REVISTA MATEMÁTICA de la
Universidad Complutense de Madrid
Volumen 10, número Suplementario: 1997
http://dx.doi.org/10.5209/rev_REMA.1997.v10.17387

Optimal degree construction of real algebraic plane nodal curves with prescribed topology, I: the orientable case.

Francisco SANTOS

Abstract

We study a constructive method to find an algebraic curve in the real projective plane with a (possibly singular) topological type given in advance. Our method works if the topological model T to be realized has only double singularities and gives an algebraic curve of degree $2N + 2K$, where N and K are the numbers of double points and connected components of T. This degree is optimal in the sense that for any choice of the numbers N and K there exist models which cannot be realized algebraically with lower degree. Moreover, we characterize precisely which models have this property.

The construction is based on a preliminar topological manipulation of the topological model followed by some perturbation technique to obtain the polynomial which defines the algebraic curve. This paper considers only the case in which T has an orientable neighborhood. The non-orientable case will appear in a separate paper.

1 Introduction

In a previous paper by the author [Santos1] it was shown that any real plane nodal curve with N singular (double) points and K connected

1991 Mathematics Subject Classification: 14P25, 14Q05.
Servicio Publicaciones Univ. Complutense. Madrid, 1997.

components in the projective plane is isotopic to a real algebraic plane curve of degree at most $4N+2K$. Also, the conjecture was raised that the degree bound could be lowered to $2N+2K$. In this paper we settle down the conjecture in the affirmative, for *orientable curves* (see the definition below). Moreover, we give a topological-combinatorial characterization of curves for which the degree bound is optimal. Our results also hold in the non-orientable case, but the proof is more intricate [Santos2] and will be detailed in a forthcoming paper [Santos3].

Let us fix some concepts and notation. Throughout this paper we will use the term *algebraic curve* as an abbreviation for *real projective algebraic plane curve.* By this we mean a non-zero real homogeneous polynomial $f \in \mathbb{R}[X,Y,Z]$ in three variables, considered up to a constant factor. Sometimes, by abuse of language, we will call algebraic curve the zero set $V(f) \subset \mathbb{RP}^2$ of the polynomial f. We will normally assume that we have an affine chart given for the projective plane. This allows us to speak of the *line at infinity* and to say, for example, that a certain conic is an ellipse.

An algebraic curve f is called *orientable* if its zero set $V(f)$ has an open neighborhood which is orientable; equivalently, if it can be moved by an isotopy to the affine chart of the projective plane. It is called *nodal* if all its singularities are order 2 singular points with two different tangents, real or complex. If the tangents are real we call the singular point a *node.* If they are complex, we will call it a *simple isolated point.*

Two algebraic curves (in general, two subsets V and W of $\mathbb{RP}^2$) are said to have the same *topological type* or to be *topologically equivalent* if there exists a global homeomorphism of the plane into itself sending V to W. Note that this condition is equivalent to V being isotopic to W, and stronger than V and W being homeomorphic. Our main result in this paper is the following, which is a re-writing of Theorem 4.3:

Theorem 1.1. *Let F be an orientable nodal algebraic curve with K connected components and N singular points (nodes or simple isolated points). Then, F is topologically equivalent to a certain nodal algebraic curve f_ϵ of degree $2N+2K$. Moreover, one can find such an f_ϵ as being a small perturbation of the form $f_\epsilon := f + \epsilon g$, where f is a product of $N+K$ ellipses (or degenerate conics) and g is the product of $2N+2K$ different lines.*

Translated to the affine plane $\mathbb{R}^2$ this gives:

Corollary 1.2. *Any compact nodal curve in $\mathbb{R}^2$ with K connected components and N singular points has the same topological type as a certain algebraic curve of degree $2N + 2K$.*

Of course, for most curves the degree bound in our theorem can be significantly lowered. The classical bounds by Bezout and Harnack indicate that for every N and K there are algebraic curves in the conditions of the theorem with degree essentially $\sqrt{2N + 2K}$. Let us remark that a construction producing the optimal degree for any topological type would provide a (constructive) answer to Hilbert's XVI problem for nodal curves, while the answer for the simpler case of non-singular curves is only known up to degree 7 (see [Gudkov, Viro, Wilson] for general information on Hilbert's XVI problem). Thus, there is no hope of obtaining such an optimal construction.

However, our bound is *generically* optimal in the following sense: for every N and every K, there are orientable nodal algebraic curves with N double points and K connected components which have not the topological type of any algebraic curve of degree lower than $2N + 2K$. This is shown in Section 5. Moreover, in that same section we give a topological characterization of algebraic curves for which the degree in our main theorem cannot be lowered (see Theorem 5.2) for the precise characterization).

The structure of the paper is as follows. In Section 2 we introduce the notion of a *topological model* for an algebraic curve and the basic concepts and results needed in our topological construction. Section 3 shows the main construction of a complicated topological model from simple pieces, in which our construction of algebraic curves is based. From this, Brussotti's theorem would inmediately give the first part of Theorem 1.1. To get the second part, an explicit perturbation technique is exhibited in Section 4. Finally, Section 5 studies the optimality of the degree obtained.

2 Topological preliminaries

We want to find an algebraic curve whose zero set has the same topological type of a certain curve given in advance. Equivalently, we can say that we are given a certain subset $T \subseteq \mathbf{RP}^2$ in the projective plane and want to find an algebraic curve f such that $V(f)$ has the same topological type as T. The conditions that such a T must satisfy for this to be possible are contained in the following definition (cf. for example [Boch-Cos-Roy]).

Definition 2.1. *Let T be a subset of $\mathbf{RP}^2$. We say that T is a topological model for an algebraic curve if it is homeomorphic to a graph with an even (possibly zero) number of edges incident to each vertex. We say that an algebraic curve f realizes a topological model T if its zero set $V(f) \subset \mathbf{RP}^2$ has the same topological type as T. By a nodal (topological) model we mean a topological model such that all of the vertices of the underlying graph G_T have 0, 2 or 4 edges. We say that a topological model is orientable if it can be isotopically moved to a position where it does not intersect the line at infinity (equivalently, if it has an orientable open neighborhood).*

Observe that the *underlying graph* G_T of a certain topological model T is not uniquely defined. In particular, an oval is homeomorphic to a cycle graph with as many edges and vertices as one wants. The points where a topological model T is locally homeomorphic to a line will be called *regular* and the rest *singular*. The singular points of a nodal topological model are the vertices with 0 and 4 edges of the underlying graph G_T and will be called *isolated points* and *double* points of T, respectively. A double point P will be said to be *disconnecting* if $T \setminus P$ has one connected component more than T.

Our basic topological operation on a topological model T will be the *desingularization* of some of its double points. Let P be a double point of T. The desingualarization of T at P consists of considering a suitable small open neighborhood U of P and substituting $T \cap U$ for two disjoint open curves in such a way that we get a new model with one double point less. This operation was called a 'flip' in [GCorb-Recio] and [GCorb-Santos]. There are exactly two ways, up to topological equivalence, of desingularizing a double point. These are shown in Figure 1. If the double point was disconnecting, one of the two desingularizations leaves the number of connected components unchanged and the other one increases it by one.

Whenever we perform a desingularization of a curve, we will mark the place where it has been done with a *bonding line* which joins the two branches which we have inserted. In all our figures bonding lines will appear as greyish dotted lines. The reason for including bonding lines is that topological models are considered modulo topological equivalence. Thus, we are allowed to transform them by global homeomorphisms. The transformed bonding lines will tell us what topological change is needed to recover the original topological type from the desingularized one.

Figure 1: Desingularization of a double point P.

We call *faces* of T the connected components of $\mathbb{RP}^2 \setminus T$. Clearly T has a unique non-orientable face F_0. We will call *depth* of an arbitrary face F of T the minimal number of crossings with T needed to go from F_0 to F (a crossing at a double point of T counts twice). The parity of the intersection number with T of a path joining F to F_0 does not depend on the path (this is not true in general for non-orientable models). Thus, adjacent faces have depths which differ by 1. Figure 2 shows the depth diagram of a certain topological model.

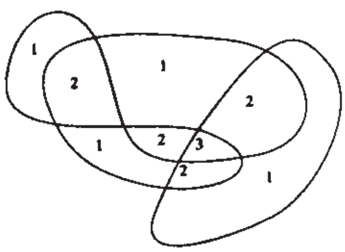

Figure 2: Depth of faces.

Let P be a double point of T. We have two possibilities for the distribution of depths in the four faces around P (see Figure 3):

- we will say that P is of Type I if the depths of faces around P are r, $r+1$, r and $r+1$, for some $r \geq 0$

Figure 3: Possible depth distributions around a double point.

- we will say that P is of Type II if the depths of faces around P are $r-1$, r, $r+1$ and r, for some $r \geq 0$.

We will say that a desingularization of T at some of its double points is *depth-consistent* if the two faces which are joined by the desingularization of each double point have the same depth.

Depth-consistency can equivalently be stated saying that each face of the desingularized model T' has the same depth as all the faces of T from which it has been obtained. Note that the different faces of the original model T which form a face of T' are still 'separated' by the bonding lines. If the desingularization is depth-consistent, then there is no ambiguity in considering the bonding lines or not for computing the depth of a face of T'. Both desingularizations of a double point of type I are depth consistent, but only one for double points of type II is.

3 Main Construction

The basis of our construction will be, given a nodal topological model T, trying to obtain a desingularization of T which consists of ellipses and with bonding lines being straight line segments. However, in order to optimize the degree we will only perform a partial desingularization of T; that is, some double points of T will not be desingularized. We first consider only connected models.

Proposition 3.1. *Let T be a connected nodal orientable topological model in $\mathbb{RP}^2$. Then, there is a connected nodal orientable topological model T' obtained as a desingularization of T in some of its double points, and with the following properties (see Figure 4):*

(i) The desingularization is depth-consistent.

(ii) Every double point of T' disconnects T' and is of type II.

(iii) Let b be a bonding line of T'. Let r be the depth of the face in which b is. Then, at least one of the two faces adjacent to the extremal points of b has depth $r-1$.

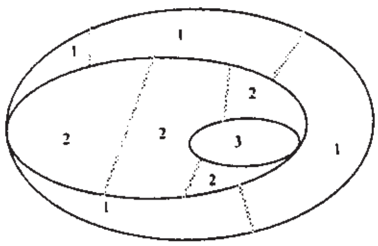

Figure 4: Desingularization of the model in Figure which satisfies the conditions in Proposition 3.1.

Proof. We first desingularize all the double points of type I in the way that joins the two faces of maximal depth. This desingularizations cannot disconnect T; otherwise, for going from one of the two faces of minimal depth to the non-orientable face it would be necessary to cross the (now unique) face of maximal depth, which is impossible.

We then proceed to desingularize non-disconnecting double points of type II one by one, until all the remaining double points are disconnecting. Condition (*iii*) is clearly satisfied for all the bonding lines obtained.

■

Proposition 3.2. *Let T be a connected nodal orientable topological model in $\mathbf{RP}^2$ which is not an isolated point. Let T' be a partial desingularization T' of T satisfying the conditions of Proposition 3.1. Then,* *the boundary of the non-orientable face of T' is an oval C. Let a certain homeomorphism h from C into an ellipse E of the proyective plane be given. Then, T' can be transformed by a global homeomorphism of $\mathbb{RP}^2$ into a topological model T'' in the following conditions:*

- *T'' is the union of a certain number of ellipses.*
- *if two of the ellipses intersect at a point P, then they do it tangentially and one is inside the other one.*
- *the bonding lines are straight line segments.*
- *the global homeomorphism which sends T' to T'' agrees with h when restricted to C.*

Proof. The fact that the boundary of the non-orientable face is an oval is guaranteed by T' being connected and having only Type II disconnecting double points. Also, there are no bonding lines in the non-orientable face because of condition (iii) in Proposition 3.1.

The rest of the proof will use induction on the maximal depth of faces in T'. If the maximal depth is 1 then T' consists of a unique oval C with some bonding lines in its interior. Thus, T' is topologically equivalent to the ellipse E with the bonding lines being straight line segments in its interior. Clearly, the homeomorphism from C to E can be prescribed in advance.

If the maximal depth of a face in T' is $r > 1$, we still have very particular properties for T': for a certain double point P of T', the depth-consistent desingularization of T' at P is precisely the one that disconnects T'. Moreover, one of the connected components which results is inside the other one, because P is of type II. Let us call the inner one the *ear* at the double point P.

Then, T' consists of an outer oval with some of these 'ears' attached to it in its inner side. Each ear itself is a topological model in the conditions of Proposition 3.1, but with maximal depth strictly less than r. Moreover, different ears are not connected to one another by bonding lines, because of condition (iii) in Proposition 3.1. However, an ear may have bonding lines connecting it to the outer oval, or there might be bonding lines connecting the outer oval to itself, through its inner face. Let us do the following:

First of all, take the outer oval as the ellipse E and realize inner bonding lines of the outer oval as the line segments joining the points on it prescribed by the homeomorphism h. Then insert a tangent ellipse at each point where an ear has to be attached (this points are again prescribed by h), small enough for not intersecting other ears or bonding lines. Then, draw the bonding lines joining the inner ellipses to the (again prescribed) points in the outer ellipse. This can be done in a unique way modulo topological equivalence. Finally, prescribe in each inner ellipse a homeomomorphism to the corresponding ear of T' in a way which agrees with the extreme points of bonding lines already drawn, and apply recursion to insert the rest.

The fact that the resulting topological model T'' and bonding lines is topologically equivalent to T' follows from the fact that each step in the 'drawing' process of T'' is unique, modulo topological equivalence.

$\blacksquare$

Theorem 3.3. *Let T be a nodal orientable topological model with N double points, M isolated points and K connected components. Then, there is a nodal orientable topological model $\mathcal{T}$ from which T can be obtained by desingularization of some double points, in the following conditions:*

- *$\mathcal{T}$ is a union of $N+K$ ellipses and M isolated points.*

- *any intersection point P between ellipses of $\mathcal{T}$ is a tangent intersection of only two ellipses. There are at most $2N$ such tangency points.*

- *The singular (i.e., double or isolated) points of $\mathcal{T}$ are in general position (no three of them on the same line).*

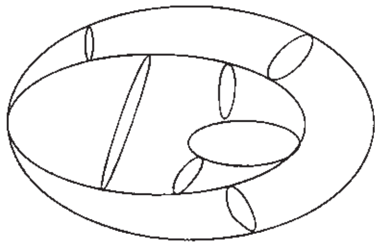

Figure 5: The topological model $\mathcal{T}$ of Theorem 3.3.

Proof. Let us first assume that T is connected and not an isolated point. Consider the topological model T' obtained from T in Proposition 3.1, embedded in the form described in Proposition 3.2. T' has one ellipse more than it has double points. In other words, T has $N_1 + 1$ ellipses and N_2 bonding lines, with $N_1 + N_2 = N$. Substitute each bonding line of T' by a sufficiently narrow ellipse joining the two ends of the bonding line and tangent to the ellipses at the ends. Let $\mathcal{T}$ be the topological model so obtained (see Figure 5).

The topological model T can be recovered (modulo topological equivalence) by desingularizing one of the two tangency points of these new ellipses. This is exhibited in Figure 6 which shows the full sequence of topological manipulations performed at a double point of T.

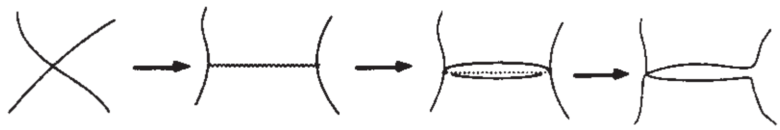

Figure 6: Topological changes at a double point.

The general position of the tangency points can be easily obtained thanks the freedom we have in Proposition 3.2 for choosing the extremal points of bonding lines and the tangency points of ellipses (by arbitrary homeomorphisms into ellipses). This finishes the connected case.

If T is not connected, let $T_1, \ldots, T_K$ be the connected components of T. Starting with the outermost ones, apply the previous construction to the connected components which have double points and realize the others by ellipses or isolated points. Place a copy of the resulting models T_i in the appropriate part of $\mathbf{RP}^2$ (reducing them as needed) in order to get T in the required conditions. Observe that in the reduction process it is essential to assume that the conics in the models are ellipses, i.e., that the models are embedded in the affine chart of the projective plane.

■

Remark 3.4. Suppose that the original nodal topological model T has a non-disconnecting double point. We claim that, in these conditions, the numbers of ellipses and double points of T in Theorem 3.3 can be decreased by one.

Indeed, if T has a non-disconnecting double point, then the desingularized model T' of Proposition 3.1 has at least one bonding line connecting two nested ellipses. In this case, the insertion of the inner ellipse (the 'ear') in the proof of Proposition 3.2 can save one bonding line with the following trick: insert the ear as an ellipse (as narrow as needed) joining the contact point of the ear to the extremal point of the bonding line in the outer ellipse (this produces two ellipses tangent to one another in two different points). Then add the other bonding lines, if any. The resulting model is not in the conditions of Proposition 3.1, but it still serves for the construction in Theorem 3.3.

4 Perturbation of algebraic curves

In order to obtain our main theorem we only need to consider the topological model T obtained in Theorem 3.3 as being an algebraic curve of degree $2(N + M) + 2K$ and algebraically perturb it in order to desingularize *some* singular points. One way to do this could be enlarging some of the ellipses in small amounts so that every tangency point becomes two transversal crossings (nodes). Then we could use the classical

Brussotti's Theorem (see for example page 12 of [Gudkov]). This result says that a singular curve having only nodal points can be perturbed to a curve of the same degree where some of the singular points are desingularized in an arbitrary prescribed way.

Nevertheless, we will show an explicit way to perturb the curve $\mathcal{T}$ of Theorem 3.3 in the desired way, in order to obtain the second part of Theorem 1.1. and to make our result more constructive. Let us first of all formalize the concept of a *perturbation* of an algebraic curve. Perturbation techniques are quite standard in the study of the topology of real algebraic curves (see [Gudkov, Viro]).

Let f be an algebraic curve with finitely many singularities. Let (f_ε), $\varepsilon \in [0, \infty)$ be a family of algebraic curves defined by polynomials f_ε of the same degree as $f = f_0$ and whose coefficients vary continuously with ε. Then, for ε sufficiently close to zero, the zero-sets $V(f_\varepsilon)$ are contained in an arbtrarily small neighborhood of $V(f)$ and their topology coincides with the topology of $V(f)$ except, maybe, at small neighborhoods of the singular points of f. Moreover, the possible changes of topology at the singular points can be predicted, if the singularities of f are sufficiently simple. Our perturbations will be explicitly given in the form $f_\varepsilon = f + \varepsilon g$, where g is a polynomial of the same degree as f and with a finite number of common zeroes with f.

The change in the topology of a curve in a neighborhood of a singular point by a small perturbation is called a *dissipation*. In our perturbations, the singular points of f will be simple isolated points or tangencies of two real non-singular branches. This two types of singularities are classified as A_1^+ and A_t^- (with $t \geq 3$ and odd) in [Viro, p. 1098 ff.] (see also [Arn-Var-GusZ]) and are diffeomorphic to the ones in $X^2 + Y^2$ and $Y^2 - X^t$, respectively. The dissipations of an A_1^+ point P of f are easy to describe: if the perturbing curve g has a singular point at P then no change in the topology appears; if g has a zero non-singular point at P then an oval passing through P appears; if g is non-zero at P then the isolated point either dissapears or becomes an oval, depending on the sign of g at P. The dissipations of an A_t^- singularity admit several other possibilities; we will be only interested in the following cases:

Lemma 4.1. *Let $f \in \mathbf{R}[X, Y, Z]$ be a homogeneous polynomial of a certain degree d and let P be an A_t^- singular point of f (with t odd). Let*

$f_\varepsilon = f + \varepsilon g$ be a perturbation of f by a certain homogeneous polynomial $g \in \mathbf{R}[X, Y, Z]$ of the same degree d with finitely many intersections with f. Then,

(i) if g is not zero at P, then the dissipation of P produced is a topological desingularization (as the ones in Section 3). Which of the two desingularizations occurs depends only on the sign of g at P.

(ii) if g has a nodal singular point at P (i.e., a double singular point with two real branches of different tangents) then the perturbed curve has a nodal or simple isolated point at P. By changing the sign of g if necessary a nodal point can be obtained, with no change in the topology of the curve.

Proof. Let us first prove a general fact: in a dissipation of a singular point P of f obtained in the form $f + \epsilon g$ with f and g having finitely many intersections, at most one new oval can appear but never in our cases (i) and (ii). In fact, observe that any new ovals (as well as the dissipated real branches of f) must collapse to P as ε goes to zero and that for different values $\varepsilon_1 \neq \varepsilon_2$ of the parameter the curves $f + \varepsilon_1 g$ and $f + \varepsilon_2 g$ do not intersect in $U \setminus P$, for a certain neighborhood U of P. From this, the perturbed curve cannot have ovals with P outside; if the perturbed curve has an oval with P inside, then the oval is unique and f cannot have any real branches at P; if the perturbed curve has an oval passing through P, then g has a non-singular zero at P and no other ovals passing through P or with P inside can appear.

Part (i) follows inmediately from the above, because in a neighborhood of P the perturbed curve will not have singular points or new ovals. For part (ii), we take P as the origin of an affine chart and study the local developements of f and g at P. The lower degree part of f is the square of a linear function l (whose zero set is the tangent line of f at P). The lower degree part of g is the product of two different linear functions $l_1 l_2$. If l coincides with one of l_1 and l_2 (say with l_1), then the perturbed curve has $l(l + \varepsilon l_2)$ as lower degree part, i.e., a nodal singularity. Otherwise, the lower degree part $l^2 + \varepsilon l_1 l_2$ decomposes in two (perhaps complex conjugate) linear factors for ε small; by changing the sign of $l_1 l_2$ if necessary the factors can be assumed to be real.

Thus, in any case a nodal singularity can be obtained (perhaps changing the sign of g). Since no new ovals can appear, there is no change in the topology.

■

Theorem 4.2. *Let f be an algebraic curve of degree d. Suppose that all the singularities of f are a certain number of A_1^+ points $P_1, \ldots, P_k$ and of A_t^- (with t odd) points $Q_1, \ldots, Q_l$. Suppose that the singular points of f are in general position, i.e. no three of them on the same line.*

Then there is a product g of d different lines such that the perturbation $f_\varepsilon = f + \varepsilon g$ with $\varepsilon > 0$ preserves all the P_i, converts a number l_1 of the Q_i in nodes (with no change in the topology), and desingularizes the other $l_2 = l - l_1$ Q_i in a prescribed way, under the assumption that $l_1 + k + l/2 \leq d$.

Proof. According to the previous lemma and what we said for A_1^+ singularities, the following conditions on g are sufficient to guarantee the desired perturbation:

- For the k points (of type A_1^+) to be preserved, that g has a singular point at each of them.
- For the l_2 points (of type A_t^-) to be desingularized, that g does not vanish at them, and has the appropriate sign.
- For the l_1 points (of type A_t^-) to be converted in nodes, that g has a nodal point at each of them, and the appropiate distribution of signs in a neighborhood of them.

Let $r_1, \ldots, r_{k+l_1}$ be different straight lines, each passing through two of the singular points to be preserved (or converted in nodes) and such that each of these points lies in two of them. Then, the product g_1 of those $k + l_1$ straight lines has a nodal singular point at each of them, because of the general position assumption on the points.

Let $s_1, \ldots, s_{d-k-l_1}$ be different lines not passing through the points P_i and so that each of the Q_i lies in exactly one of them. These lines exist, because of the condition $l/2 \leq d - k - l_1$. Then, the lines can be slightly moved (as shown in Figure 7) in such a way that the product g_2 of them has a prescribed sign at each point Q_i.

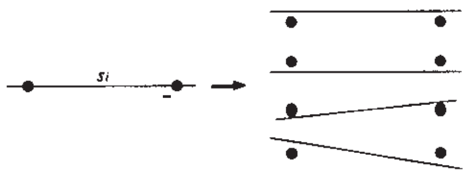

Figure 7: Obtention of the adequate sign at a point by moving s_i.

So, just make the signs of g_2 at the points Q_i be the ones that we need in order to obtain $g = g_1 g_2$ with the appropriate signs, and take ε sufficiently small and positive.

∎

This, together with Theorem 3.3, gives our main theorem:

Theorem 4.3. *Let T be an orientable nodal topological model with N singular (double or isolated) points and K connected components. Then, T can be algebraically realized by a curve $f_\varepsilon := f + \varepsilon g$ of degree $2N + 2K$, with f being a product of $N + K$ ellipses or degenerate conics and g being a product of $2N + 2K$ lines.*

Proof. Let f be the product of the ellipses obtained in the model $\mathcal{T}$ of Theorem 3.3, and a factor of the form $(cX - aZ)^2 + (bX - aY)^2 + (cY - bZ)^2$ for each isolated point $(a, b, c) \in \mathbf{RP}^2$ of $\mathcal{T}$. Each connected component T_i of T contributes $N_i + 1$ ellipses or degenerate conics to

f, where N_i is the number of singular points in T_i. Thus, f is as in the statement.

Let k and l be the number of isolated and double points of $\mathcal{T}$, and $l = l_1 + l_2$ with l_1 and l_2 being the numbers of double points to be preserved and desingularized, respectively. Then $l_1 + k = N$ and $l = l_1 + l_2 \leq 2N$. Thus, $l_1 + k + l/2 \leq 2N < d$ and Theorem 4.2 applies.

■

5 Optimality of the construction

The purpose of this section is to show in what cases the degree in our construction is optimal. This will give us the somehow surprising result that the only obstructions to lowering the degree in the construction are those which are obvious, as the one in the following example. Consider the model drawn in Figure 8 with three double points, or its obvious generalization to an arbitrary number N of double points. Insert $K - 1$ additional ellipses inside the innermost face, one inside another. The resulting model cannot be realized by any algebraic curve of degree lower than $2N + 2K$ because in any realization of the model any straight line passing through the innermost face intersects the curve (at least) $2N + 2K$ times, counted with multiplicity.

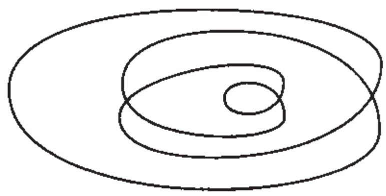

Figure 8: A simple model, not realizable with degree lower than eight.

As a first result, in remark 3.4 we mentioned that if the topological model T has a non-disconnecting double point, then the degree of the construction can be lowered, at least, by two. Thus, we only need to consider the case of topological models with only disconnecting double points. This condition is necessary but not sufficient: for example, the seven topological models in Figure 9 can easily be constructed with degree 4 (as we will see in the proof of Lemma 5.1).

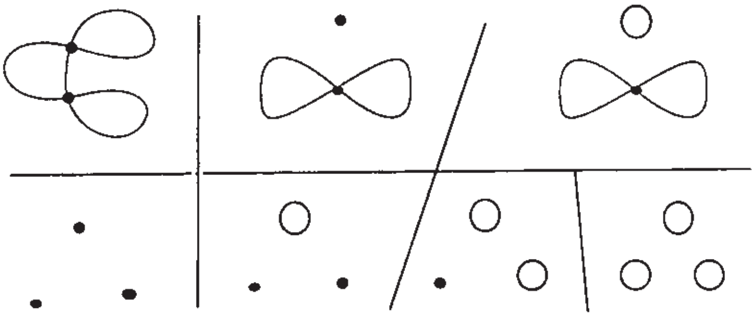

Figure 9: Some models which can be realized with degree 4.

Let T be a nodal orientable topological model, all of whose double points disconnect it. Let N be the number of double points of T and K the number of connected components. If we desingularize every double point of T in the way that disconnects T we get a non-singular topological model T_0 with $N + K$ connected components. The topological structure of T_0 can be represented in a rooted tree, with a node for each connected component of T_0 and an extra node (the root of the tree) 'at infinity'. A component C_1 is a son of a second component C_2 in the tree if and only if C_1 is inmediately inside C_2. The sons of the root node are the outermost components.

The interesting point is that a sufficient condition for the model not being realizable with degree lower than $2N + 2K$ is that the tree of connected components of T_0 has at most two *leaves* (innermost connected components): if this is the case, then for any algebraic realization of T_0 any line intersecting the two innermost components will cut every connected component of T_0 at least twice (counted with multiplicities). If T was realizable with degree lower than $2N + 2K$, then T_0 would also be, by means of a small perturbation. Thus, T itself cannot be realized with degree lower than $2N+2K$. We will see that this sufficient condition turns out to be also necessary.

For this observe that the tree structure of T_0 suggests a different construction procedure for an algebraic realization of the topological

[Santos3] F. Santos, *Optimal degree construction of real algebraic plane nodal curves with prescribed topology II: the non-orientable case*, in preparation.

[Viro] O. Ya. Viro, Real algebraic plane curves: constructions with controlled topology, *Algebra i Analiz* **1**(5) (1989), 1–73; English transl., *Leningrad Math. J.* **1**(5) (1990), 1059–1134.

[Wilson] G. Wilson, Hilbert's sixteenth problem, *Topology* **17** (1978), 53–74.

Dpto. de Matemáticas Estadística y Computación
Universidad de Cantabria
E-39071 Santander, Spain
santos@matesco.unican.es