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The topological complexity of sets of convex differentiable functions.

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Abstract

Let $\mathcal{C}(X)$ be the set of all convex and continuous functions on a separable infinite dimensional Banach space X , equipped with the topology of uniform convergence on bounded subsets of X . We show that the subset of all convex Fréchet-differentiable functions on X , and the subset of all (not necessarily equivalent) Fréchet-differentiable norms on X , reduce every coanalytic set, in particular they are not Borel-sets.

1 Introduction

Let X be an infinite dimensional Banach space. We denote by $\mathcal{N}(X)$ the set of all equivalent norms on X . This space is topologically metrizable complete when equipped with the uniform convergence on bounded subsets of X . Recently, the topological nature of some collections of norms has been investigated: In [1], the collection of all uniformly rotund norms in every direction on a separable space with a basis was shown to be coanalytic non Borel for the Effros-Borel structure, as well as the set of all weakly locally uniformly rotund norms (see [2], [3]).

More recently, it is showed in [12] that the set of all Gâteaux-smooth norms reduces any coanalytic subset of a Polish space M through a continuous function in $\mathcal{N}(X)$. This implies of course that this set is coanalytic complete when $\mathcal{N}(X)$ is equipped with the Effros-Borel structure (see [6], [16] for the definition of this notion).

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In a third article [4], the locally uniformly rotund norms are investigated. Recall that an equivalent norm $\|\cdot\|$ on a Banach space X is locally uniformly rotund (in short, L.U.R.) if whenever $x \in X$ and a sequence (x_n) in X satisfy :

$$\lim_{n \rightarrow +\infty} 2(\|x\|^2 + \|x_n\|^2) - \|x + x_n\|^2 = 0,$$

one has

$$\lim_{n \rightarrow +\infty} \|x - x_n\| = 0.$$

In [4] it was shown by combining the methods of [12] with a classical topological theorem due to Hurewicz [11] and its extensions ([13],[14]), that a similar result is obtained for the L.U.R. norms when X is separable. Moreover for the case when X^* is separable we have the following result for L.U.R. dual-norms (see [4]) :

Theorem 1. *Let X be an infinite dimensional Banach space such that X^* is separable. Let M be a Polish space, and A an analytic subset of M . Then there exists a continuous map $\Lambda : M \rightarrow \mathcal{N}(X)$ such that:*

(i) *If $t \in A$, then the norm $\Lambda(t) = \|\cdot\|_t$ is not everywhere Gâteaux-differentiable.*

(ii) *If $t \notin A$, then the dual norm $\Lambda^*(t) = \|\cdot\|_t^*$ is L.U.R.*

In particular, since a norm whose dual norm is L.U.R. is Fréchet-differentiable ([9], p.43), it follows that if X is a separable infinite dimensional Banach space such that X^* is separable, then the set of all equivalent Fréchet-differentiable norms on $X \setminus \{0\}$ and the set of all convex continuous and Fréchet-differentiable functions on X are not Borel subsets of $\mathcal{C}(X)$.

The aim of this work is to show a similar result for every separable infinite dimensional Banach space (theorem 2, corollary 7 and 8).

2 Notations and definitions

Let X be a Banach space. We denote by $\mathcal{C}(X)$ the set of all convex and continuous functions on X . This set is topologically metrizable and complete when equipped with the topology of uniform convergence on bounded subsets of X . By $S(X)$ (resp. $B(X)$) we denote the unit sphere (resp. the unit ball) of X .

A norm $\|\cdot\|$ on X is Gâteaux-differentiable at a non zero point x of X , if for every $h \in X$,

$$\lim_{t \rightarrow 0} \frac{\|x + th\| - \|x\|}{t} \text{ exists.}$$

An equivalent reformulation for $x \in S(X)$, is that there is a unique $f \in S(X^*)$ such that $f(x) = 1$ ([9], p. 5). In this case we say that f is the differential of the norm $\|\cdot\|$ at x . This norm will be said Fréchet-differentiable at x if the limit above exists uniformly for $h \in S(X)$.

A norm is said Gâteaux-differentiable (resp. Fréchet-differentiable), if it is Gâteaux-differentiable (resp. Fréchet-differentiable) at all non zero points of X .

3 Results

Theorem 2 is the main result of this work, its proof relies in part on the proof of Theorem 1 mentioned above.

Theorem 2. *Let X be a separable infinite dimensional Banach space. Let M be a Polish space and A an analytic subset of M . Then there is a continuous map $\Gamma : M \rightarrow \mathcal{C}(X)$ such that:*

(i) *If $t \in A$, then the function $\Gamma(t)$ is not everywhere Gâteaux differentiable.*

(ii) *If $t \notin A$, then the function $\Gamma(t)$ is Fréchet-differentiable.*

Proof of Theorem 2:

The proof is divided into two cases according to whether X^* has or has not the Schur property. Remember that a Banach space has the Schur property if every weakly convergent sequence is norm convergent.

1st case : *X^* has the Schur property.*

Then in particular X^* contains an isomorphic subspace to $\ell_1(\mathbb{N})$ ([10], p. 112). Moreover, since X is separable we deduce that $c_0(\mathbb{N})$ is isomorphic to a quotient space of X ([15], p.104). Hence, there exists a surjective operator T from X onto $c_0(\mathbb{N})$.

On the other hand, Theorem 1 applied to the Banach space $c_0(\mathbb{N})$ implies that there exists a continuous map $\Lambda : M \rightarrow \mathbb{N}(c_0(\mathbb{N}))$ such that:

(i) If $t \notin A$, the norm $\Lambda(t) = \|\cdot\|_t$ is Fréchet-differentiable on $c_0(\mathbb{N}) \setminus \{0\}$.

(ii) If $t \in A$, the norm $\Lambda(t) = \|\cdot\|_t$ is not Gâteaux-differentiable at some points of $c_0(\mathbb{N}) \setminus \{0\}$.

We define the map $\Gamma : M \rightarrow \mathcal{C}(X)$ by ,

$$\Gamma(t) = \|T(\cdot)\|_t^2.$$

Fact 1. *If $t \notin A$, the function $\Gamma(t)$ is Fréchet-differentiable on X .*

Indeed, it is the composition of Fréchet-differentiable functions since the function $\|\cdot\|_t^2$ is Fréchet-differentiable on $c_0(\mathbb{N})$.

Fact 2. *If $t \in A$, the function $\Gamma(t)$ is not Gâteaux-differentiable at some points of X .*

Let $t_0 \in A$ and let z_0 be a non zero point of $c_0(\mathbb{N})$ where the norm $\|\cdot\|_{t_0}$ is not Gâteaux-differentiable. T being surjective, we consider $x_0 \in X$ such that $T(x_0) = z_0$. Assume that the function $\Gamma(t_0) = \|T(\cdot)\|_{t_0}^2$ is Gâteaux-differentiable at x_0 . Then the function $\|T(\cdot)\|_{t_0}$ is also Gâteaux-differentiable at x_0 since $T(x_0) = z_0$ is non zero. It follows that for all $h \in X$,

$$\lim_{t \rightarrow 0} \frac{\|T(x_0 + th)\|_{t_0} - \|T(x_0)\|_{t_0}}{t} \text{ exists.}$$

Hence for all $k \in c_0(\mathbb{N})$

$$\lim_{t \rightarrow 0} \frac{\|z_0 + tk\|_{t_0} - \|z_0\|_{t_0}}{t} \text{ exists,}$$

since T is surjective.

In other words, the norm $\|\cdot\|_{t_0}$ is Gâteaux-differentiable at z_0 . It is a contradiction , thus the function $\Gamma(t_0)$ is not Gâteaux-differentiable at x_0 . ■

2nd case : X^* has not the Schur property.

In this case, we use the same idea as in the proof of the first case, namely, to find a Banach space Y whose dual is separable and an appropriate operator T from X to Y . Then we consider the functions

$\|T(\cdot)\|_t$ on X where the collection of the norms $\|\cdot\|_t$ on Y is appropriately given by Theorem 1. To have (ii), T has to be “nearly surjective” (with dense range) and if the norm $\|\cdot\|_t$ is not everywhere Gâteaux-differentiable, it is necessarily not Gâteaux-differentiable at some points of $T(X)$.

Fact 3. *There exists a separable reflexive Banach space Y and a non compact operator T from X to Y with dense range.*

Proof. Since X^* fails the Schur property, there exists a subset C in X which is w -compact but not norm-compact. We put

$$K = \overline{\text{conv}}(C \cup (-C))$$

which is also w -compact ([18], II.c.8) but not norm-compact. Let E_K be the vector space generated by K equipped with the norm j_K (the Minkowski functional of K). Then E_K is a Banach space since K is w -compact. Indeed, let $(x_n)_{n>0}$ be a j_K -Cauchy sequence in E_K . Then for every integer p , there exists an integer $N(p)$ such that $x_p \in (x_{N(p)} + p^{-1}K)$ for every integer $q \geq N(p)$. We put

$$K_p = x_{N(p)} + p^{-1}K.$$

It is clear that every finite intersection of K_p is non empty, and so by compactness, $\bigcap_{p>0} K_p \neq \emptyset$. It is easy to check that $\bigcap_{p \geq 0} K_p = \{x_\infty\}$ and that $j_K - \lim_n x_n = x_\infty$. Thus, (E_K, j_K) is a Banach space.

Consider now the canonical injection $i : E_K \rightarrow X^*$, which is w -compact and not norm-compact because $i(B(E_K)) = K$. By the interpolation theorem of Davis-Figiel-Johnson-Pelczynski (see [7] ou [18], II.c.5), there exists a reflexive space R and two operators $\alpha : E_K \rightarrow R$ and $\beta : R \rightarrow X^*$ such that $i = \beta\alpha$.

In particular, β is not norm-compact since the operator $i = \beta\alpha$ is not. Moreover β is w^* -continuous since R is reflexive. Hence, there exists an operator $\beta_0 : X \rightarrow R^*$ such that $\beta = \beta_0^*$ and then, like β , β_0 is not norm-compact (see [18], I.A.15).

We put $Y = \beta_0(X)$ and $T = \beta_0 : X \rightarrow Y$. It easily checked that Y and T work. This prove the fact 3. ■

Let Z be a fixed closed hyperplane of Y . We write $Y = \mathbb{R} \oplus Z$. Since Z is a separable reflexive Banach space, we may and do assume that Z is equipped with an equivalent L.U.R. norm $|\cdot|$ whose dual norm $|\cdot|^*$ is also L.U.R. (see [9], p. 55).

Fact 4. $X_0 = T^{-1}(Z)$ is a hyperplane of X .

Indeed, let Q be the quotient map from Y onto Y/Z . If we consider the operator $\tilde{T} = Q \circ T$ defined from X to Y/Z , then since $\dim(Y/Z) = 1$ and T has dense range, the operator \tilde{T} corresponds to a non zero element of X^* and therefore, $\ker \tilde{T} = T^{-1}(Z) = X_0$ is a hyperplane of X .

■

Now if we call T_0 the restriction of T on the hyperplane X_0 , it is clear that T_0 is also a non-compact operator.

From now on, we denote $(0, z) \in Y$ the element $z \in Z$

Fact 5. *There exists a weakly closed subset F in the unit sphere $S(Z)$ such that $F \subseteq S(Z) \cap T_0(X_0)$ and F is weakly homeomorphic to $\mathbb{N}^{\mathbb{N}}$.*

To prove that fact, we need the following theorem (see [17], [14], [13] p. 133) :

Theorem 3. *Let E be a metrizable compact set, A an analytic subset of E and B a subset of E having an empty intersection with A . If there exists no F_σ subset of E containing A and having empty intersection with B , then there exists a subset K of E included in $A \cup B$ and homeomorphic to $\mathbb{N}^{\mathbb{N}}$ such that $(K \cap B)$ is countable and dense in K .*

We put:

- . $E = (B(Z), w)$.
- . $A = S(Z) \cap T_0(X_0)$.
- . $B = B(Z) \setminus S(Z)$.

E is compact metrizable since Z is separable and reflexive. The set A is a Borel subset (hence analytic) of E . Indeed, the sphere $S(Z)$ is w -Borelian since it is a G_δ of $(B(Z), w)$. Moreover, $T_0(X_0)$ is a $|\cdot|$ -Borel subset since it is the injective and continuous image of $X_0 / \ker T_0$ (see [6]). Moreover, since Z is separable, any norm-Borel subset of Z is w -Borel. Thus $T_0(X_0)$ is w -Borel subset.

Assume now that there is a F_σ subset G of E such that $A \subseteq G$ and $B \cap G = \emptyset$, i.e. that $G = \bigcup_{n \geq 0} K_n$ where K_n are closed subsets in $(B(Z), w)$ and $T_0(X_0) \cap S(Z) \subseteq G \subseteq S(Z)$. In particular, for any integer n , K_n is w -compact in $S(Z)$. Moreover, since the norm $|\cdot|$ is L.U.R. on Z , the weak and norm topologies agree on the sphere $S(Z)$ of Z , and hence K_n is $|\cdot|$ -compact for any integer n .

We consider the map $\varphi : \mathbb{R} \times Z \rightarrow Z$

$$(\lambda, z) \mapsto \lambda z.$$

If for every integers n and p we put $K_{n,p} = \varphi([0, p] \times K_n)$, then the subsets $K_{n,p}$ are $|\cdot|$ -compact since φ is continuous and moreover we have:

$$T_0(X_0) \subseteq \bigcup_{n \geq 0, p \geq 0} K_{n,p}.$$

Thus, the subsets $F_{n,p} = T_0^{-1}(K_{n,p})$ of X_0 are closed and,

$$X_0 = \bigcup_{n \geq 0, p \geq 0} F_{n,p}.$$

According to Baire theorem, there exists two integers n_0 and p_0 such that the interior F_{n_0, p_0}° of F_{n_0, p_0} is non-empty. Let B_0 a non-empty open ball included in F_{n_0, p_0}° . Then $T(B_0) \subseteq K_{n_0, p_0}$, and hence $T_0(B_0)$ is $|\cdot|$ -relatively compact in Z , and finally by translation and homothety, we deduce that $T_0(B(X_0))$ is $|\cdot|$ -relatively compact. This contradicts the fact that T_0 is not a compact operator.

Thus, by the theorem 3 above, there exists a subset K of E satisfying:

- . $K \subseteq (T(X_0) \cap S(Z)) \cup B$.
- . K w -homeomorphic to $2^{\mathbb{N}}$.
- . $K \cap B$ is countable and w -dense in K .

It follows that $F = K \cap (T(X_0) \cap S(Z))$ is w -homeomorphic to $\mathbb{N}^{\mathbb{N}}$ and w -closed in $S(Z)$.

■

The next fact can be also seen as a particular case of the lemma 10 in [8].

Fact 6. Let $\sigma : (S(Z), w) \rightarrow (S(Z^*), w)$ be the map which associate to $x \in S(Z)$ the element $f = \sigma(x)$ of $S(Z^*)$ such that $f(x) = 1$. Then, since Z is reflexive and the norms $|\cdot|$ and $|\cdot|^*$ are L.U.R., the map σ is an homeomorphism.

Indeed, for $x \in S(Z)$, $\sigma(x)$ is the differential of the norm $|\cdot|$ at x . Since the norm $|\cdot|$ is Fréchet-differentiable, σ is a $(|\cdot|, |\cdot|^*)$ -continuous map (see [9], p. 7) and surjective because Z is reflexive. Similarly, for $f \in S(Z^*)$, $\sigma^{-1}(f)$ is the differential of the norm $|\cdot|^*$ at x . Since the norm $|\cdot|^*$ is Fréchet-differentiable, σ^{-1} is a map from $S(Z^*)$ onto $S(Z^{**}) = S(Z)$ which is moreover $(|\cdot|^*, |\cdot|)$ -continuous.

Hence σ is a $(|\cdot|, |\cdot|^*)$ -homeomorphism, and thus, a (w, w) -homeomorphism because the weak and the strong topologies agree on the sphere when the norm is L.U.R. .

■

In particular, the subset $\sigma(F) = F^*$ is ω -closed in the sphere $S(Z^*)$ and ω -homeomorphic to $\mathbb{N}^{\mathbb{N}}$.

By using the same arguments as in the proof of the Theorem 1 of Bossard-Godefroy-Kaufman ([4], theorem 3), but applied to our reflexive space Y and for our special subset F^* of Z , we get:

Lemma 4. Let M be a Polish space and let A be an analytic subset of M . Then there exists a continuous map $\Lambda : M \rightarrow \mathbb{N}(Y)$ such that:

(i) If $t \notin A$, then the norm $\Lambda(t) = \|\cdot\|_t$ is Fréchet-differentiable.

(ii) If $t \in A$, then the norm $\Lambda(t) = \|\cdot\|_t$ is not Gâteaux-differentiable.

More precisely, it is not Gâteaux-differentiable at some points of the subset $(\{0\} \times F)$ of $Y = \mathbb{R} \oplus Z$.

Proof of lemma 4: For the proof, we need the following lemma ([4], lemma 2).

Lemma 5. Let (S, d) be a metric space which contains a closed subset E homeomorphic to $\mathbb{N}^{\mathbb{N}}$. Let A be an analytic subset of a Polish space (M, d') . Then, there exists an uniformly continuous map φ from $(M \times S, d' + d)$ to $[0, 1]$ such that:

(i) If $t \notin A$, then $\varphi(t, y) < 1$ for all $y \in S$.

(ii) If $t \in A$, then $\varphi(t, y_0) = 1$ for some $y_0 \in E$.

We apply lemma 5 to $S = E = (F^*, d)$ where d is the restriction to F^* of a distance on $B(Z^*)$ which defines the w -topology $((B(Z^*), w)$ is

metric compact since Z is reflexive separable). There exists a uniformly continuous function $\varphi : (M \times F^*, d' + d) \rightarrow [0, 1]$ such that:

(i) If $t \notin A$, then $\varphi(t, f) < 1$ for all $f \in F^*$.

(ii) If $t \in A$, then $\varphi(t, f) = 1$ for some $f_0 \in F^*$.

We put $L = \overline{F^*}^w$ which is a compact of $(B(Z^*), w)$. Observe that since φ is $(d' + d)$ -uniformly continuous, it has an unique uniformly continuous extension Φ to the completion $(M \times L)$ of $(M \times F^*)$. Moreover, $t \in A$ if and only if there is $f_0 \in F^*$ such that $\Phi(t, f_0) = 1$.

We consider the following w -compact subsets of $Y^* = \mathbb{R} \oplus Z^*$:

$$R(t) = (\{0\} \times B(Z^*)) \cup \{\pm(1, \Phi(t, f).f) : f \in L\}$$

and

$$K(t) = \overline{\text{conv}}^w(R(t)).$$

Clearly, $K(t)$ is the unit ball of an equivalent dual norm on Y^* which we denote by $|\cdot|_t^*$. Finally, we define an equivalent dual norm $\|\cdot\|_t^*$ by ,

$$\|(s, f)\|_t^{*2} = |(s, f)|_t^{*2} + |f|^{*2},$$

whose predual norm is denoted by $\Lambda(t) = \|\cdot\|_t$. This give us the continuous map $\Lambda : M \rightarrow \mathbb{N}(Y)$, which satisfies (i) and (ii) of the lemma 4. Indeed ,

(i) Let $t \notin A$. Assume that the norm $\|\cdot\|_t$ is not Fréchet-differentiable. Then the dual norm $\|\cdot\|_t^*$ is not L.U.R. ([9], p.43), and hence, there exists $f_0 \in S(Z^*)$ and $s_0 > 0$ such that ([4], proof of theorem 3),

$$|(0, f_0)|_t^* = 1 = |(s_0, f_0)|_t^*.$$

Since $t \notin A$, we have $\varphi(t, f) < 1$ for all $f \in F^*$. Since F^* is w -closed in $S(Z^*)$, $L \cap S(Z^*) = F^*$. It follows that $|\Phi(t, f).f|_t^* < 1$ for all $f \in L$.

Let μ a probability measure on $R(t)$ such that (s_0, f_0) is the barycenter of μ . The function $h((s_0, g)) = |g|^*$ is convex and w^* -lower semi-continuous on Y^* , hence (see [5] proposition 26-19) :

$$1 = |f_0|^* = h((s_0, f_0)) \leq \int_{R(t)} |g|^* d\mu((s, g)).$$

Since $|\Phi(t, f).f|_t^* < 1$ for all $f \in L$, it follows that μ is supported by $(\{0\} \times B(Z^*))$ and hence $s_0 = 0$. This contradiction shows (i).

(ii) Let $t \in A$ and let then $f_0 \in F^*$ such that $\Phi(t, f_0) = 1$. We consider $z_0 = \sigma^{-1}(f_0) \in F$, so $f_0(z_0) = |z_0| = 1$. For any $f \in Z^*$ and $s \in \mathbb{R}$, we have :

$$\|(s, f)\|_t^* \geq \|(0, f)\|_t^* = \sqrt{2}|f|^*.$$

Hence for all $z \in Z$,

$$\|(0, z)\|_t = \frac{|z|}{\sqrt{2}}.$$

On the other hand, since $\Phi(t, f_0) = 1$, we have for all $s \in [-1, 1]$,

$$|(s, f_0)|_t^* = 1.$$

Hence for all $s \in [-1, 1]$,

$$\langle (s, f_0), (0, z_0) \rangle = 1 = \|(s, f_0)\|_t^* \cdot \|(0, z_0)\|_t.$$

And thus, the norm $\|\cdot\|_t$ is not Gâteaux-differentiable at a point $(0, z_0)$ of $(\{0\} \times F)$. This finishes the proof of lemma 4 .

■

We define the map $\Gamma : M \rightarrow \mathcal{C}(X)$ where for $t \in M$ and $x \in X$, $\Gamma(t)(x) = \|T(x)\|_t^2$. Let us check that Γ is the map we are looking for.

First, if $t \notin A$, according to lemma 4, the function $\|\cdot\|_t^2$ is Fréchet-differentiable on Y , and then $\Gamma(t)$ is also Fréchet-differentiable as composition of Fréchet-differentiable functions.

Fact 7. *If $t \in A$, the map $\Gamma(t)$ is not Gâteaux-differentiable at some points of X .*

Let $t_0 \in A$. The lemma 4 provides the existence of $z_0 \in F$ such that the norm $\|\cdot\|_{t_0}$ is not Gâteaux-differentiable at $(0, z_0)$. Since $F \subseteq S(Z) \cap T_0(X_0)$ (see Fact 5), we consider $x_0 \in X_0$ such that $T_0(x_0) = z_0$. Recall that T_0 is the restriction of the operator $T : X \rightarrow Y$ to the hyperplane X_0 and that an element z of the hyperplane Z of Y is written in Y as $(0, z)$. Then since $z_0 \in Z$, we have :

$$T(x_0) = (0, z_0) \in Y.$$

Let us check that $\Gamma(t_0)$ is not Gâteaux-differentiable at x_0 . Indeed, if not, since $\Gamma(t_0)(x_0) = \|T(x_0)\|_{t_0}^2 = \|(0, z_0)\|_{t_0}^2 = \frac{1}{2} \neq 0$, the semi-norm

$\|T(\cdot)\|_{t_0}$ will be also Gâteaux-differentiable at x_0 . Thus, the norm $\|\cdot\|_{t_0}$ will be Gâteaux-differentiable at $T(x_0) = (0, z_0)$ along the directions subset $T(X)$. It follows that the norm $\|\cdot\|_{t_0}$ is Gâteaux-differentiable at $(0, z_0)$ since $T(X)$ is dense in Y . With this contradiction we finish the proof of theorem 2 .

■

Now we turn to the norm case. Let $\mathcal{N}(X)$ be the set of all continuous norms on a Banach space X , equipped with the topology induced by $\mathcal{C}(X)$.

Corollary 6. *Let X be a separable infinite dimensional Banach space. Let M be a Polish space and A an analytic subset of M . Then, there exists a continuous map $\Psi : M \rightarrow \mathcal{N}(X)$ such that:*

- (i) *If $t \in A$, then the norm $\Psi(t)$ is not everywhere Gâteaux-differentiable on $X \setminus \{0\}$.*
- (ii) *If $t \notin A$, then the norm $\Psi(t)$ is Fréchet-differentiable on $X \setminus \{0\}$.*

Proof. We consider the semi-norms $(\Gamma(t))^{\frac{1}{2}} = \|T(\cdot)\|_t$, defined in the proof of theorem 2 satisfying :

- . If $t \in A, \Gamma(t) = \|T(\cdot)\|_t^2$ is not everywhere Gâteaux-differentiable.
- . If $t \notin A, \Gamma(t) = \|T(\cdot)\|_t^2$ is Fréchet-differentiable on X .

X being separable infinite dimensional, we have $\overline{S(X^*)}^{w^*} = B(X^*)$ and $(B(X^*), w^*)$ is metrizable. Hence, there exists a sequence $(x_n^*)_{n \geq 1}$ in $S(X^*)$ which is w^* -dense in the unit ball $B(X^*)$. Then we define the injective operator J from X in $c_0(\mathbb{N})$ by ,

$$J(x) = \left(\frac{x_n^*(x)}{n} \right)_{n \geq 1} .$$

Let $\|\cdot\|$ be a Fréchet-differentiable norm on $c_0(\mathbb{N})$. we define $\|\|\cdot\|\|$ a Fréchet-differentiable norm on X by ,

$$\|\|x\|\| = \|J(x)\| .$$

Let now the norm $\|\|\cdot\|\|_t$ defined on X by :

$$\|\|x\|\|_t^2 = \|\|x\|\|^2 + \|T(x)\|_t^2 .$$

It is easy to check that the map $\Psi : M \rightarrow \mathcal{N}(X)$, defined by $\Psi(t) = \|\cdot\|_t$, works. ■

Since any uncountable Polish space M (for example $\mathbb{N}^{\mathbb{N}}$) contains an analytic non-Borel subset A , theorem 2 and corollary 6 imply :

Corollary 7. *Let X be a separable infinite dimensional Banach space. Then, the set of continuous convex and Fréchet-differentiable functions on X and the set of all Fréchet-differentiable (and not necessarily equivalent) norms on $X \setminus \{0\}$, are non Borel subsets of $\mathcal{C}(X)$.*

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