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Monotone coefficients and monotonicity of Orlicz spaces.

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Abstract

The criteria for uniform monotonicity, locally uniformly monotonicity and monotonicity of of Orlicz spaces with Luxemburg and Orlicz norms are given. The monotone coefficients of a point and of the spaces are computed.

The monotonicity and the uniform monotonicity are important properties of Banach lattices. In 1985, Akcoglu and Sucheston [1] showed how these properties are related to ergodic theory. Moreover, in 1992 Kurc [8] discovered that the role of monotonicity properties in Banach lattices is similar to the role of rotundity properties in Banach spaces. For example, he proved that for any Banach lattice X the following statements are equivalent: (1) X is monotone; (2) for every $x \in X$ and every order interval $[y, z]$ in X satisfying $x \geq [y, z]$, Card $\left(P_{[y,z]}(x)\right) \leq 1$, where $P_K(x) = \{y \in K :\| x - y \|= d(x, K)\}$; (3) for every sublattice subset K of E and every $x \geq K$, $\mathrm{Card}(P_K(x)) \leq 1$. H. Hudzik and W. Kurc obtained the following important result recently: Let X be an uniformly monotone Banach lattice, then the dominated best approximation problem with respect to any closed sublattice subset E of X is strongly solvable (i.e. for any $x \in X$, $x \geq K$, we have $\mathrm{Card}(P_K(x)) = 1$ and $\lim_n d(x_n, P_K(x)) = 0$ for any minimizing sequence (x_n)). For the discussion of monotonicity, uniform monotonicity and locally uniform monotonicity see [1-5, 7-8]. The uniformly monotone coefficient of the space X and of a point of X are the quantitative characteristics of

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monotonicity. We expect that they will be useful in estimations of the errors of the approximation.

Let X be a Banach lattice and let X^+ denote the positive cone of X.

If $\| x \|= 1$ and $y \neq 0$ imply $\| x+y \|> 1$ for any $x, y \in X^+$ then X is said to have the monotonicity. Moreover, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x, y \in X^+$ satisfying $\| x \|= 1$ and $\| y \|\geq \varepsilon$ we have $\| x+y \|> 1+\delta$, then X is said to have the uniform monotonicity. For $\varepsilon \in [0,1]$, define

$$\eta_X(\varepsilon) = \inf\{\| x+y \| -1 : x, y \in X^+, \| x \|= 1, \| y \|\geq \varepsilon\}.$$

We call

$$m(X) = \sup\{\varepsilon \in [0,1] : \eta_X(\varepsilon) = 0\}$$

the monotone coefficient of X. Obviously, X has uniform monotonicity if and only if $m(X) = 0$. Moreover, for a point x of the unit sphere $S(X)$ we define

$$\eta(x,\varepsilon) = \inf\{\| x+y \| -1 : y \in X^+, \| y \|\geq \varepsilon\}.$$

We call $m(x) = \sup\{\varepsilon \in [0,1] : \eta(x,\varepsilon) = 0\}$ the monotone coefficient of x. If $m(x) = 0$ then x is called an uniformly monotone point. If every point of $S(X^+)$ is an uniformly monotone point then X is said to have locally uniform monotonicity.

In this paper, we will discuss the criteria for monotonicity, uniform monotonicity and locally uniform monotonicity of Orlicz spaces and monotone coefficients of these spaces will be computed. Let $M(u)$ be a N−function, $N(V)$ its complemented function and $p(u)$ the right derivative of $M(u)$. We say $M(u)$ satisfies Δ_2−condition ($M \in \Delta_2$) for large u provided that there exist $u_0 > 0$ and $k > 2$ such that $M(2u) \leq kM(u)$ for any $u \geq u_0$. Let (G, Σ, μ) be a nonatomic finite measure space and L_0 the space of all equivalence classes of Σ−measurable functions equal μ−almost everywhere. The linear set

$$\left\{x \in L_0 : \exists a > 0, \rho_M(ax) = \int_G M(ax(t))d\mu < \infty\right\}$$

endowed with Luxemburgo norm

$$\| x \|= \inf\{c > 0 : \rho_M(x/c) \leq 1\}$$

or Orlicz norm

$$\| x \|^{\circ} = \inf_{k>0} \frac{1}{k} (1 + \rho_M(kx))$$

are all Banach spaces, we call them Orlicz spaces and denote them by L_M or L_M^0 respectively. We have known from Th 1.31 of [6] that for every $0 \neq x \in L_M^0, \| x \|^{\circ} = \frac{1}{k}(1 + \rho_M(kx))$ if and only if $k \in K(x) = [k_x^*, k_x^{**}]$, where $k_x^* = \inf\{k > 0 : \rho_N(p(k \mid x \mid)) \geq 1\}$, and $k_x^{**} = \sup\{k > 0 : \rho_N(p(k \mid x \mid)) \leq 1\}$.

Theorem 1. *For an Orlicz space* L_M, $m(L_M) = \begin{cases} 0, & \text{if} \quad M \in \Delta_2; \\ 1, & \text{if} \quad M \notin \Delta_2. \end{cases}$

Proof. If $M \in \Delta_2$, by Th. 1.39 in [6], for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\| z \| \geq \varepsilon$ implies $\rho_M(z) \geq \delta$. For this δ, we have $\theta > 0$ such that $\rho_M(\mathbf{z}) \geq 1 + \delta$ implies $\| z \| \geq 1 + \theta$.

Fix $x, y \in L_M^+$ satisfying $\| x \| = 1$ and $\| y \| \geq \varepsilon$. Since $M \in \Delta_2$, it is easy to check that $\rho_M(x) = 1$. Noticing $\mid M(u) - M(\nu) \mid \geq M(u - \nu)$ for any $u \geq 0$ and $\nu \geq 0$, we have

$$\rho_M(x+y) - 1 = \rho_M(x+y) - \rho_M(x) = \int_G (M(x(t) + y(t)) - M(x(t)))\, d\mu$$

$$\geq \int_G M(y(t))\, d\mu = \rho_M(y) \geq \delta$$

i.e. $\rho_M(x + y) \geq 1 + \delta$. Therefore, $\| x + y \| \geq 1 + \theta$, $\eta_{L_M}(\varepsilon) \geq \theta > 0$. So, we have $m(L_M) = 0$.

If $M \notin \Delta_2$, take $E \subset G$ satisfying $0 < \mu E < \mu G$. For every $\varepsilon \in (0, 1)$, by Th. 1.13 in [6], we have $u_n \uparrow \infty$ such that $M(u_1)\mu E > \varepsilon$ and $M((1 + 1/n) u_n) > 2^n M(u_n) (n = 1, 2, \cdots)$. Let $\{E_n\}_{n=1}^{\infty} \subset E$ be such that $E_i \cap E_J = \phi(i \neq j)$ and $\mu E_n = \frac{\varepsilon}{2^n M(u_n)} (n = 1, 2, \cdots)$. If

$$x(t) = \sum_{n=1}^{\infty} u_n \chi_{E_n}(t),$$

then $\rho_M(x) = \sum_{n=1}^{\infty} M(u_n)\mu E_n = \varepsilon < 1$. And for any $\lambda > 0$ we have

$$\begin{aligned} \rho_M((1 + \lambda)x) &= \sum_{n=1}^{\infty} M((1 + \lambda)u_n)\, \mu E_n \geq \sum_{n>\frac{1}{\lambda}} M((1 + 1/n)\, u_n)\, \mu E_n \\ &\geq \sum_{n>\frac{1}{\lambda}} 2^n M(u_n)\mu E_n = \infty. \end{aligned}$$

So $\| x \|= 1$. Choose $u_0 > 0$ such that $M(u_0)\mu(G\backslash E) = 1 - \varepsilon$. If

$$y(t) = u_0\chi_{G\backslash E}(t),$$

then $\rho_M(y) = 1 - \varepsilon$. Hence $\rho_M\left(\frac{y}{1-\varepsilon}\right) \geq 1$ and $\| y \| \geq 1 - \varepsilon$. Since $\rho_M(x+y) = \rho_M(x) + \rho_M(y) = 1$, we obtain $\| x + y \|= 1$. This shows $\eta_{L_M(1-\varepsilon)} = 0$. Moreover $m(L_M) \geq 1 - \varepsilon$. By the arbitrariness of ε, we have $m(L_M) = 1$.

Theorem 2. *The following three statements are equivalent:*

1) *The space L_M has the uniform monotonicity,*

2) *The space L_M has the monotonicity,*

3) $M \in \Delta_2$.

Proof. 1)$\Rightarrow$ 2) is trivial. Let us suppose 3) does not hold, then from the proof of the Theorem 1, we can get $x, y \in L_M^+$ such that $\| x \|= 1, y \neq 0$ and $\| x + y \|= 1$. This contradicts 2), so 2)$\Rightarrow$ 3) is true. 3)$\Rightarrow$ 1) is the Theorem 1.

Theorem 3. *The space L_M^0 has the monotonicity.*

Proof. For any $x, y \in L_M^{0+}$ satisfying $\| x \|^o= 1, y \neq 0$. Taking $k \in K(x+y)$, we obtain

$$\begin{aligned} \| x + y \|^o - 1 &= \| x + y \|^o - \| x \|^o = \frac{1}{k}(1 + \rho_M(k(x+y))) - \| x \|^o \\ &\geq \frac{1}{k}(1 + \rho_M(k(x+y))) - \frac{1}{k}(1 + \rho_M(kx)) \\ &\geq \frac{1}{k}\rho_M(ky) > 0. \end{aligned}$$

Theorem 4. *For space $L_M^0, m(L_M^0) = \begin{cases} 0 & if \quad M \in \Delta_2, \\ 1 & if \quad M \notin \Delta_2. \end{cases}$*

Proof. If the first statement is not true, then there exists $\varepsilon > 0$ such that $\eta_{L_M^0}(\varepsilon) = 0$. So we can find $x_n, y_n \in L_M^{0+}$ satisfying $\| x_n \|^o= 1, \| y_n \|^o \geq \varepsilon$ and $\| x_n + y_n \|^o \to 1$.

Since $M \in \Delta_2$, by Th. 1.39 of [6], there exists $\delta > 0$ such that $\| z \|^{\circ} \geq \frac{\varepsilon}{2}$ implies $\rho_M(z) \geq \delta$.
Taking $k_n \in K(x_n + y_n)$, such that $\| x_n + y_n \|^{\circ} = \frac{1}{k_n}(1 + \rho_M(k_n(x_n + y_n)))$ for $n = 1, 2, \cdots$ Since $\| x_n + y_n \|^{\circ} \leq 2$ we have $k_n > \frac{1}{2}$ for large n. Hence

$$\begin{aligned} \| x_n + y_n \|^{\circ} - 1 &= \| x_n + y_n \|^{\circ} - \| x_n \|^{\circ} \geq \frac{1}{k_n}(1 + \rho_M(k_n(x_n + y_n))) \\ &\quad - \frac{1}{k_n}(1 + \rho_M(k_n x_n)) \\ &\geq \frac{1}{k_n}\rho_M(k_n y_n) \geq 2\rho_M\left(\frac{y_n}{2}\right) \geq 2\delta, \end{aligned}$$

which contradicts the condition $\| x_n + y_n \|^{\circ} \to 1$.

Now, let us prove the second statement. Since $M \notin \Delta_2$, then by §2.3 (1) of [7], for every $\varepsilon \in \left(0, \frac{1}{2}\right)$ there exists large $u > 0$. Such that $M(u) < \varepsilon u p(u)$. Without loss of generality, we may assume that $N(p(u))\mu G > 1$. Take a subset E of G satisfying $N(p(u))\mu E = 1$. Define

$$k = 1 + M(u)\mu E$$

and

$$x(t) = \frac{1}{k} u \chi_E(t).$$

It is easy to check that $k \in K(x)$, whence we have $\| x \|^{\circ} = \frac{1}{k}(1 + \rho_M(kx)) = 1$.

Since

$$\begin{aligned} k &= 1 + M(u)\mu E < 1 + \varepsilon u p(u)\mu E = 1 + \varepsilon\left(N(p(u))\mu E + M(u)\mu E\right) \\ &= 1 + \varepsilon(1 + M(u)\mu E) = 1 + \varepsilon k, \end{aligned}$$

we obtain that $k < \frac{1}{1-\varepsilon}$.

Since $M \notin \Delta_2$, for every $\theta \in (0, \varepsilon)$, there exists a large $v > 0$ such that $M((1+\theta)\nu) > \frac{M(\nu)}{\theta}$. We can assume $M(v)\mu(G \backslash E) > \theta$ and take $F \subset G \backslash E$ satisfying $M(\nu)\mu F = \theta$. Denote

$$y(t) = \frac{1}{k}\nu \chi_F(t).$$

Noticing that $\rho_M((1+\theta)ky) = M((1+\theta)\nu)\mu F > \frac{1}{\theta}M(\nu)\mu F = 1$, we have $\| y \|^o \geq \| y \| \geq \frac{1}{(1+\theta)k} > \frac{1-\varepsilon}{1+\varepsilon}$. Since $k > 1$, it follows that

$$\begin{aligned}\| x+y \|^o - 1 &= \| x+y \|^o - \| x \|^o \leq \frac{1}{k}(1+\rho_M(k(x+y))) \\ &\quad - \frac{1}{k}(1+\rho_M(kx)) \\ &= \frac{1}{k}\rho_M(ky) < \rho_M(ky) = M(\nu)\mu E = \theta\end{aligned}$$

This shows $\eta_{L_M^0}\left(\frac{1-\varepsilon}{1+\varepsilon}\right) < \theta$. By the arbitrariness of θ we get $\eta_{L_M^0}\left(\frac{1-\varepsilon}{1+\varepsilon}\right) = 0$. Moreover $m(L_M^0) \geq \frac{1-\varepsilon}{1+\varepsilon}$. By the arbitrariness of ε we get $m(L_M^0) = 1$.

Next, we will discuss the monotone coefficient of a point of the unit sphere.

Theorem 5. *For every* $x \in S\left(L_M^+\right)$, $m(x) = \begin{cases} 0 & \text{if } M \in \Delta_2, \\ 1 & \text{if } M \notin \Delta_2. \end{cases}$

Proof. The first statement is a consequence of Theorem 1.

Take $c > 0$ such that $E = \{t \in G : x(t) \leq c\}$ has positive measure. Since $M \notin \Delta_2$, for any $\varepsilon \in (0, 1/3)$ and $\delta > 0$, there exists a large u such that $u > \frac{c}{\varepsilon}$ and $M((1+\varepsilon)u) > \frac{1}{\delta}M(u)$. Assume $M(u)\mu E > \delta$ and take $E_0 \subset E$ such that $M(u)\mu E_0 = \delta$. Put

$$y(t) = (u - x(t))\,\chi_{E_0}(t).$$

Then

$$\begin{aligned}\rho_M((1+3\varepsilon)y) &= \int_{E_0} M((1+3\varepsilon)(u - x(t)))\,d\mu \\ &\geq \int_{E_0} M((1+3\varepsilon)(u-c))\,d\mu \\ &\geq \int_{E_0} M((1+3\varepsilon)(1-\varepsilon)u)\,d\mu \\ &> \int_{E_0} M((1+\varepsilon)u)\,d\mu > \frac{1}{\delta}M(u)\mu E_0 = 1.\end{aligned}$$

This shows $\| y \| \geq \frac{1}{1+3\varepsilon}$. But

$$\rho_M(x+y) = \int_{G\setminus E_0} M(x(t))d\mu + \int_{E_0} M(u)d\mu \leq 1 + \delta.$$

Hence $\| x+y \| \leq 1+\delta, \| x+y \| -1 \leq \delta$. So $\eta\left(x, \frac{1}{1+3\varepsilon}\right) \leq \delta$. By the arbitrariness of δ we get $\eta\left(x, \frac{1}{1+3\varepsilon}\right) = 0$ and $m(x) \geq \frac{1}{1+3\varepsilon}$. By the arbitrariness of ε we have $m(x) = 1$.

Corollary 1. *Every $x \in X\left(L_M^+\right)$ is a uniformly monotone point if and only if $M \in \Delta_2$.*

Corollary 2. *The space L_M has locally uniformly monotonicity if and only if $M \in \Delta_2$.*

Now let us discuss these properties of Orlicz spaces with Orlicz norm.

Theorem 6. *For every*

$$x \in S\left(L_M^{0+}\right), m(x) = \begin{cases} 0 & if \quad M \in \Delta_2, \\ \frac{1}{k_x^*} & if \quad M \notin \Delta_2. \end{cases}$$

Proof. The first statement is a consequence of Theorem 4.

If $M \notin \Delta_2$, take $c > 0$ such that the set $E = \{t \in G : k_x^* x(t) \leq c\}$ has positive measure. For every $\varepsilon \in (0, 1/3)$ and $\delta > 0$ there exists $u > \frac{c}{\varepsilon}$ such that $M((1+\varepsilon)u) > \frac{1}{\delta} M(u)$. We can assume $M(u)\mu E > \delta$ and take $E_0 \subset E$ such that $M(u)\mu E_0 = \delta$. Put

$$y(t) = \left(\frac{u}{k_x^*} - x(t)\right) \chi_{E_0}(t).$$

Since

$$\begin{aligned} \rho_M\left((1+3\varepsilon)k_x^* y\right) &= \int_{E_0} M\left((1+3\varepsilon)\left(u - k_x^* x(t)\right)\right) d\mu \\ &> M\left((1+3\varepsilon)(u-c)\right)\mu E_0 > M\left((1+\varepsilon)u\right)\mu E_0 > 1. \end{aligned}$$

we have $\| y \|^\circ \geq \| y \| > \frac{1}{(1+3\varepsilon)k_x^*}$. But

$$\begin{aligned} \| x+y \|^\circ - 1 &= \| x+y \|^\circ - \| x \|^\circ \leq \frac{1}{k_x^*}\left(1 + \rho_M\left(k_x^*(x+y)\right)\right) \\ &\quad - \frac{1}{k_x^*}\left(1 + \rho_M\left(k_x^* x\right)\right) \\ &\leq \frac{1}{k_x^*} M(u)\mu E_0 = \frac{\delta}{k_x^*} < \delta. \end{aligned}$$

So $\eta\left(x, \frac{1}{(1+3\varepsilon)k_x^*}\right) < \delta$. By the arbitrariness of δ we have $\eta\left(x, \frac{1}{(1+3\varepsilon)k_x^*}\right) = 0$. Moreover $m(x) \geq \frac{1}{(1+3\varepsilon)k_x^*}$. By the arbitrariness of ε we get $m(x) \geq \frac{1}{k_x^*}$.

We prove $m(x) > \frac{1}{k_x^*}$ is false. Assuming there exists $\varepsilon_0 > 0$ satisfying $m(x) > \frac{1}{k_x^*} + \varepsilon_0$, we can find $\{y_n\}_{n=1}^{\infty} \subset L_M^{0+}$ such that $\| y_n \|^o \geq \frac{1}{k_x^*} + \varepsilon_0$ and $\| x + y_n \|^o < 1 + \frac{1}{n}$ for $n = 1, 2, \cdots$ For

$$k_n = k_{x+y_n}^*; \text{ there holds}$$

$$\| x + y_n \|^o = \frac{1}{k_n}\left(1 + \rho_M\left(k_n(x + y_n)\right)\right) (n = 1, 2, \cdots).$$

Obviously $k_n \leq k_x^*$ for $n = 1, 2, \cdots$ We show that $\lim_{n\to\infty} k_n = k_x^*$. In fact, suppose $k_n < k_x^* - \delta\,(0 < \delta < k_x^*)$. Since the function $F(k) = \frac{1}{k}\left(1 + \rho_M(kx)\right)$ strictly decreases on $(0, k_x^*)$, we have

$$\frac{1}{k_n}\left(1 + \rho_M(k_n x)\right) \geq \frac{1}{k_x^* - \delta}\left(1 + \rho_M\left(\left(k_x^* - \delta\right) x\right)\right) = \frac{1}{k_x^*}\left(1 + \rho_M\left(k_x^* x\right)\right) + \theta$$

where $\theta > 0$. Hence

$$\begin{aligned} \| x + y_n \|^o - 1 &= \| x + y_n \|^o - \| x \|^o = \frac{1}{k_n}\left(1 + \rho_M\left(k_n\left(x + y_n\right)\right)\right) \\ &\quad - \frac{1}{k_x^*}\left(1 + \rho_M\left(k_x^* x\right)\right) \geq \theta, \end{aligned}$$

which contradicts the condition $\| x + y_n \|^o < 1 + \frac{1}{n}$ for $n = 1, 2, \cdots$ So $\lim_{n\to\infty} k_n = k_x^*$. By the Fatou Lemma (passing to a subsequence if necessary)

$$\rho_M(k_n x) > \rho_M\left(k_x^* x\right) - \frac{1}{n} = k_x^* - 1 - \frac{1}{n}$$

when n is large enough. Thus

$$k_n\left(1 + \frac{1}{n}\right) \geq k_n \| x + y_n \|^o = 1 + \rho_M\left(k_n(x + y_n)\right) \geq 1 + \rho_M(k_n x) + \rho_M(k_n y_n)$$

$$\geq \| k_n y_n \|^o + \rho_M(k_n x) \geq k_n\left(\frac{1}{k_x^*} + \varepsilon_0\right) + k_x^* - 1 - \frac{1}{n},$$

and so $k_x^* \geq k_x^*(1 + \varepsilon_0)$, a contradiction. Hence $m(x) = \frac{1}{k_x^*}$.

Corollary 3. *Every $x \in S\left(L_M^{0+}\right)$ is an uniformly monotone point if and only if $M \in \Delta_2$.*

Corollary 4. *The space L_M^0 has locally uniformly monotonicity if and only if $M \in \Delta_2$.*

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