REVISTA MATEMÁTICA COMPLUTENSE
Volumen 11, número 2: 1998
http://dx.doi.org/10.5209/rev_REMA.1998.v11.n2.17254

A summability condition on the gradient ensuring BMO.

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Abstract

It is well-known that if $u \in W^{1,1}(\Omega)$, $\Omega \subset \mathbb{R}^N$ satisfies $| Du | \in L^N(\Omega)$, then u belongs to $BMO(\Omega)$, the John-Nirenberg Space. We prove that this is no more true if $| Du |$ belongs to an Orlicz space $L_A(\Omega)$ when the N-function $A(t)$ increases less than t^N. In order to obtain $u \in BMO(\Omega)$, we impose a suitable uniform L_A condition for $| Du |$.

1 Introduction

In a recent paper Fusco-Lions-Sbordone ([FLS]) gave imbeddings of Orlicz-Sobolev spaces $W^{1,A}(\Omega)$, Ω a cube in $\mathbb{R}^N$, in Orlicz spaces with exponential growth, when the Young function A is of type $A(t) = t^N \log^{-\sigma}(e+t)$. If $\sigma = 0$, the space $W^{1,A}(\Omega)$ reduces to $W^{1,N}(\Omega)$ and it is well-known that such space is imbedded in $BMO(\Omega)$. If $\sigma = 1$ there are some counterexamples (see [GISS]) showing that $W^{1,A}(\Omega)$ is not imbedded in $BMO(\Omega)$.

In this paper first we show, adapting an example appeared in [GISS], that for any Young function $A(t)$ which growths essentially less than t^N, the space $W^{1,A}(\Omega)$ is not imbedded in $BMO(\Omega)$. Such a result has been recently proved, in a different way, in a paper by Cianchi-Pick [CP]. Moreover, if we require that, in some sense, the gradient of a function u is in $L_A(\Omega, \mathbb{R}^N)$ *uniformly with respect to the cubes contained in* Ω, then

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1991 Mathematics Subject Classification: 46E30, 46E35.
*This work has been partially performed as a part of a National Research Project supported by M.U.R.S.T.
Servicio Publicaciones Univ. Complutense. Madrid, 1998.

we get the imbedding in $BMO(\Omega)$, even if the Young function $A(t)$ has a growth essentially less than t^N. Namely, let us introduce the *uniform Orlicz spaces*

$$f \in \mathcal{U}_A(\Omega, \mathbb{R}^N) \Longleftrightarrow \mid f \mid_{\mathcal{U}_A(\Omega,\mathbb{R}^N)} = \sup_{Q \subset \Omega} \mid Q \mid^{\frac{1}{N}} \| \mid f \mid \|_{L_A(Q)} < +\infty$$

where the supremum is extended to all cubes Q contained in Ω with sides parallel to the coordinate axis. If $A(t) = t^N$, then $\mathcal{U}_A(\Omega, \mathbb{R}^n)$ reduces to $L^N(\Omega, \mathbb{R}^N)$; if $A(t) = \frac{t^N}{\log^\sigma(e+t)}$, $\sigma > 0$, then $\mathcal{U}_A(\Omega, \mathbb{R}^N)$ contains $L^N(\Omega, \mathbb{R}^N)$. We show that for such A if $\nabla u \in \mathcal{U}_A(\Omega, \mathbb{R}^N)$ then $u \in BMO(\Omega)$ (see Corollary 3.4) and, more generally, following [IS], if we introduce the space

$$f \in \mathcal{U}_\sigma^{N)}(\Omega, \mathbb{R}^N) \Longleftrightarrow \sup_{Q \subset \Omega} \mid Q \mid^{\frac{1}{N}} \sup_{0<\epsilon\leq 1} \left(\epsilon^\sigma \fint_Q \mid f \mid^{N-\epsilon} dx \right)^{\frac{1}{N-\epsilon}} < +\infty$$

we have that $\mathcal{U}_A(\Omega, \mathbb{R}^N) \subset \mathcal{U}_\sigma^{N)}(\Omega, \mathbb{R}^N)$ (see Proposition 3.2) and, if $\nabla u \in \mathcal{U}_\sigma^{N)}(\Omega, \mathbb{R}^N)$, then $u \in BMO(\Omega)$ (see Theorem 3.3).

Finally, following [FLS], we will prove also some imbedding results in Orlicz spaces for the Riesz Potential Operator in the critical case (see Theorem 3.5).

2 Notation and Preliminary results

Let us fix notation and recall basic concepts. For our purposes, a *Young function* will be any nonnegative, even, convex function $\Phi : \mathbb{R}^1 \to \mathbb{R}^1$ such that Φ is (strictly) increasing on $[0, \infty)$, and $\lim_{t\to 0} \Phi(t)/t = 0$, $\lim_{t\to\infty} \Phi(t)/t = \infty$.

Let Ω be a bounded open set in $\mathbb{R}^N$. The Orlicz space $L_\Phi(\Omega)$ is defined to be the smallest vector space containing the set of all measurable functions f defined on Ω such that $\Phi(\mid f \mid) \in L^1(\Omega)$. It may be checked that $L_\Phi(\Omega)$ is a Banach space with respect to the norm

$$\|f\|_\Phi = \inf\left\{ \lambda > 0 : \fint_\Omega \Phi\left(\frac{\mid f \mid}{\lambda}\right) dx \leq 1 \right\}$$

where the symbol $\displaystyle\fint_\Omega$ stands for $\displaystyle\frac{1}{|\Omega|}\int_\Omega$. A special case is $\Phi(t) = \frac{t^p}{p}$ $(p \geq 1)$, in which $L_\Phi(\Omega)$ reduces to $L^p(\Omega)$. If $\Phi(t) = \frac{t^p}{\log^\sigma(e+t)}$ $(p > 1, \sigma \geq 0)$ then the corresponding Orlicz space will be denoted by $L^p \log^{-\sigma} L(\Omega)$. Following [IS], we will consider also a space larger than $L^p \log^{-\sigma} L(\Omega)$, namely $L^{p)}_\sigma(\Omega)$ $(p > 1, \sigma \geq 0)$, defined as the Banach space of all measurable functions on Ω such that

$$\|f\|_{L^{p)}_\sigma} = \sup_{0<\epsilon\leq 1} \left(\epsilon^\sigma \fint_\Omega |f|^{p-\epsilon}\, dx\right)^{\frac{1}{p-\epsilon}} < +\infty.$$

Following [G], the closure of $L^\infty(\Omega)$ in $L^{p)}_\sigma(\Omega)$ will be denoted by $\Sigma^p_\sigma(\Omega)$ (by $\Sigma^p(\Omega)$ if $\sigma = 1$), and it is characterized as the space of all measurable functions on Ω such that

$$\lim_{\epsilon\to 0} \left(\epsilon^\sigma \fint_\Omega |f|^{p-\epsilon}\, dx\right)^{\frac{1}{p-\epsilon}} = 0.$$

In [FLS] it is proved the following extension of Trudinger's imbedding theorem ([T]) for $W^{1,N}_0(\Omega)$ functions:

Theorem 2.1. *If $u \in W^{1,1}_0(\Omega)$ is such that $|Du| \in L^{N)}_\sigma(\Omega)$ for some $\sigma \geq 0$, then there exist $c_1 = c_1(N,\sigma)$, $c_2 = c_2(N,\sigma)$ such that*

$$\fint_\Omega \exp\left(\left(\frac{|u|}{c_1\|Du\|_{L^{N)}_\sigma} |\Omega|^{\frac{1}{N}}}\right)^{\frac{N}{N-1+\sigma}}\right) dx \leq c_2$$

We remark that if Ω is convex, then an inequality of the same type is true also for functions $u \in W^{1,1}(\Omega)$, provided $|u|$ is replaced by $|u - \fint_\Omega u dx|$. In fact, giving a closer look to the proof of Theorem 2.1, the assumption $u \in W^{1,1}_0(\Omega)$ has been used only to write the inequality

$$|u(x)| \leq C(N) \int_\Omega |Du|\, |x-y|^{1-N}\, dy$$

If $u \in W^{1,1}(\Omega)$ and Ω is convex, replacing $|u|$ by $|u - \fint_\Omega u dx|$, this inequality is true with the constant in the right hand side depending only on N and the shape of Ω, but independently on the measure of Ω ([GT]). In the proof of Theorem 3.3 we will use such inequality with Ω replaced by a cube, therefore the constants will depend only on N.

In [FLS] it is proved also that if $u \in W_0^{1,1}(\Omega)$ and $| Du | \in \Sigma^N(\Omega)$ then $u \in exp(\Omega)$, that is the closure of $L^\infty(\Omega)$ in the Banach space

$$EXP(\Omega) = \left\{ f \in L^1(\Omega) : \exists \lambda > 0 \text{ such that} \fint_\Omega exp\left(\frac{| f |}{\lambda}\right) dx < \infty \right\}.$$

More generally, we will denote by $exp_\alpha(\Omega)$, $\alpha > 0$, the closure of $L^\infty(\Omega)$ in $EXP_\alpha(\Omega)$, the Orlicz space generated by the function $\Phi(t) = \exp(t^\alpha) - 1$.

Finally, let us recall that $BMO(\Omega)$ is defined (see [S] for instance) as the space of the measurable functions u such that

$$\|u\|_{BMO} = \sup_{Q \subset \Omega} \fint_Q | u - u_Q | dx < +\infty$$

where the supremum is taken over all cubes Q with sides parallel to the coordinate axes, and u_Q stands for $\fint_Q u dx$. We would get an equivalent definition of $BMO(\Omega)$ if we replace the family of all cubes by the family of all balls. It is possible to prove (see [KJF] for instance) that if Ω is a cube then $BMO(\Omega)$ functions can be characterized by the following property:

$$\exists \lambda > 0 : \sup_{Q \subset \Omega} \fint_Q \exp\left(\frac{| u - u_Q |}{\lambda}\right) dx < +\infty.$$

3 The main results

Let us recall that by Moser's inequality ([M]) $W^{1,N}(\Omega)$ functions are $exp_{\frac{n}{n-1}}(\Omega)$ functions, and if $| Du | \in L^N \log^{-\sigma} L(\Omega)$ then $u \in exp_{\frac{n}{n-1+\sigma}}(\Omega)$

We now study imbeddings in $BMO(\Omega)$. While $W^{1,N}(\Omega)$ functions are $BMO(\Omega)$ functions, if $A(t)$ is a Young function with a growth essentially less than t^N, then the Orlicz-Sobolev space $W^{1,A}(\Omega)$ is not imbedded in $BMO(\Omega)$. In fact we have the following example (see [GISS] for the case $A(t) = t^N \log^{-\sigma}(e+t)$):

Example 3.1. *Let Ω be a bounded open set in $\mathbb{R}^N$, and let A be a Young function of the type $A(t) = t^N \varphi(t)$, $\varphi(+\infty) = 0$. Then there exists a measurable function u such that $| Du | \in L_A(\Omega)$ and $u \notin BMO(\Omega)$.*

Let $\{a_j\}_{j \in \mathbb{N}}$ be such that

$$\sum_j a_j^N j^{-2} < +\infty \tag{3.1}$$

$$\lim_j a_j = +\infty \tag{3.2}$$

and let $\{r_j\}_{j \in \mathbb{N}}$ be such that

$$\sum_j r_j < +\infty \tag{3.3}$$

$$\varphi(t) \leq \frac{1}{j^2 \log 2} \qquad \forall t > \frac{a_j}{r_j}, \quad \forall j \in \mathbb{N} \tag{3.4}$$

Let us note that by (3.3) we can find a sequence of points $x_j \in \Omega$ such that the balls $B(x_j, r_j)$ are pairwise disjoint and contained in Ω (at least for j large enough). Let us define

$$h_j(x) = a_j h\left(\frac{x - x_j}{r_j}\right) \quad \forall x \in \Omega, \quad \forall j \in \mathbb{N}$$

where

$$h(x) = \begin{cases} 0 & \text{if } |x| \geq 1 \\ -\log |x| & \text{if } \frac{1}{2} \leq |x| \leq 1 \\ \log 2 & \text{if } |x| \leq \frac{1}{2} \end{cases} \quad \forall x \in \mathbb{R}^n$$

and let $u = \sum_j h_j$. Notice that $u(x) = h_j(x)$ if $| x - x_j | < r_j$.

Hence, we have

$$\|u\|_{BMO} \geq \fint_{B_j} | h_j - (h_j)_{B_j} | \, dx = a_j \fint_B | h - (h)_B | \, dx \quad \forall j \in \mathbb{N}$$

where B is the unit ball of $\mathbb{R}^n$, and therefore, by (3.2), $u \notin BMO(\Omega)$.

On the other hand

$$| Dh_j | \leq \begin{cases} \frac{a_j}{|x-x_j|} & \text{if } \frac{r_j}{2} \leq | x - x_j | \leq r_j \\ 0 & \text{if } | x - x_j | \leq \frac{r_j}{2} \end{cases}$$

and therefore, by (3.4),

$$\begin{aligned} \int_{|x-x_j|\leq r_j} A(| Dh_j |)dx &\leq \int_{\frac{r_j}{2}\leq|x-x_j|\leq r_j} A\left(\frac{a_j}{| x - x_j |}\right) dx \\ &= N\omega_N \int_{\frac{r_j}{2}}^{r_j} A\left(\frac{a_j}{\rho}\right) \rho^{N-1}d\rho \\ &= N\omega_N a_j^N \int_{\frac{r_j}{2}}^{r_j} \frac{1}{\rho}\varphi\left(\frac{a_j}{\rho}\right) d\rho \\ &\leq N\omega_N a_j^N \int_{\frac{r_j}{2}}^{r_j} \frac{1}{\rho} \cdot \frac{1}{j^2 \log 2} d\rho \\ &= N\omega_N a_j^N \frac{1}{j^2} \end{aligned}$$

where ω_N denotes the measure of the unit ball in $\mathbb{R}^n$, from which, summing over j and using (3.1), we get $| Du | \in L_A(\Omega)$.

■

We remark that if $| Du |$ belongs to some suitable spaces containing $L^N(\Omega)$ (for instance, $weak-L^N(\Omega)$) then it is known that $u \in BMO(\Omega)$. Now we introduce some new spaces having this property, which represent a variant of the classical Orlicz spaces. Namely, we consider the functions $f \in L_A(\Omega)$ such that

$$| f |_{p,A,\Omega} = \sup_{Q\subset\Omega} | Q |^{\frac{1}{p}} \|f\|_{L_A(Q)} < +\infty$$

If $A(t) = t^p$, then $| f |_{p,A,\Omega}$ reduces to the classical norm in L^p spaces. If $p = N$ and $A(t) = \dfrac{t^N}{\log^\sigma(e+t)}$ $(N > 1, \sigma > 0)$ then $| f |_{p,A,\Omega}$ is a norm defining a Banach space and it is different from $\|f\|_{L_A(\Omega)}$. The following result hold:

Proposition 3.2. *Let* $A(t) = \dfrac{t^N}{\log^\sigma(e+t)}$ $(N > 1, \sigma > 0)$. *If*

$$\sup_{Q\subset\Omega} \mid Q \mid^{\frac{1}{p}} \|f\|_{L_A(Q)} < +\infty$$

then

$$\sup_{\substack{Q\subset\Omega \\ 0<\epsilon\leq 1}} \mid Q \mid^{\frac{1}{N}} \left(\epsilon^\sigma \fint_Q \mid f \mid^{N-\epsilon} dx \right)^{\frac{1}{N-\epsilon}} < +\infty$$

Proof. Let $f \in L_A(\Omega)$, $f \geq T_\sigma$ where $A(T_\sigma) = 1$. By using the elementary inequality

$$(e+t)^\varepsilon < e + t^\varepsilon \quad (0 < \varepsilon < 1, t \geq 0)$$

we obtain

$$\begin{aligned} \varepsilon^\sigma f^{N-\varepsilon} &= \frac{\log^\sigma[(e+f)^\varepsilon]}{f^\varepsilon} \frac{f^N}{\log^\sigma(e+f)} \leq \frac{\log^\sigma(e+f^\varepsilon)}{f^\varepsilon} \frac{f^N}{\log^\sigma(e+f)} \\ &\leq C_\sigma \frac{f^N}{\log^\sigma(e+f)} \end{aligned}$$

for some $C_\sigma > 0$, therefore

$$\sup_{0<\epsilon\leq 1} \left(\epsilon^\sigma \fint_Q \mid f \mid^{N-\epsilon} dx \right)^{\frac{1}{N-\epsilon}} \leq C_\sigma \fint_Q \frac{f^N}{\log^\sigma(e+f)} dx$$

If we drop the condition $f \geq T_\sigma$, applying the previous estimate to $\max(\mid f \mid, T_\sigma)$ we get

$$\begin{aligned} \sup_{0<\epsilon\leq 1} \left(\epsilon^\sigma \fint_Q \mid f \mid^{N-\epsilon} dx \right)^{\frac{1}{N-\epsilon}} &\leq C_\sigma \fint_Q \frac{\max(\mid f \mid, T_\sigma)^N}{\log^\sigma(e + \max(\mid f \mid, T_\sigma))} dx \\ &\leq C_\sigma \fint_Q \frac{f^N}{\log^\sigma(e+f)} dx + D_\sigma \end{aligned}$$

for some $D_\sigma \geq 0$.

Replacing f by $\frac{f}{\|f\|_{L_A(Q)}}$, the right hand side is majorized by a constant depending only on σ, and independent of Q, therefore the assertion follows multiplying by $| Q |^{\frac{1}{N}} \|f\|_{L_A(Q)}$ and taking the supremum over all cubes Q contained in Ω.

∎

Theorem 3.3. *If $u \in W^{1,1}(\Omega)$, Ω cube in $\mathbb{R}^N$ $(N > 1)$, is such that $| Du |$ verifies the condition*

$$| Du | \in \mathcal{U}^N_\sigma(\Omega, \mathbb{R}^N) \iff \sup_{Q \subset \Omega} | Q |^{\frac{1}{N}} \sup_{0<\epsilon\leq 1} \left(\epsilon^\sigma \fint_Q | Du |^{N-\epsilon} \, dx \right)^{\frac{1}{N-\epsilon}} =$$

$$M_{u,\sigma} < +\infty, \tag{3.5}$$

for some $\sigma > 0$, then $u \in BMO(\Omega)$.

Proof. Without loss of generality we can assume $0 < \sigma \leq 1$. Let us fix $Q \subset \Omega$ and let us apply Theorem 2.1 with Ω replaced by Q, and u replaced by $u - u_Q$. We have

$$\fint_Q \exp\left(\left(\frac{| u - u_Q |}{c_1 M_{u,\sigma}} \right)^{\frac{N}{N-1+\sigma}} \right) dx \leq$$

$$\fint_Q \exp\left(\left(\frac{| u - u_Q |}{c_1 \|Du\|_{L^{N)}_\sigma} \, | Q |^{\frac{1}{N}}} \right)^{\frac{N}{N-1+\sigma}} \right) dx \leq c_2(N, \sigma),$$

from which

$$\int_Q \exp\left(\frac{| u - u_Q |}{c_1 M_{u,\sigma}} \right) dx =$$

$$\int_{\frac{|u-u_Q|}{c_1 M_{u,\sigma}} > 1} \exp\left(\frac{| u - u_Q |}{c_1 M_{u,\sigma}} \right) dx + \int_{\frac{|u-u_Q|}{c_1 M_{u,\sigma}} \leq 1} \exp\left(\frac{| u - u_Q |}{c_1 M_{u,\sigma}} \right) dx$$

$$
\begin{aligned}
&\leq \int_Q \exp\left(\left(\frac{|u-u_Q|}{c_1 M_{u,\sigma}}\right)^{\frac{N}{N-1+\sigma}}\right) dx + \int_Q \exp(1) dx \\
&\leq c_2(N,\sigma)\,|Q| + e\,|Q|
\end{aligned}
$$

and therefore

$$
\sup_{Q\subset\Omega} \fint_Q \exp\left(\frac{|u-u_Q|}{c_1 M_{u,\sigma}}\right) dx < +\infty.
$$

Since Ω is a cube, then $u \in BMO(\Omega)$.

■

We prove now the following

Corollary 3.4. *Let* $A(t) = \dfrac{t^N}{\log^\sigma(e+t)}$ $(N>1, \sigma>0)$. *If* $|\,Du\,|_{N,A,\Omega} < +\infty$, *then* $u \in BMO(\Omega)$.

Proof. For any $Q \subset \Omega$ we have $\|f\|_{L_A(Q)} < +\infty$ and therefore (see [BFS] lemma 3; see also [G])

$$
\lim_{\epsilon\to 0+} \left(\epsilon^\sigma \fint_Q |\,Du\,|^{N-\epsilon} dx\right)^{\frac{1}{N-\epsilon}} = 0
$$

from which

$$
\sup_{0<\epsilon\leq 1} \left(\epsilon^\sigma \fint_Q |\,Du\,|^{N-\epsilon} dx\right)^{\frac{1}{N-\epsilon}} < +\infty \quad \forall Q \subset \Omega.
$$

We have

$$
\begin{aligned}
\sup_{Q\subset\Omega} |\,Q\,|^{\frac{1}{N}} \sup_{0<\epsilon\leq 1} & \left(\epsilon^\sigma \fint_Q |\,Du\,|^{N-\epsilon} dx\right)^{\frac{1}{N-\epsilon}} \\
&\leq \sup_{Q\subset\Omega} |\,Q\,|^{\frac{1}{N}} \sup_{0<\epsilon\leq 1} \epsilon^{\frac{\sigma}{N-\epsilon}} c(N,\sigma)\|Du\|_{L_A(Q)} \\
&\leq c(N,\sigma) \sup_{Q\subset\Omega} |\,Q\,|^{\frac{1}{N}} \|Du\|_{L_A(Q)} \\
&= c(N,\sigma)\,|\,Du\,|_{A,\Omega} < +\infty
\end{aligned}
$$

and therefore by Theorem 3.3 the assertion follows. ■

By Corollary 3.4, the function f of Example 3.1 is such that $| Df |_{N,A,\Omega} = +\infty$. This fact could be also verified directly, by proving that

$$| B_j |^{\frac{1}{N}} \sup_{0<\epsilon\leq 1} \left(\epsilon \fint_{B_j} | Dh_j |^{N-\epsilon} \, dx \right)^{\frac{1}{N-\epsilon}} = c(N) a_j \qquad \forall j \in \mathbb{N}.$$

Let us note also that the BMO function $u(x) = \log | x |$ ($| x | \leq 1$) verifies the condition (3.5), and is such that $u \notin L^\infty$, $| Du | \notin L^N$.

We remark that by using the same arguments to prove Theorem 2.1 it is possible to give an alternative proof of a well-known result by Adams [A] (see Corollary 4.2) about the Riesz Potential Operator defined by

$$I_{\frac{N}{p}} f = \int_\Omega | x - y |^{\frac{N}{p} - N} f(y) dy.$$

Theorem 3.5. *Let* $1 < p < +\infty$, $\sigma > 0$. *If* $f \in L^{p)}_\sigma(\Omega)$, *then* $I_{\frac{N}{p}} f \in EXP_{\frac{p}{p-1+\sigma}}(\Omega)$

Proof. Let us start again from the inequality

$$\| I_{\frac{N}{p}} f \|_q \leq q^{\frac{p-\epsilon-1}{p-\epsilon}} \cdot q^{\frac{1}{q}} \cdot \omega_N^{\frac{p-\epsilon-1}{p-\epsilon}} \cdot | \Omega |^{\frac{1}{q}} \cdot \| f \|_{p-\epsilon} \quad \forall q \geq p, \qquad \forall 0 < \epsilon \leq 1.$$

We have

$$\begin{aligned} \epsilon^{\frac{\sigma}{p-\epsilon}} \| I_{\frac{N}{p}} f \|_q &\leq q^{\frac{p-\epsilon-1}{p-\epsilon}} \cdot q^{\frac{1}{q}} \cdot \omega_N^{\frac{p-\epsilon-1}{p-\epsilon}} \cdot | \Omega |^{\frac{1}{q}} \cdot \epsilon^{\frac{\sigma}{p-\epsilon}} \| f \|_{p-\epsilon} \\ &\leq q^{\frac{p-\epsilon-1}{p-\epsilon}} \cdot q^{\frac{1}{q}} \cdot \omega_N^{\frac{p-\epsilon-1}{p-\epsilon}} \cdot | \Omega |^{\frac{1}{q}} \cdot \| f \|_{L^{p)}_\sigma} \end{aligned}$$

and therefore

$$\sup_{0<\epsilon\leq 1} \epsilon^{\frac{p-1+\sigma}{p}} \left(\fint_\Omega \left| \frac{I_{\frac{N}{p}} f}{\| f \|_p} \right|^{\frac{1}{\epsilon}} dx \right)^\epsilon < c(n)$$

from which the assertion follows.

■

Corollary 3.6. *Let $1 < p < +\infty$. There exist constant $c_0 = c_0(N)$, $c_1 = c_1(N,p)$ such that for any $f \in L^p(\Omega)$ the following inequality holds:*

$$\fint_\Omega \exp\left(\left(\frac{|I_{\frac{N}{p}} f|}{c_0\|f\|_p}\right)^{\frac{p}{p-1}}\right) dx \le c_1$$

Applying to the Theorem 3.5 the same density argument as in [CS], if a function f is in the closure of $L^\infty(\Omega)$ of $L^{p)}_\sigma(\Omega)$ then the image of f by the linear continuous operator $I_{\frac{N}{p}}$ must be in the closure of $L^\infty(\Omega)$ of $EXP_{\frac{p}{p-1+\sigma}}(\Omega)$, therefore we have also the following

Corollary 3.7. *Let $1 < p < +\infty$, $\sigma > 0$. If $f \in \Sigma^p_\sigma(\Omega)$, then $I_{\frac{N}{p}} f \in exp_{\frac{p}{p-1+\sigma}}(\Omega)$*

We remark that, in the same way, as a corollary of Theorem 3.5, we get that if $f \in L^p(\Omega)$, then $I_{\frac{N}{p}} f \in exp_{\frac{p}{p-1}}(\Omega)$.

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Recibido: 14 de Abril de 1997
Revisado: 3 de Diciembre de 1997