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On prequojections and their duals.

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Abstract

The paper is devoted to the class of Fréchet spaces which are called prequojections. This class appeared in a natural way in the structure theory of Fréchet spaces. The structure of prequojections was studied by G. Metafune and V.B. Moscatelli, who also gave a survey of the subject. Answering a question of these authors we show that their result on duals of prequojections cannot be generalized from the separable case to the case of spaces of arbitrary cardinality. We also introduce a special class of prequojections, we call them canonical, and show that in the main result of G. Metafune and V.B. Moscatelli on the existence of a prequojection with a given dual we may require this prequojection to be a canonical one.

The main purpose of this paper is to investigate a class of Fréchet spaces which are called prequojections. This class appeared naturally in the development of the structure theory of Fréchet spaces.

Let us recall some definitions. A Fréchet space F is called a *quojection* if there exists a sequence $\{F_n\}_{n=1}^{\infty}$ of Banach spaces and a sequence of linear continuous surjective mappings $R_n : F_{n+1} \to F_n$ $(n \in \mathbb{N})$ such that F is isomorphic to the projective limit $\text{proj}_n(F_n, R_n)$. According to [6] quojections were first introduced by A. Grothendieck [5]. Since then, quojections have been intensively studied by many authors (see surveys [10], [6]). A Fréchet space whose strong bidual is a quojection is called a *prequojection*. This class of spaces was introduced by S. Bellenot and E. Dubinsky [2] in their study of Fréchet spaces with nuclear Köthe

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quotients. More precisely, S. Bellenot and E. Dubinsky [2] proved that a separable Fréchet space F (with fundamental system of seminorms $\{p_n\}$) has a nuclear Köthe quotient if and only if it satisfies the following condition: there exists k such that for every l there is an j with

$$\sup\{||x'||_l : \ x' \in F'_k \text{ and } ||x'||_j \leq 1\} = \infty, \qquad (*)$$

where $||\cdot||_n$ is the dual norm of the seminorm p_n on F_n and $F'_n := \{x' \in F' : \ ||x'||_n < \infty\}$. S. Önal and T. Terzioğlu [11] removed the assumption of separability in this result. D. Vogt [13] proved that F doesn't satisfy condition (*) if and only if F is a prequojection. S. Bellenot and E. Dubinsky [2] posed the problem which in ourdays terminology can be stated as follows: whether every prequojection is a quojection? This problem was solved in negative by E. Behrends, S. Dierolf and P. Harmand [1]. (The solution relies heavily on the results of S. Dierolf and V.B. Moscatelli [4].) Now, the following terminology is used. A prequojection is called *nontrivial* if it is not a quojection.

The structure of prequojections was investigated in [8] and [7]. D. Vogt [14] shows the relevance of quojections and prequojections in connection with the splitting of exact sequences of Fréchet spaces. In the present paper we continue investigations of [8], [7] of the structure of strong duals of prequojections with continuous norms. In addition we introduce a class of prequojections which we call canonical and study the structure of strong duals of prequojections of this class.

The restriction of existence of continuous norm is justified to some extent by the following result.

Theorem 1. [4] *Let F be a nontrivial prequojection. Then there exists a quotient of F which is a nontrivial prequojection with a continuous norm.*

Let us introduce some definitions and notation. Let X be a Banach space and M be a total subspace of X^*. The completion of X under the norm

$$||x||_M = \sup\{|f(x)| : f \in M, ||f|| \leq 1\}$$

will be denoted by X_M. If M and N are total subspaces of X^* and $M \subset N$ then the natural embedding of X_N into X_M will be denoted by $C_{N,M}$. The set of all limits of weak* convergent and bounded nets in M is called the *derived set* of M and is denoted by M^1. *Derived set of* *order* n ($n \in \mathbb{N}$) is defined inductively by the equality $M^n = (M^{n-1})^1$. By $B(X)$ and $S(X)$ we denote the closed unit ball and the unit sphere of a Banach space X.

We shall use the following characterization of nontrivial prequojections with continuous norm.

Theorem 2. [4] (a) *Let F be a nontrivial prequojection with a continuous norm. Then there exists a Banach space X and a sequence of proper total subspaces $G(n) \subset X^*$ ($n \in \mathbb{N}$) such that for every $n \in \mathbb{N}$ we have $(G(n))^1 \subset G(n+1)$, F is isomorphic to*

$$proj_n(X_{G(n)},\ C_{G(n+1),G(n)})$$

and $(F', \beta(F', F))$ is isomorphic to $ind_n G(n)$.

(b) Let X be a Banach space and $\{G(n)\}_{n=1}^{\infty}$ be a sequence of proper total subspaces of X^* such that $(G(n))^1 \subset G(n+1)$. Then

$$proj_n(X_{G(n)},\ C_{G(n+1),G(n)})$$

is a nontrivial prequojection with a continuous norm and $(F', \beta(F', F))$ is isomorphic to $ind_n G(n)$.

Let $\{H(n)\}_{n=1}^{\infty}$ be an increasing sequence of Banach spaces. We are going to show that the "moderate" growth of density characters of the sequence $\{H(n)\}_{n=1}^{\infty}$ is necessary for $\text{ind}_n H(n)$ to be the strong dual of a prequojection with a continuous norm.

Let us introduce cardinal numbers:

$$\alpha = \limsup_{n\to\infty} \text{dens}(H(n+1)/H(n)),$$

$$\beta = \sup_n \inf_U \text{dens}(H(n)/U),$$

where the infimum is taken over all reflexive subspaces $U \subset H(n)$.

Theorem 3. *If $\alpha > 2^{2^\beta}$ then $ind_n H(n)$ is not isomorphic to the strong dual of a prequojection with a continuous norm.*

Proof. Let us assume the contrary. Using part (a) of Theorem 2 we find a Banach space X and an increasing sequence of total subspaces

$G(n) \subset X^*$ ($n \in \mathbb{N}$) such that $\mathrm{ind}_n G(n)$ is isomorphic to $\mathrm{ind}_n H(n)$. Therefore there exists an injective linear continuous mapping of the space $\mathrm{ind}_n H(n)$ into X^* satisfying the following conditions:

1) Its restriction to every $H(n)(n \in \mathbb{N})$ is an isomorphism.

2) The images of $H(n)$ (we shall still denote them by $H(n)$) are eventually total.

Using the condition of Theorem 3 we find $n \in \mathbb{N}$ and a reflexive subspace $U \subset H(n)$ such that $H(n)$ is total over X and

$$2^{2^{\mathrm{dens}(H(n)/U)}} < \mathrm{dens}(H(n+1)/H(n)).$$

The space $H(n)$ can be represented in form $H(n) = \mathrm{cl}(\mathrm{lin}(U \cup V))$ where V is a subspace of $H(n)$ with $\mathrm{dens}V = \mathrm{dens}(H(n)/U)$. Since $H(n)$ is total over X then V is total over $U^\top \subset X$. Therefore

$$\mathrm{dens}U^\top \leq 2^{dens\ V} = 2^{dens\ (H(n)/U)}.$$

Hence

$$\mathrm{dens}(U^\top)^* \leq 2^{dens\ (U^\top)} \leq 2^{2^{dens\ (H(n)/U)}}.$$

Since U is reflexive then we have the canonical identity $(U^\top)^* = X^*/U$. In addition

$$\mathrm{dens}(H(n+1)/H(n)) \leq \mathrm{dens}(H(n+1)/U) \leq \mathrm{dens}(X^*/U).$$

Therefore we have

$$\mathrm{dens}(H(n+1)/H(n)) \leq 2^{2^{dens\ (H(n)/U)}}.$$

This contradiction completes the proof.

Remark 1. Theorem 3 gives the negative answer to the following question posed in Remark 5 of [7, p. 224]:

Let quojection $E = \mathrm{proj}_n(E_n, R_n)$ be such that the spaces $X_n = \ker R_n$ are nonquasireflexive and $\mathrm{dens}X_n = const$. Does there exist a prequojection F with a continuous norm such that $F' = E'$?

In fact, let E_n be the n-fold direct sum of the space $(Y \oplus Z)$, where Z is a nonquasireflexive Banach space with separable dual and Y is a reflexive space with $\mathrm{dens}Y > 2^c$ (where c is the cardinality of continuum).

For $R_n : E_{n+1} \to E_n$ we take the natural projections. It is clear that the sequence $\{E_n^*\}_{n=1}^{\infty}$ satisfies the conditions of Theorem 3. Hence there does not exist a prequojection F with a continuous norm such that $F' = E'$. On the other hand

$$\text{dens } \ker R_n = \text{dens}(Y \oplus Z) = const.$$

The strongest known result on the structure of the strong duals of prequojections with continuous norms looks as follows.

Theorem 4. [7] *Let quojection $E = \text{proj}_n(E_n, R_n)$ be such that the spaces $X_n = \ker R_n$ $(n \in \mathbb{N})$ are eventually separable. A nontrivial prequojection F with a continuous norm for which $F' = E'$ exists if and only if the spaces X_n $(n \in \mathbb{N})$ are nonquasireflexive for infinitely many n.*

In [8] when proving Theorem 4 in the particular case $E = (c_0)^{\mathbb{N}}$ V.B. Moscatelli proved the following result:

There exists a closed total subspace $M \subset (c_0)^* = l_1$ such that

1) $(\forall n \in \mathbb{N})(M^n \neq (c_0)^*)$.

2) For every $n \in \mathbb{N}$ the subspace $\text{cl}(M^n) \subset l_1$ is isomorphic to l_1 and is complemented.

By part (b) of Theorem 2 these conditions imply that

$$F = \text{proj}_n((c_0)_{M^n}, C_{M^{n+1},M^n}) \tag{1}$$

is a nontrivial prequojection with a continuous norm and that $(F', \beta(F', F))$ is isomorphic to $(l_1)^{(\mathbb{N})}$.

Prequojections of the form (1) we call canonical. Let us give the formal definition. Following [9] we call a total subspace $M \subset X^*$ *strongly nonnorming* if $M^n \neq X^*$ for every $n \in \mathbb{N}$. We call a prequojection F *canonical* if it is isomorphic to $\text{proj}_n(X_{M^n}, C_{M^{n+1},M^n})$ for some Banach space X and some total strongly nonnorming subspace $M \subset X^*$.

We prove that the class of canonical prequojections is a proper subclass of the class of prequojections with a continuous norm.

Theorem 5. *There exists a nontrivial prequojection with a continuous norm which is not canonical.*

Lemma 1. *Let X be a Banach space and K be a subspace of X^*. There exists a continuous linear operator $T : K^{**} \to X^*$ such that $T|_K = I$ and $T(K^{**}) = K^1$.*

Proof. Let us introduce the set $\Lambda = \{(A, \varepsilon) : A$ is a finite subset of $K^*, \varepsilon > 0\}$ and the order relation $\succ$ on it in the following way:

$$((A_1, \varepsilon_1) \succ (A_2, \varepsilon_2)) \Leftrightarrow ((A_1 \supset A_2) \wedge (\varepsilon_1 < \varepsilon_2)).$$

The set Λ with this order relation is a directed set. Let $\mathcal{U}$ be an ultrafilter on Λ dominating the filter generated by this order relation. It is easy to see that for every $z^{**} \in B(K^{**})$ we can find a collection $\{z_\lambda\}_{\lambda \in \Lambda} \subset B(K)$ such that

$$z^{**} = \sigma(K^{**}, K^*) - \lim_{\mathcal{U}} z_\lambda.$$

On the other hand there exists a limit $x^* = \sigma(X^*, X) - \lim_{\mathcal{U}} z_\lambda$. Let us introduce an operator $T : K^{**} \to X^*$ by the equality $Tz^{**} = x^*$. It is easy to verify that T is well-defined. Furthermore, we have $||T|| = 1, T(K^{**}) = K^1$ and $T|_K = I$. Lemma 1 is proved.

Lemma 2. [12] *Let Y be a separable nonquasireflexive Banach space. Then for every countable ordinal γ there exists a subspace N of Y^* and a bounded sequence $\{h_n\}_{n=1}^{\infty}$ in Y^{**} such that:*

A. If a weak convergent and bounded net $\{x_\lambda^*\}_{\lambda \in \Lambda}$ is contained in N^β for some $\beta < \gamma$ and $x^* = w^* - \lim_\lambda x_\lambda^*$ then*

$$(\forall n \in \mathbb{N})(h_n(x^*) = \lim_\lambda h_n(x_\lambda^*)). \tag{2}$$

B. There exists a collection $\{x_{n,m}^*\}_{n=1,m=1}^{\infty\ \ \infty}$ of vectors in N^γ such that for every $k, n \in \mathbb{N}$ we have

$$w^* - \lim_{m \to \infty} x_{n,m}^* = 0;$$

$$(\forall m \in \mathbb{N})(h_k(x_{n,m}^*) = \delta_{k,n}).$$

Remark 4. Part A of this lemma was proved in [12] for sequences only, but the same proof is valid for nets.

Proof of Theorem 5. Let G be arbitrary quojection with dens$G \leq c$. Let Y be some nonquasireflexive Banach space and $N \subset Y^*$ be a subspace satisfying the conditions of Lemma 2 when γ is the first infinite ordinal. Let us show that there exists a non-trivial prequojection F with a continuous norm such that $(F', \beta(F', F))$ is isomorphic to $(G', \beta(G', G)) \oplus \text{ind}_n(\text{cl}N^n)$.

Let $G = \text{proj}_n(G_n, Q_n)$. We may assume without loss of generality that Q_n ($n \in \mathbb{N}$) are quotient maps. Let us introduce the Banach space H as a subspace in the Banach space direct sum $(\sum_{n=1}^{\infty} \oplus G_n)_\infty$ defined in the following way:

$$H = \{\{g_n\}_{n=1}^{\infty} : g_n \in G_n, Q_n g_{n+1} = g_n\}.$$

It is clear that H^* contains an increasing sequence of weak* closed subspaces isometric to G_n^* and Q_n^* embeds G_n^* isometrically into G_{n+1}^*.

Let Γ be a set of cardinality continuum. Let $\phi : l_1(\Gamma) \to H \oplus Y$ be some quotient mapping. Let $\{s_n^*\}_{n=1}^{\infty}$ be some total sequence in $(l_1(\Gamma))^*$. (Such sequence exists by the following well-known fact: if card$\Gamma \leq c$ then $l_1(\Gamma)$ is isometric to a subspace of l_∞).

Let us choose numbers $\nu_n > 0$ ($n \in \mathbb{N}$) in such a way that

$$\min\{1/3, \text{dist}(S(\phi^* Y^*), \phi^* H^*)\} > (1/2) \sum_{n=1}^{\infty} |\nu_n| \cdot ||h_n|| \cdot ||s_n^*|| \qquad (3)$$

Since ϕ^* is an isometric embedding of $H^* \oplus Y^*$ into $(l_1(\Gamma))^*$ we may (and shall) identify spaces H^* and Y^* with their images under ϕ^*.

Let us introduce operator $R : Y^* \to l_\infty(\Gamma) = (l_1(\Gamma))^*$ by the equality

$$R(y^*) = y^* + \sum_{n=1}^{\infty} \nu_n h_n(y^*) s_n^*.$$

Let $M := R(N) \subset l_\infty(\Gamma)$. Let us prove by induction that for every $n \in \mathbb{N}$ we have

$$M^n = R(N^n). \qquad (4)$$

For $n = 0$ this equality is valid by definition. Let us suppose that it has been proved for $n = k$ and prove it for $n = k + 1$. Let $x^* \in M^{k+1}$, i.e. $x^* = w^* - \lim_\lambda x^*_\lambda$ for some bounded net $\{x^*_\lambda\}_{\lambda\in\Lambda} \subset M^k$. By the induction hypothesis there exist $\{y^*_\lambda\}_{\lambda\in\Lambda}$ such that

$$x^*_\lambda = y^*_\lambda + \sum_{n=1}^{\infty} \nu_n h_n(y^*_\lambda) s^*_n.$$

Inequality (3) implies that R is an isomorphic embedding and hence the net $\{y_\lambda\}_{\lambda\in\Lambda}$ is also bounded. Therefore it contains a weak* convergent subnet, say $\{y_\lambda\}_{\lambda\in\Theta}$. Let us denote by y^* its weak* limit. The subnet is of course bounded and by (2) we obtain

$$(\forall n \in \mathbb{N})(h_n(y^*) = \lim_{\lambda\in\Theta} h_n(y^*_\lambda)).$$

This inequality immediately implies that $x^* = R(y^*)$. Hence $M^{k+1} \subset R(N^{k+1})$.

Let us prove the inverse inclusion. Let $\{y_\lambda\}_{\lambda\in\Lambda}$ be a bounded weak* convergent net in N and y^* be its limit. Condition (2) implies that

$$R(y^*) = w^* - \lim_{\lambda\in\Lambda} R(y^*_\lambda).$$

Hence we have $R(N^{k+1}) \subset M^{k+1}$. Therefore equality (4) is proved. It immediately implies that

$$M^\omega = R(N^\omega),$$

where ω is the first infinite ordinal. Therefore we obtain $M^\omega \ni R(x^*_{n,m}) = x^*_{n,m} + \nu_n s^*_n$. Since $w^* - \lim_{m\to\infty} x^*_{n,m} = 0$ then $s^*_n \in M^{\omega+1}$. Therefore subspace $M \subset (l_1(\Gamma))^*$ is total.

Let us introduce an increasing sequence $\{V(n)\}_{n=1}^\infty$ of subspaces of $(l_1(\Gamma))^*$ by the equality:

$$V(n) = \mathrm{lin}(G^*_n \cup \mathrm{cl}(M^n)).$$

By (3) and (4) the subspace $V(n)$ is a direct sum of G^*_n and $\mathrm{cl}(M^n)$. Therefore

$$(V(n))^1 = \mathrm{lin}(G^*_n \cup M^{n+1}) \subset V(n+1).$$

Let us introduce prequojection F as

$$F = \mathrm{proj}_n(l_1(\Gamma)_{V(n)}, C_{V(n+1),V(n)}).$$

Part (b) of Theorem 2 implies that F is a nontrivial prequojection with a continuous norm and $(F', \beta(F', F)) = \text{ind}_n V(n)$. It is easy to see that this space is isomorphic to $\text{ind}_n G_n^* \oplus \text{ind}_n \text{cl}(M^n)$, and therefore is isomorphic to

$$(G', \beta(G', G)) \oplus \text{ind}_n \text{cl} N^n.$$

We finish the proof of Theorem 5 in the following way. Let $G = \prod_{n=1}^{\infty} U_n$, where U_n $(n \in \mathbb{N})$ are nonseparable reflexive Banach spaces and let Y be a nonquasireflexive Banach space with separable third conjugate space. By what we have proved above we can find a nontrivial prequojection with a continuous norm such that

$$(F', \beta(F', F)) = \oplus_{n=1}^{\infty} U_n^* \oplus \text{ind}_n(\text{cl} N^n),$$

where N is some total subspace of Y^*. Let us show that F is not canonical. Assume the converse. Then the space $(F', \beta(F', F))$ is isomorphic to $\text{ind}_n(\text{cl} K^n)$, where K is some total strongly non-norming subspace in the dual of some Banach space.

By properties of inductive limits it follows that for every $n \in \mathbb{N} \cup \{0\}$ there exists $k(n) \in \mathbb{N}$ such that

$$\text{cl} K^n \subset \oplus_{m=1}^{k(n)} U_m^* \oplus \text{cl}(N^{k(n)}).$$

By reflexivity of $\{U_n\}_{n=1}^{\infty}$ and separability of Y^{***} it follows that the spaces

$$((\text{cl}(K^n))^{**}/\text{cl}(K^n))$$

are separable. Applying Lemma 1 we obtain that the spaces $\text{cl} K^{n+1}/\text{cl} K^n$ $(n \in \mathbb{N})$ are separable. Since

$$K \subset \oplus_{m=1}^{k(0)} U_m^* \oplus \text{cl}(N^{k(0)})$$

and $U_{k(0)+1}^* \subset \text{cl}(K^m)$ for some $m \in \mathbb{N}$, it follows that $U_{k(0)+1}^*$ is separable, a contradiction. Theorem 5 is proved.

It is not known whether the prequojections constructed by G. Metafune and V.B. Moscatelli in their proof of Theorem 4 are canonical. We prove the following generalization of the result of V.B. Moscatelli [8] mentioned above.

Theorem 6. *Let quojection $E = proj_n(E_n R_n)$ be such that the spaces $U_n = ker R_n$ are separable and nonquasireflexive. Then there exists a canonical prequojection F such that $F' = E'$.*

In the proof of this theorem we use the following definition.

Definition 1. *Let $Z_0, Z_1, \dots, Z_n, \dots$ be an increasing sequence of Banach spaces. (Note that by increasing sequence of Banach spaces we mean not only the sequence of spaces but also the sequence of isometric embeddings). The sequence $\{Z_n\}_{n=0}^{\infty}$ is said to be* **representable by weak* derived sets** *in the dual Banach space X^* if there exists a total subspace $M \subset X^*$ such that the following diagramm commutes for some sequence $\{\psi_n\}_{n=0}^{\infty}$ of isomorphisms.*

$$\begin{array}{ccccccccccc} Z_0 & \longrightarrow & Z_1 & \longrightarrow & Z_2 & \longrightarrow & \dots & \longrightarrow & Z_n & \longrightarrow & \dots \\ \psi_0 \downarrow & & \psi_1 \downarrow & & \psi_2 \downarrow & & \downarrow & & \psi_n \downarrow & & \downarrow \\ M & \longrightarrow & cl(M^1) & \longrightarrow & cl(M^2) & \longrightarrow & \dots & \longrightarrow & cl(M^n) & \longrightarrow & \dots \end{array}$$

Part (b) of Theorem 2 implies that in order to prove Theorem 6 it is sufficient to find a Banach space X and an increasing sequence $\{Z_n\}_{n=0}^{\infty}$ which is representable by weak* derived sets in X^* and is such that $\mathrm{ind}_n Z_n$ is isomorphic to E'. In order to do this we need the following result.

Theorem 7. *Let X be a Banach space and let $\{W_i\}_{i=0}^{\infty}$ be an increasing sequence of subspaces of X^* such that the following conditions are satisfied:*

1) $W_0 \subset X_0^{\perp} \subset W_1 \subset X_1^{\perp} \subset \dots \subset W_i \subset X_i^{\perp} \subset \dots$.

2) $(W_i)^1 = X_i^{\perp}$ $(i = 0, 1, 2, \dots)$.

3) *The quotients $W_i/(X_{i-1}^{\perp})$ are separable.*

4) *The strong closure of the natural image of X_{i-1} in $(W_i/(X_{i-1}^{\perp}))^*$ is of infinite codimension.*

5) *The strong closure of the natural image of X in W_0^* is of infinite codimension.*

6) $\cap_{i=0}^{\infty} X_i = 0$.

Then the sequence $\{W_i\}_{i=0}^{\infty}$ is representable by weak derived sets in X^*.*

First we prove Theorem 6 using Theorem 7. We may assume without loss of generality that E_1 is nonquasireflexive and that R_n $(n \in \mathbb{N})$ are quotient mappings.

Let X be a subspace of $(\sum_{n=1}^{\infty} \oplus E_n)_\infty$ defined in the following way:

$$X = \{\{e_n\}_{n=1}^{\infty} : (\forall n \in \mathbb{N})\ (R_n e_{n+1} = e_n)\}.$$

Let X_k $(k = 0, 1, 2, \ldots)$ be subspaces of X defined in the following way:

$$X_k = \{\{e_n\}_{n=1}^{\infty} :\ e_1 = \ldots = e_{k+1} = 0\}.$$

It is clear that $\cap_{k=0}^{\infty} X_k = \{0\}$.

Let $\hat{e}$ be an element of X/X_k. Let $\{e_n\}_{n=1}^{\infty}$ be its representation. It is easy to verify that the correspondence $\hat{e} \leftrightarrow e_{k+1}$ is bijective and defines an isometry between X/X_k and E_{k+1}. Therefore the following diagramm is commutative:

$$\begin{array}{ccc} X/X_k & \longrightarrow & X/X_{k-1} \\ \downarrow & & \downarrow \\ E_{k+1} & \longrightarrow & E_k, \end{array}$$

where $X/X_k \to X/X_{k-1}$ is the natural quotient mapping. Therefore $X_0^\perp$ is isometric to $(E_1)^*$. By nonquasireflexivity of E_1 this implies that the closure of the canonical image of X in $(X_0^\perp)^*$ is of infinite codimension. Hence the spaces $W_0 := X_0^\perp$ satisfy condition 5 of Theorem 7.

We have identifications

$$X_{k+1}^\perp / X_k^\perp = ((X/X_{k+1})^* / (X/X_k)^*) = (X_k/X_{k+1})^* = (\ker R_{k+1})^* = U_{k+1}^*.$$

Since U_{k+1} is nonquasireflexive then the natural image of X_k in $(X_{k+1}^\perp / X_k^\perp)^* = U_{k+1}^{**}$ is a closed norming subspace of infinite codimension.

In order to continue the proof we need the following lemma. Recall that a subspace M of a dual Banach space X^* is called 1-*norming* if

$$(\forall x \in X)(\|x\| = \sup\{|f(x)| :\ f \in S(M)\}).$$

Lemma 3. *Let U be a separable nonquasireflexive Banach space. Then there exists a separable 1-norming subspace $M \subset X^*$ such that the natural image of U in M^* is of infinite codimension.*

Proof. Let $\{x_i\}_{i=1}^{\infty}$ be a sequence dense in $S(U)$ and let $\{x_i^*\}_{i=1}^{\infty} \subset S(U^*)$ be such that $x_i^*(x_i) = 1$.

Using [3, p. 360] we find a weak* null sequence $\{u_n^*\}_{n=1}^{\infty} \subset U^*$, a bounded sequence $\{u_k^{**}\}_{k=1}^{\infty} \subset U^{**}$ and a partition $\{I_k\}_{k=1}^{\infty}$ of the integers into pairwise disjoint infinite subsets such that

$$u_k^{**}(u_n^*) = \begin{cases} 1, & \text{for } n \in I_k; \\ 0, & \text{for } n \notin I_k. \end{cases}$$

Let us define M in the following way:

$$M = \text{cl lin}(\{u_n^*\}_{n=1}^{\infty} \cup \{x_n^*\}_{n=1}^{\infty}).$$

It is clear that M is separable and 1-norming. Furthermore, the restrictions $\{u_k^{**}|_M\}_{k=1}^{\infty}$ are linearly independent and since $\{u_n^*\}_{n=1}^{\infty}$ is weak* null then no linear combination of $\{u_k^{**}|_M\}_{k=1}^{\infty}$ is contained in the natural image of U. Lemma 3 is proved.

Remark 5. It is worth mentioning that since M is norming then the natural image of U is closed.

We apply Lemma 3 to $U = U_k$ $(k \in \mathbb{N})$ and denote the obtained subspace of $U_k^* = (X_k^{\perp}/X_{k-1}^{\perp})$ by M_k. Let N_k be a separable subspace of $X_k^{\perp}$ such that the strong closure of its image under the quotient mapping $X_k^{\perp} \to X_k^{\perp}/X_{k-1}^{\perp}$ coincides with M_k. Let $W_k = \text{cl lin}(N_k \cup X_{k-1}^{\perp})$ $(k \in \mathbb{N})$. It is easy to verify that conditions 1, 3 and 4 of Theorem 7 are satisfied.

Let us show that $(W_k)^1 = X_k^{\perp}$ $(k \in \mathbb{N})$. Since $w^* - \text{cl}B(W_k)$ is a weak* compact subset of $X_k^{\perp}$ then its image under the quotient mapping $\phi_k : X_k^{\perp} \to X_k^{\perp}/X_{k-1}^{\perp}$ is also weak* compact. Since the image is dense in the unit ball of M_k and M_k is 1-norming then

$$\phi_k(w^* - \text{cl}B(W_k)) = B(X_k^{\perp}/X_{k-1}^{\perp}).$$

Let us show that $w^*-\text{cl}B(W_k) \supset (1/3)B(X_k^{\perp})$.

In fact let $x \in (1/3)B(X_k^{\perp})$. Then

$$\phi_k(x) \in (1/3)B(X_k^{\perp}/X_{k-1}^{\perp}) = (1/3)\phi_k(w^* - \text{cl}B(W_k)).$$

Hence there exists $z \in (1/3)w^*-\text{cl}B(W_k)$ such that $x - z \in X_{k-1}^{\perp}$. It is clear that $\|x - z\| \leq 2/3$. Since $B(X_{k-1}^{\perp}) \subset B(W_k)$ then

$$x = (x - z) + z \in (2/3)B(X_{k-1}^{\perp}) + (1/3)w^* - \text{cl}B(W_k) \subset w^* - \text{cl}B(W_k).$$

Thus we proved that the condition 2 of Theorem 7 is also satisfied. By Theorem 7 and part (b) of Theorem 2 there exists a canonical prequojection F such that $(F', \beta(F', F))$ is isomorphic to $\text{ind}_n W_n$. This space is in turn isomorphic to $\text{ind}_n X_n^\perp \approx E'$. Theorem 6 is proved.

In order to prove Theorem 7 we need the following lemma.

Lemma 4. *Let X be a Banach space such that X^* contains a closed norming subspace M of infinite codimension. Then there exist a sequence $\{f_i^*\}_{i=1}^\infty \subset X^*$ and a countable collection of nets $\{x_\lambda^i\}_{\lambda\in\Lambda} \subset X$ $(i \in \mathbb{N})$ (over the same directed set) such that the following conditions are satisfied:*

1) $\sup_{\lambda,i} ||x_\lambda^i|| < \infty$;

2) $(\forall i, k \in \mathbb{N})(\lim_\lambda f_i^*(x_\lambda^k) = \delta_{i,k})$;

3) $(\forall i \in \mathbb{N})(\forall m \in M)(\lim_\lambda m(x_\lambda^i) = 0)$.

Proof. The quotient X^*/M is infinite dimensional. Therefore there exists a basic sequence $\{z_i\}_{i=1}^\infty \subset X^*/M$. We may and shall suppose that this sequence is bounded and is bounded away from zero. Let $\{g_i\}_{i=1}^\infty \subset (X^*/M)^* = M^\perp \subset X^{**}$ be some biorthogonal functionals of this sequence. It is clear that we may assume that the sequence $\{g_i\}_{i=1}^\infty$ is bounded. Let us denote by Q the quotient mapping $Q : X^* \to X^*/M$. Let vectors $\{f_k^*\}_{k=1}^\infty$ be such that $||f_k^*|| \leq 2||z_k||$ and $Qf_k^* = z_k$ $(k \in \mathbb{N})$.

Let us introduce a directed set Λ in the following way. Pairs (k, F), where $k \in \mathbb{N}$ and F is a finite dimensional subspace of M are its elements. We define an order on Λ in the following way:

$$((k_1, F_1) \prec (k_2, F_2)) \Leftrightarrow ((k_1 \leq k_2) \& (F_1 \subset F_2)).$$

By Helly's theorem we can for every triple (k, F, i) find a vector $x_{k,F}^i \in X$ which coincides with g_i on $\text{lin}(F \cup \{f_j^*\}_{j=1}^k)$ and $||x_{k,F}^i|| \leq 2||g_i||$.

It is easy to verify that these vectors satisfy all conditions of Lemma 4.

Proof of Theorem 7. Condition 4 allows us to apply Lemma 4 to $X = W_k/(X_{k-1}^\perp)$ and M defined as the natural image of X_{k-1} in

$(W_k/(X_{k-1}^{\perp}))^*$ $(k \in \mathbb{N})$. By Lemma 4 we find nets

$$\{x_\lambda^{i,k}\}_{\lambda\in\Lambda(k)} \subset W_k/(X_{k-1}^{\perp})\ (i,k \in \mathbb{N})$$

and sequences $\{f_{i,k}^*\}_{i=1}^{\infty} \subset (W_k/(X_{k-1}^{\perp}))^*$ in such a way that the following conditions are satisfied:

1) $(\forall x \in X_{k-1})(\lim_{\lambda\in\Lambda(k)} x_\lambda^{i,k}(x) = 0)$;

2) $(\forall i,j,k \in \mathbb{N})(\lim_{\lambda\in\Lambda(k)} f_{i,k}^*(x_\lambda^{j,k}) = \delta_{i,j})$.

Let $\{g_{i,k}\}_{i=1}^{\infty}$ be norm-preserving extensions of functionals $\{f_{i,k}^*\}_{i=1}^{\infty}$ onto the whole $(X^*/X_{k-1}^{\perp})$. Using canonical identifications we may consider $\{g_{i,k}\}_{i=1}^{\infty}$ as elements of $X_{k-1}^{\perp\perp} \subset X^{**}$.

Let $\psi_k : X^* \to X^*/X_{k-1}^{\perp}$ $(k \in \mathbb{N})$ be the quotient mappings. Since $W_k \supset X_{k-1}^{\perp}$ then all extensions of $x_\lambda^{i,k} \in X^*/X_{k-1}^{\perp}$ onto X are contained in W_k. This observation implies that for some directed sets $\Gamma(k)$ $(k \in \mathbb{N})$ there exist nets

$$\{v_{(\lambda,\gamma)}^{i,k}\}_{(\lambda,\gamma)\in\Lambda(k)\times\Gamma(k)} \subset W_k,$$

which are bounded and weak* convergent to zero and are such that

$$\psi_k v_{(\lambda,\gamma)}^{i,k} = x_\lambda^{i,k}.$$

Therefore for every i,j,k,λ and γ we have

$$f_{i,k}^*(x_\lambda^{j,k}) = g_{i,k}(v_{(\lambda,\gamma)}^{j,k}).$$

In other words for some system of nets $\{v_\theta^{i,k}\}_{\theta\in\Theta(k)} \subset W_k$ and some collection of functionals $\{g_{i,k}\}$ $(i,k \in \mathbb{N})$ we have

$$(\forall i,k \in \mathbb{N})(d_{i,k} = \sup_\theta ||v_\theta^{i,k}|| < \infty); \tag{5}$$

$$(\forall i,j,k \in \mathbb{N})(\lim_\theta g_{i,k}(v_\theta^{j,k}) = \delta_{i,j}); \tag{6}$$

$$g_{i,k} \in X_{k-1}^{\perp\perp}. \tag{7}$$

Similarly using condition 5 of Theorem 7 and Lemma 4, we find in W_0 a countable collection of weak* convergent to zero and bounded nets

$\{v_\theta^{j,0}\}_{\theta\in\Theta(0)}$ ($j \in \mathbb{N}$) such that for some sequence $\{f_{i,0}^*\}_{i\in\mathbb{N}} \subset W_0^*$ we have

$$\lim_{\theta\in\Theta(0)} f_{i,0}^*(v_\theta^{j,0}) = \delta_{i,j}.$$

Let us denote by $\{g_{i,0}\}_{i\in\mathbb{N}}$ some norm-preserving extensions of $\{f_{i,0}^*\}_{i\in\mathbb{N}}$ onto the whole X^*.

Since the quotients $W_k/X_{k-1}^\perp$ ($k \in \mathbb{N}$) are separable then we can for every $k \in \mathbb{N}$ find a sequence $\{w_i^k\}_{i=1}^\infty \subset W_k$ such that $W_k = \mathrm{cl}(\mathrm{lin}(X_{k-1}^\perp \cup \{w_i^k\}_{i=1}^\infty))$.

Let reals a_i^k ($i \in \mathbb{N}, k = 0,1,2,\dots$) be such that

$$\sum_{i=1}^\infty a_i^k ||g_{i,k}|| \cdot ||w_i^{k-1}|| \max\{d_{i,k}, 1\} < \infty.$$

Later on we shall impose two more conditions on numbers $\{a_i^k\}$.

Introduce operators $T_k : X^* \to X^*$ by the equality

$$T_k(x^*) = \sum_{i=1}^\infty a_i^k g_{i,k}(x^*) w_i^{k+1}.$$

For every pair (n,k) of integers such that $0 \le n \le k$ let

$$S_n^k = (I + T_k)(I + T_{k-1}) \cdot \ldots \cdot (I + T_n).$$

We shall suppose that the numbers $\{a_i^k\}$ are so small that the following conditions are satisfied:

1) For every $n \in \mathbb{N}$ the sequence $\{S_n^k\}_{k=n}^\infty$ converges in the uniform topology.

2) For every $0 \le n \le k$ we have

$$||S_n^k - I|| \le 1/2.$$

The limit of the sequence $\{S_n^k\}_{k=n}^\infty$ will be denoted by S_n ($n = 0,1,2,\dots$). It is clear that for every $n = 0,1,2,\dots$ we have $||S_n - I|| \le 1/2$ and therefore S_n is an isomorphism. It is clear that $T_n(X) \subset W_{n+1}$. Condition (7) implies that T_n vanishes on $X_{n-1}^\perp$ ($n \in \mathbb{N}$).

Let $M = S_0(W_0)$. Let us show by induction that for every $i = 0, 1, 2, \dots$ we have

$$\mathrm{cl}(M^i) = S_i(W_i). \tag{9}$$

For $i = 0$ this equality is satisfied by definition. Let us suppose that we have proved that $\mathrm{cl}(M^n) = S_n(W_n)$ and prove the equality $\mathrm{cl}(M^{n+1}) = S_{n+1}(W_{n+1})$.

Let $x^* \in M^{n+1}$. It means that for some bounded net $\{x^*_\lambda\}_{\lambda\in\Lambda} \subset M^n$ we have $x^* = w^* - \lim_\lambda x^*_\lambda$. Using the induction hypothesis we find $\{y^*_\lambda\}_{\lambda\in\Lambda} \subset W_n$ such that $x^*_\lambda = S_n y^*_\lambda$ $(\lambda \in \Lambda)$. Since S_n is an isomorphism then the net $\{y^*_\lambda\}_{\lambda\in\Lambda}$ is also bounded. Therefore it contains a weak* convergent subnet $\{y^*_\lambda\}_{\lambda\in\Theta}$. Since T_n is compact then we may assume that the net $\{T_n y^*_\lambda\}_{\lambda\in\Theta}$ is strongly convergent. Let us introduce vectors $y^* = w^* - \lim_{\lambda\in\Theta} y^*_\lambda$ and $z^* = \lim_{\lambda\in\Theta} T_n y^*_\lambda$.

We have

$$x^* = w^* - \lim_{\lambda\in\Theta} S_n y^*_\lambda = w^* - \lim_{\lambda\in\Theta}(S_{n+1}y^*_\lambda + S_{n+1}T_n y^*_\lambda).$$

Observe that $y^*_\lambda \in W_n \subset X_n^\perp$ and the restriction $S_{n+1}|_{X_n^\perp}$ is the identity operator. The latter assertion implies that $x^* = y^* + S_{n+1}z^*$. Since $y^*_\lambda \in W_n \subset X_n^\perp$ and $X_n^\perp$ is weak* closed then $y^* \in X_n^\perp$. By $T_n y^*_\lambda \in W_{n+1}$ it follows $z^* \in W_{n+1}$. Therefore $x^* \in \mathrm{lin}(X_n^\perp \cup S_{n+1}(W_{n+1})) = S_{n+1}(W_{n+1})$. Since $S_{n+1}(W_{n+1})$ is closed then it follows that $\mathrm{cl}(M^{n+1}) \subset S_{n+1}(W_{n+1})$.

Let us prove the converse inclusion. At first we prove that for every $i \in \mathbb{N}$ we have $S_{n+1}w_i^{n+1} \in M^{n+1}$. It is clear that $M^{n+1} = (\mathrm{cl}(M^n))^1$. Therefore it is sufficient to prove that

$$S_{n+1}w_i^{n+1} \in (S_n(W_n))^1. \tag{10}$$

Let $\{v_\theta^{i,n}\}_{\theta\in\Theta(n)} \subset W_n$ $(i, n \in \mathbb{N})$ be the nets introduced above. We have

$$S_n(v_\theta^{i,n}) = S_{n+1}(v_\theta^{i,n} + \sum_{j=1}^{\infty} a_j^n g_{j,n}(v_\theta^{i,n}) w_j^{n+1}) =$$

$$v_\theta^{i,n} + \sum_{j=1}^{\infty} a_j^n g_{j,n}(v_\theta^{i,n}) S_{n+1} w_j^{n+1}.$$

Taking weak* limits over $\theta \in \Theta$ and using conditions (5), (6), (8) and weak* convergence of $\{v_\theta^{i,n}\}_{\theta\in\Theta}$ to zero we obtain $a_i^n S_{n+1}(w_i^{n+1}) \in M^{n+1}$. Since $a_i^n > 0$ we obtain (10).

Our next step is the proof of the inclusion

$$X_n^{\perp} \subset \mathrm{cl}((S_n(W_n))^1). \tag{11}$$

Let $x^* \in X_n^{\perp}$. Since $(W_n)^1 = X_n^{\perp}$ there exists a bounded net $\{x_\lambda^*\}_{\lambda\in\Lambda} \subset W_n$ such that $x^* = w^* - \lim_\lambda x_\lambda^*$. We have $S_n x_\lambda^* = S_{n+1}(x_\lambda^* + T_n x_\lambda^*) = x_\lambda^* + S_{n+1}T_n x_\lambda^*$. By compactness of T_n (passing to a subnet if necessary) we may assume that the net $\{T_n x_\lambda^*\}_{\lambda\in\Lambda}$ is strongly convergent. The definition of T_n implies that its limit is contained in $\mathrm{cl}(\mathrm{lin}\{w_i^{n+1}\}_{i=1}^{\infty})$. Therefore the net $\{S_n x_\lambda^*\}_{\lambda\in\Lambda}$ weak* converges to the sum of x^* and some vector of $\mathrm{cl}(\mathrm{lin}\{S_{n+1}w_i^{n+1}\}_{i=1}^{\infty})$. It follows that

$$x^* \in (S_n(W_n))^1 + \mathrm{cl}\ \mathrm{lin}(\{S_{n+1}w_i^{n+1}\}_{i=1}^{\infty}).$$

By (10) the second summand is contained in the strong closure of the first. Therefore (11) is proved.

Using the definition of $\{w_i^{n+1}\}$ we obtain

$$S_{n+1}(W_{n+1}) = \mathrm{cl}\ \mathrm{lin}(S_{n+1}X_n^{\perp} \cup \{S_{n+1}w_i^{n+1}\}_{i=1}^{\infty}) =$$

$$\mathrm{cl}\ \mathrm{lin}(X_n^{\perp} \cup \{S_{n+1}w_i^{n+1}\}_{i=1}^{\infty}) \subset \mathrm{cl}((S_n(W_n))^1) = \mathrm{cl}(M^{n+1}).$$

Therefore (9) is proved. Let us write out the commutative diagramm which is required in Definition 1. For the sake of convenience we write it in three lines.

$$\begin{array}{ccccccccccc}
W_0 & \longrightarrow & W_1 & \longrightarrow & W_2 & \longrightarrow & \dots & \longrightarrow & W_i & \longrightarrow & \dots \\
I\downarrow & & S_0^0\downarrow & & S_0^1\downarrow & & \downarrow & & S_0^{i-1}\downarrow & & \downarrow \\
W_0 & \xrightarrow{I+T_0} & W_1 & \xrightarrow{I+T_1} & W_2 & \xrightarrow{I+T_2} & \dots & \xrightarrow{I+T_{i-1}} & W_i & \xrightarrow{I+T_i} & \dots \\
S_0\downarrow & & S_1\downarrow & & s_2\downarrow & & \downarrow & & S_i\downarrow & & \downarrow \\
M & \longrightarrow & \mathrm{cl}(M^1) & \longrightarrow & \mathrm{cl}(M^2) & \longrightarrow & \dots & \longrightarrow & \mathrm{cl}(M^i) & \longrightarrow & \dots
\end{array}$$

Some comments to this diagramm.

1) In the first and in the third lines arrows mean the natural embeddings.

2) We assume that S_0^{i-1} is an isomorphism of W_i since $T_0, \dots, T_{i-1}$ map W_i into itself and $\|I - S_0^{i-1}\| < 1/2$. By (9) operators S_k are isomorphisms of the spaces from the second and the third lines. It is clear that the diagram is commutative.

It remains to prove that M is total. But this assertion easily follows from the following facts:

$$\cap_{n=0}^{\infty} X_n = \{0\}, \ \{X_n\} \text{ is decreasing;}$$

$$\cup_{n=1}^{\infty} \mathrm{cl}(M^n) \supset \cup_{n=0}^{\infty} X_n^{\perp}.$$

Theorem 7 is proved.

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