1/5}(\theta_{j,4})$. Thus $A(1) = S_1(0)$. All points of the sphere $S_1(0)$ are attracted by $a_1 = 1$.

2). Let $p = 7$ and $n = 21$, i.e. $q = 3$. Then $m_1 = (q-1, p-1) = 2$. Hence there are two attractors; these are $a_1 = 1$ and $a_2 = -1$. As all $m_j = (q^j - 1, p-1) = 2, j = 1, 2, ...$, there are no k-cycles, $k \geq 2$. To find the basins of attraction, we compute $(q, p-1) = 3$. Thus $A(1)$ is the union of balls $\mathcal{U}_{1/7}(\theta_{j,3}), j = 1, 2, 3$. It is evident that $\theta_{2,3} = 2, \theta_{3,3} = 4 \bmod 7$. Therefore $A(1) = \mathcal{U}_{1/7}(1) \cup \mathcal{U}_{1/7}(2) \cup \mathcal{U}_{1/7}(4)$ and $A(-1) = \mathcal{U}_{1/7}(-1) \cup \mathcal{U}_{1/7}(3) \cup \mathcal{U}_{1/7}(5)$ (since $2(-1) = 5, 4(-1) = 3 \bmod 7$).

3). Let $p = 7$ and $n = 14$, i.e. $(q, p-1) = 2$. Thus $\Gamma_2 \cap \mathbf{Q}_7 = \Gamma^{(2)}$ and $A(1) = \mathcal{U}_{1/7}(1) \cup \mathcal{U}_{1/7}(6)$. There $INT_{p_{14}} = \cup_{j=2}^{5} \mathcal{U}_{1/7}(j)$. As $m_2 = 3$, there exist 2-cycles in $INT_{p_{14}}$. It is easy to see that that the 2-cycle is unique and $\gamma = (b_1, b_2)$ with $b_1 = 2, b_2 = 4 \bmod 7$. This cycle generates the cycle of balls on the sphere $S_1(1)$; $\gamma^{(f)} = (\mathcal{U}_{1/7}(2), \mathcal{U}_{1/7}(4))$ ("fuzzy cycle"). The other two balls on $S_1(1)$: $\mathcal{U}_{1/7}(3), \mathcal{U}_{1/7}(5)$, are attracted by $\gamma^{(f)}$ (by the balls $\mathcal{U}_{1/7}(2)$ and $\mathcal{U}_{1/7}(4)$ respectively).

The last example shows us that sometimes it can be interesting to study, not only cycles of points, but also cycles of balls. We propose the following general definition.

Let $x \to g(x), x \in \mathbf{Q}_p$, be a dynamical system. If there exists balls $U_r(a_j), j = 1, ..., n$, such that iterations of the dynamical system generate the cycle of balls $\gamma^{(f)} = (U_r(a_1), ..., U_r(a_n))$, $(r = p^l, l = 0, \pm 1, ...)$ then it is called a *fuzzy cycle* of the length n and the radius r. Of course, we suppose that the balls in the fuzzy cycle do not coincide.

Proposition 5.3. *There is a one to one correspondence between cycles and fuzzy cycles of radius $r = 1/p$ of the dynamical system $p_n(x)$ in $\mathbf{Q}_p$.*

Proof. 1.Let $\gamma = (a_1, ..., a_k)$ be a k-cycle of the dynamical system $p_n(x)$. Then $p_n : \mathcal{U}_{1/p}(a_j) \to \mathcal{U}_{1/p}(a_{j+1})$ (with $a_{k+1} = a_1$), i.e. $\gamma^{(f)} = (\mathcal{U}_{1/p}(a_1), ..., \mathcal{U}_{1/p}(a_k))$ is a fuzzy cycle. 2.Let $\gamma^{(f)} = (\mathcal{U}_{1/p}(b_1), ..., \mathcal{U}_{1/p}(b_k))$ be a fuzzy cycle. Then $b_1^{n^k - 1} = 1 \bmod p$, i.e. b_1 satisfies the equation $\phi(x) = x^{m_k} - 1 = 0 \bmod p$ (see the remark after Corollary 2.2.). By the Hensel lemma (see appendix) there exists $a = b_1 \bmod p$ such that $\phi(a) = 0$ in $\mathbf{Q}_p$. As $a \in \mathcal{U}_{1/p}(b_1)$, then $a^{n^j} \in \mathcal{U}_{1/p}(b_j)$. Since these balls have empty intersections, the point a generates the cycle of the length k.

The situation with fuzzy cycles of radius $r < 1/p$ is more complicated.

Examples. Let $p = 3, n = 2$. There exist 2-cycles of radius $r = 1/9$ which do not correspond to any ordinary cycle. For example, $\gamma^{(f)} = (4,7)$. Further there exist fuzzy 6-cycles with $r = 1/27$; fuzzy 18-cycles with $r = 1/81,...$

Proposition 5.4. *Any point $x \in INT_{p_n}$ is attracted by some cycle.*

Proof. By Proposition 5.3 it suffices to show that any point $x_0 \in INT_{p_n}$ is attracted by a fuzzy cycle. We have $INT_{p_n} = \cup_{l=1}^{p-m_1-1} \mathcal{U}_{1/p}(q_l)$ where $m_1 = (n-1, p-1)$ and q_l are natural numbers $2 \leq q_l \leq p-1$. Thus we can reduce our considerations to iterations of q_l mod p. However, there is only a finite number of different $q_l^{n^k}$ mod p.

Examples. Let $n = 2$ in all following examples.

1). Let $p = 2$. There is only one fixed point $a_1 = 1$ on $S_1(0)$. It is an attractor and $A(1) = \mathcal{U}_{1/2}(1) = S_1(0)$. Thus $\mathbf{Q}_2 = A(0) \cup A(1) \cup A(\infty)$ and $INT_{p_2} = \emptyset$.

2). Let $p \neq 2$. The point $a_1 = 1$ is the center of the Siegel disk $\mathcal{U}_{1/p}(1)$. So $INT_{p_2} = S_1(0) \setminus \mathcal{U}_{1/p}(1) =$

$$\{x = \alpha_0 + \alpha_1 p + \cdots + \alpha_n p^n + \cdots : \alpha_j = 0, 1, ..., p-1, \alpha_0 \neq 0, 1\} \quad (9)$$

The behaviour of the dynamical system on the intermediate set is not described by our general theorems. It must be investigated in each concrete case.

3). As l_k are odd numbers, then m_k must also be an odd number. Therefore there are no k-cycles ($k > 1$) for $p = 3, 5, 17$ and for any prime number which has the form $p = 2^k + 1$ [14]. In these cases INT_{p_2} does not contain any periodic point. By Proposition 5.4 INT_{p_2} is attracted by the Siegel disk $\mathcal{U}_{1/p}(1)$.

4). Let $p = 7$. There m_k can be equal to 1 or 3. As $m_2 = 3$, there are only 2-cycles. It is easy to show that the 2-cycle is unique. By Proposition 5.4 INT_{p_2} is attracted by the Siegel disk $\mathcal{U}_{1/p}(1)$ and the fuzzy 2-cycle $\gamma^{(f)} = (\mathcal{U}_{1/p}(2), \mathcal{U}_{1/p}(4))$ which corresponds to the ordinary two cycle.

5). Let $p = 11$. There $m_k = 1$ or 5. As $m_2 = m_3 = 1$ and $m_4 = 5$, there exist only 4-cycles. There is only one 4-cycle : $\gamma(\xi_5)$.

6). Let $p = 13$. There $m_k = 1$ or 3. As $m_2 = 3$, there exists only the (unique) 2-cycle.

7). Let $p = 19$. There $m_k = 1$ or 3, or 9. As $m_2 = 3$, there is a (unique) 2-cycle. However, although $m_4 = 3$, there are no 4-cycles, because m_4 divides m_3. As $m_6 = 9$ does not divide $m_2, ..., m_5$, there exist 6-cycles and there are no k-cycles with $k > 6$. There is only one 6-cycle : $\gamma(\xi_9)$.

8). Let $p = 23$. There $m_k = 1$ or 11. Direct computations show that there are no k-cycles for the first $k = 2, ..., 9$. As $m_{10} = 11$, there is a (unique) 10-cycle. There are no k-cycles with $k > 10$.

9). Let $p = 29$. There $m_k = 1$ or 7. As $m_3 = 7$ and $m_2 = 1$, there exist only 3-cycles. It is easy to show that there are two 3-cycles : $\gamma(\xi_7)$ and $\gamma(\xi_7^3)$.

10). Let $p = 31$. There $m_k = 1, 3, 5, 15$. As $m_2 = 3$, there exits a (unique) 2-cycle. As $m_4 = 15$ and $m_3 = 1$, there exist 4-cycles : $\gamma(\xi_{15}), \gamma(\xi_{15}^3), \gamma(\xi_{15}^7)$. There are no k-cycles with $k \neq 2, 4$.

11). Let $p = 37$. There $m_k = 1, 3, 9$. As $m_2 = 3$, there exists a (unique) 2-cycle. As $m_6 = 9$ and $m_2 = m_4 = 3, m_3 = m_5 = 1$, there exist 6-cycles. It is easy to show that there is the unique 6-cycle: $\gamma(\xi_9)$. There are no k-cycles, $k \neq 2, 6$.

12). Let $p = 41$. There $m_k = 1, 5$. As $m_4 = 5$ and all previous $m_j = 1$, there exist 4-cycles. It is easy to show that this cycle is unique : $\gamma(\xi_5)$. There are no k-cycles with $k \neq 4$.

It is sufficiently complicated to continue these computations for large p.

It is interesting that we have never observed any 5 or 7-cycle. Do they exist? According to our formalism, this question is reduced to the following question of number theory : Are there prime numbers p such that $p = 1 \bmod 31$ or $\bmod 127$? As 31 and 127 are prime numbers, then by the Dirichlet theorem (see, for example, [14], p.129) there exist an infinite number of such prime numbers p.

Further we study the behaviour of the dynamical system $p_2(x)$ on the intermediate set. We consider the case $p \neq 2$ where this set is described by (9). We shall use the following result.

Proposition 5.5. *Let $p \neq 2$ and $n = 2$. Let the canonical expansion of a p-adic number $x_0 \in INT_{p_2}$ has the form $x = ...\alpha_m...\alpha_0$ where $\alpha_0 = ... = \alpha_k = p - 1$ and $\alpha_{k-1} \neq p - 1$. Then the first iteration of the initial point x_0 belongs to the sphere $S_{1/p^{k+1}}(1)$.*

Proof. We have $x_0 = (1 + \alpha_{k+1})p^{k+1} - 1 \bmod p^{k+2}$. Thus $x_1 = x_0^2 =$

$1-2(1+\alpha_{k+1})p^{k+1} \mod p^{k+2}$. Finally we remark that $2(1+\alpha_{k+1}) \neq p$.

By this proposition the set $\mathcal{U}_{1/p}(p-1)$ is attracted by the Siegel disk $SI(1)=\mathcal{U}_{1/p}$ after the first iteration.

Examples. Let $n=2$ in all further examples.

1). Let $p=3$. There all points of the intermediate set satisfy the condition of Proposition 5.5. Thus after the first iteration INT_{p_2} is attracted by $SI(1)$.

2). Let $p=5$, i.e. $INT_{p_2}=\mathcal{U}_{1/5}(2)\cup\mathcal{U}_{1/5}(3)\cup\mathcal{U}_{1/5}(4)$. The last ball will be attracted by $SI(1)$ after the first iteration. Direct computations show that the balls $\mathcal{U}_{1/5}(j), j=2,3$, will be attracted by $SI(1)$ after the second iteration. Thus after two iterations the configuration space of the dynamical system on $S_1(0)$ will be reduced to $SI(1)$.

3). Let $p=7$. There we consider the balls $\mathcal{U}_{1/7}(j), j=2,3,4,5$ (the case $j=6$ is described by Proposition 5.5.). It is evident that there exists the fuzzy cycle : $\gamma^{(J)}=(\mathcal{U}_{1/7}(2),\mathcal{U}_{1/7}(4))$. Thus after the first iteration any point $x_0\in INT_{p_2}$ arrives to one of the Siegel disks : $\mathcal{U}_{1/7}(1)$ or $\mathcal{U}_{1/7}(2)\cup\mathcal{U}_{1/7}(4)$.

It should be noted that the field of complex p-adic numbers $\mathbf{C}_p$ is an infinite dimensional linear space over $\mathbf{Q}_p$. Thus dynamical systems in $\mathbf{C}_p$ are, in fact, infinite dimensional dynamical systems over $\mathbf{Q}_p$.

6 Computer calculations for Fuzzy cycles

The following results were obtained with the aid of the package p-adic arithmetic, which was created by S. De Smedt [2] on the basis of the standard program packet MATHEMATICA [18].

Example. Consider the function $p_3(x)=x^3$ in $\mathbf{Q}_5$. Then we found among others the following fuzzy cycles.

Cycles of length 2 :

$$U_{\frac{1}{5}}(2)-U_{\frac{1}{5}}(3)\ ;\ U_{\frac{1}{25}}(7)-U_{\frac{1}{25}}(18)\ ;\ U_{\frac{1}{125}}(57)-U_{\frac{1}{125}}(68)$$

Cycles of length 4 :

$$U_{\frac{1}{25}}(2)-U_{\frac{1}{25}}(8)-U_{\frac{1}{25}}(12)-U_{\frac{1}{25}}(3)$$

$$U_{\frac{1}{25}}(9)-U_{\frac{1}{25}}(4)-U_{\frac{1}{25}}(14)-U_{\frac{1}{25}}(19)$$

$$U_{\frac{1}{25}}(6) - U_{\frac{1}{25}}(16) - U_{\frac{1}{25}}(21) - U_{\frac{1}{25}}(11)$$

$$U_{\frac{1}{25}}(22) - U_{\frac{1}{25}}(23) - U_{\frac{1}{25}}(17) - U_{\frac{1}{25}}(13)$$

$$U_{\frac{1}{125}}(7) - U_{\frac{1}{125}}(93) - U_{\frac{1}{125}}(107) - U_{\frac{1}{125}}(43)$$

$$U_{\frac{1}{125}}(24) - U_{\frac{1}{125}}(74) - U_{\frac{1}{125}}(99) - U_{\frac{1}{125}}(49)$$

$$U_{\frac{1}{125}}(26) - U_{\frac{1}{125}}(76) - U_{\frac{1}{125}}(101) - U_{\frac{1}{125}}(51)$$

$$U_{\frac{1}{125}}(18) - U_{\frac{1}{125}}(82) - U_{\frac{1}{125}}(112) - U_{\frac{1}{125}}(32)$$

Cycles of length 20 :

$$U_{\frac{1}{125}}(6) - U_{\frac{1}{125}}(91) - U_{\frac{1}{125}}(71) - U_{\frac{1}{125}}(36) - U_{\frac{1}{125}}(31) - U_{\frac{1}{125}}(41) - U_{\frac{1}{125}}(46) -$$
$$U_{\frac{1}{125}}(86) - U_{\frac{1}{125}}(56) - U_{\frac{1}{125}}(116) - U_{\frac{1}{125}}(21) - U_{\frac{1}{125}}(11) - U_{\frac{1}{125}}(81) - U_{\frac{1}{125}}(66) -$$
$$U_{\frac{1}{125}}(121) - U_{\frac{1}{125}}(61) - U_{\frac{1}{125}}(106) - U_{\frac{1}{125}}(16) - U_{\frac{1}{125}}(96) - U_{\frac{1}{125}}(111)$$

As we have said in the previous section, if $p_2(x) = x^2$ in $\mathbf{Q}_3$, there are fuzzy cycles of the length 2,6,18,... On the basis of these examples and Propositions 5.2. and 5.3. we propose the following :

Let us consider the function $f(x) = x^n$ in $\mathbf{Q}_p$. To find cycles of length m, we have to solve $f^m(x) = x$. Now $f^m(x) = x \iff x = 0$ or $x^{n^m-1} = 1$. So η is the member of a cycle if and only if η is a root of unity of degree $n^m - 1$ or $\eta = 0$. Considering the Hensel Lemma, we thus need to solve the congruence $x^{n^m-1} = 1 \bmod p$. This congruence has exactly $(n^m - 1, p - 1)$ roots.

So we might conclude that there exist $(n^m - 1, p - 1)$ members of cycles of length m. Of course, we have each time to subtract the number of members of cycles of length a divisor of m.

Let us take an example to explain it. Let $n = 3$ and $p = 17$.
There should be $(2, 16) = 2$ members of cycles of length 1 and thus 2 cycles of length 1. Indeed $U_{\frac{1}{17}}(1)$ and $U_{\frac{1}{17}}(16)$ are.
There should be $(8, 16) = 8$ members of cycles of length 2. Subtracting the previous 2 (1 and 16) there remain 6 members, which allows us to construct $3 = 6/2$ cycles of length 2. And indeed we find

$$U_{\frac{1}{17}}(2) - U_{\frac{1}{17}}(8) \ ; \ U_{\frac{1}{17}}(4) - U_{\frac{1}{17}}(13) \text{ and } U_{\frac{1}{17}}(9) - U_{\frac{1}{17}}(15)$$

There should be $(26, 16) = 2$ members of cycles of length 3. It is 1 and 16 which we already found as cycles of length 1. So there are no cycles of length 3.
There should be $(80, 16) = 16$ members of cycles of length 4. If we delete the 8 previously founded cycles of length 1 and 2, there remain 8 members which allows us to construct 2 cycles af length 4 :

$$U_{\frac{1}{17}}(3) - U_{\frac{1}{17}}(10) - U_{\frac{1}{17}}(14) - U_{\frac{1}{17}}(7)$$

and

$$U_{\frac{1}{17}}(5) - U_{\frac{1}{17}}(6) - U_{\frac{1}{17}}(12) - U_{\frac{1}{17}}(11)$$

And there are no more cycles with balls of radius $\frac{1}{17}$ since we used the 17 possible centers.

In general we obtain the following proposition :

Proposition 6.1. *The number of cycles of length* $m = p_1^{r_1} p_2^{r_2} \ldots p_k^{r_k} \geq 2$ *with radius* $\frac{1}{p}$ *for the function* $f(x) = x^n$ *in* Q_p *is equal to*

$$\frac{(n^m - 1, p - 1) - (n^{p_1^{r_1 - 1}} - 1, p - 1)}{m} \; for\ k = 1$$

$$\frac{(n^m - 1, p - 1) + (k - 1) * (n - 1, p - 1) - \sum_{i=1}^{k} (n^{p_i^{r_i}} - 1, p - 1)}{m} \; for\ k \geq 2$$

As we mentioned already in the previous section, the situation for cycles with radius $r < \frac{1}{p}$ is more complicated.

Example. Consider the function $p_7(x)$ in $\mathbf{Q}_7$ This dynamical system has the following attractors on $S_1(0) : x_1 = 1; x_2 = -1;$

$$x_3 = 2.46302624344521214611\ldots;\ x_4 = 3.46302624344521214611\ldots;$$

$$x_5 = 4.20364042322145452055\ldots;\ x_6 = 5.20364042322145452055\ldots$$

p-adic arithmetic gives the possibility to study more complicated dynamical systems. However, we cannot find exact cycles with the aid of the computer. As a consequence, of a finite exactness, we could find only fuzzy cycles. Therefore we shall study fuzzy cycles and their behaviour. We define *fuzzy attractors* and *fuzzy Siegel disks* by direct generalization of the corresponding definitions for point cycles. Let $f(x) = x^2 + x$ The following fuzzy cycles were found in case p is a prime less than 100.

Cycles of length 2 for $p = 5, 13, 17, 29, 37, 41, 53, 61, 73, 89, 97$. Moreover, we proved the following general statement.

Proposition 6.2. *Let $p = 1 \bmod 4$. Then the dynamical system $f(x) = x^2 + x$ has fuzzy cycles of the length 2. In case $p = 5$ these are fuzzy cyclic attractors, in the other cases these are Siegel disks.*

Cycles of length 3 for $p = 11, 41, 43, 59, 67, 89$ (twice), 97. In case $p = 89$ one of the fuzzy cycles are attractors, all others are Siegel disks.

Cycles of length 4 for $p = 19, 43, 47, 71$ (all Siegel disks).

Cycles of length 5 for $p = 23, 41, 71, 73$ (all Siegel disks)

Cycles of length 6 for $p = 47, 83, 89$ (all Siegel disks)

Cycles of length 7 for $p = 29, 53, 59, 67$ (cyclic attractors in case $p = 29$)

Cycles of length 8 for $p = 61$ (all Siegel disks)

Cycles of length 9 for $p = 31$ (all Siegel disks)

Remark that for some primes we have fuzzy cycles of different lengths. There are fuzzy cycles of length 2, 3 and 6, for example, for $p = 89$ There are fuzzy cycles of length 2, 3 and 5 for $p = 41$

Some of these cycles (we suppose all of them, but we did not prove this) contain subcycles. For example in case $p = 11$, we have the cycle of length 3:

$$U_{1/11}(2) - U_{1/11}(6) - U_{1/11}(9)$$

which contains subcycles of length 15 :

$$U_{1/121}(112) - U_{1/121}(72) - U_{1/121}(53) - U_{1/121}(79) - U_{1/121}(28) -$$
$$U_{1/121}(86) - U_{1/121}(101) - U_{1/121}(17) - U_{1/121}(64) - U_{1/121}(46) -$$
$$U_{1/121}(105) - U_{1/121}(119) - U_{1/121}(2) - U_{1/121}(6) - U_{1/121}(42)$$

and

$$U_{1/121}(35) - U_{1/121}(50) - U_{1/121}(9) - U_{1/121}(90) - U_{1/121}(83) -$$
$$U_{1/121}(75) - U_{1/121}(13) - U_{1/121}(61) - U_{1/121}(31) - U_{1/121}(24) -$$
$$U_{1/121}(116) - U_{1/121}(20) - U_{1/121}(57) - U_{1/121}(39) - U_{1/121}(108)$$

and ...

In case $p = 13$, we have the cycle of length 2 :

$$U_{1/13}(4) - U_{1/13}(7)$$

which contains among others the subcycle of length 8 :

$$U_{1/169}(4) - U_{1/169}(20) - U_{1/169}(82) - U_{1/169}(36) - U_{1/169}(134)-$$
$$U_{1/169}(7) - U_{1/169}(56) - U_{1/169}(150)$$

which contains among others subcycles of length 104.

One of the problems, which arise in our computer investigations of p-adic dynamical systems, is that we could not propose a reasonable way to create p-adic pictures which could illustrate our numerical results. However, this is a general problem of the p-adic framework, since human brain could understand only pictures in real space.

7 Appendix

Theorem 7.1. (Hensel lemma) *Let $F(x), x \in \mathbf{Z}_p$, be a polynomial with coefficients $F_i \in \mathbf{Z}_p$ Let there exists $\gamma \in \mathbf{Z}_p$ such that*

$$F(\gamma) = 0 \bmod p^{2\delta+1} \text{ and } F'(\gamma) = 0 \bmod p^{\delta}, F'(\gamma) \neq 0 \bmod p^{\delta+1}$$

where δ is a natural number. Then there exists a p-adic integer α such that

$$F(\alpha) = 0 \text{ and } \alpha = \gamma \bmod p^{\delta+1}.$$

Some results of this paper were published in the preprint [8] (see also [9] for applications).

References

[1] Aref'eva I. Ya., Dragovich B., Frampton P. H. and Volovich I. V., 1991 The wave function of the Universe and p-adic gravity, Int. J. of Modern Phys. A, 6, No 24, 4341-4358 .

[2] De Smedt S., p-adic arithmetic, Submitted to Mathematica in Education and Research (electronical available at MathSource, item 0208-392).

[3] Ed. Bunde A. and Havlin S., 1994 Fractals in Science, (Heidelberg-New York: Springer Verlag).

[4] Escassut A., 1995 Analytic elements in p-adic analysis (Singapore : World Scientic).

[5] Freund P. G. O., Olson M. and Witten E., 1987 Adelic string amplitudes , Phys.Letters B, 199, 191-195.

[6] Khrennikov A. Yu., 1991 p-adic quantum mechanics with p-adic valued wave functions, J.Math.Physics, 32, 932-937.

[7] Khrennikov A. Yu., 1994 p-adic valued distributions in mathematical physic (Dordrecht : Kluwer Academic).

[8] Khrennikov A. Yu., 1995 p-adic classification of fractals and chaos, Preprint Ruhr-University Bochum SFB n. 288 .

[9] Khrennikov A. Yu., p-adic model of population growth, Proc. Int. Conf. "Fractals in Biology and Medicine" Ascona March 1996 (to be published).

[10] Kolmogoroff A. N., 1933 Grundbegriffe der Wah (Berlin) English translation by N. Morrison, New-York (1950).

[11] Marinari E. and Parisi G., 1988 Adelic approach to the string theory, Phys.Lett. B, 203, 52-56.

[12] Peitgen H. -O., Jürgens H. and Saupe D., 1992 Chaos and Fractals (Heidelberg-New York : Springer Verlag).

[13] Schikhof W., 1984 Ultrametric Calculus, Cambridge Studies in Adv. Math. 4 (Cambridge : Univ. Press.).

[14] Sierpinski W., 1988 Elementary theory of numbers (Amsterdam: North-Holland).

[15] Vladimirov V. S., Volovich I. V. and Zelenov E. I., 1994 p-adic numbers in mathematical physics (Singapore : World Sc. Publ).

[16] Volovich I. V., 1987 Number theory as the ultimate physical theory. Preprint TH.4781/87.

[17] Volovich I. V., 1987 p-adic string, Classical Quantum Grav., 4, L83-L87.

[18] Wolfram S., 1989 Mathematica : A system for doing mathematics by computer (Addison-Wesley Publishing Company).

Faculty of Applied Sciences,
Vrije Universiteit Brussel,
Pleinlaan 2,
1050 Brussel,
Belgium
e-mail: `stany.desmedt@kb.be`

Department of Mathematics,
Rikkyo University,
Ikebukuro,
Toshima-ku,
Tokyo 171,
Japan

Recibido: 20 Septiembre de 1997
Revisado: 1 de Febrero de 1999