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Topology of real algebraic T-surfaces.

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Abstract

The paper is devoted to algebraic surfaces which can be obtained using a simple combinatorial procedure called the T-construction. The class of T-surfaces is sufficiently rich: for example, we construct T-surfaces of an arbitrary degree in $\mathbf{R}P^3$ which are M-surfaces. We also present a construction of T-surfaces in $\mathbf{R}P^3$ with $dim\ H_1(\mathbf{R}X;\mathbf{Z}/2) > h^{1,1}(\mathbf{C}X)$, where $\mathbf{R}X$ and $\mathbf{C}X$ are the real and the complex point sets of the surface.

1 Introduction

The subject of the paper is T-surfaces, i. e. real algebraic surfaces which can be constructed in a simple combinatorial fashion : one can patchwork them from the pieces which essentially are planes.

The construction of combinatorial patchworking (or T-construction) works in any dimension. We restrict ourself here by the case of surfaces. The general T-construction can be formulated in a completely similar way (the combinatorial patchwork construction in the case of curves is described in [I-V], [I1], [I2]). The T-construction is a particular case of the Viro theorem (see [V2], [V3], [V5], [V6], [Ri]).

The results on topology of T-surfaces presented in the paper are concentrated around the following conjecture proposed by O. Viro ([V4]): let X be a nonsingular simply connected compact complex surface with

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an antiholomorphic involution $c : X \to X$; then $dim\ H_1(\mathbf{R}X;\mathbf{Z}/2) \le h^{1,1}(X)$, where $\mathbf{R}X$ is the fixed point set of the involution c (for a detailed information on real algebraic surfaces see [Kh], [Si], [Wi]).

This conjecture is related to the Ragsdale conjecture (see [Ra]) concerning the topology of real algebraic curves. To formulate the Ragsdale conjecture, let us denote the number of even ovals of a nonsingular real algebraic plane projective curve of degree $2k$ by p (an oval of a nonsingular curve of an even degree is called *even* (resp. *odd*), if it lies inside of even (resp. odd) number of other ovals of this curve), and denote the number of odd ovals by n.

Ragsdale conjecture. For a nonsingular real algebraic plane projective curve of degree $2k$

$$p \le \frac{3k^2 - 3k + 2}{2}, \quad n \le \frac{3k^2 - 3k}{2}.$$

Any counter-example to the inequality $p \le \frac{3k^2-3k+2}{2}$ produces a counter-example to Viro's conjecture: one can take a double plane ramified along the complex point set of a counter-example to the Ragsdale conjecture with an appropriate choice of a lifting of the involution of complex conjugation. Thus, the counter-examples to Ragsdale conjecture obtained in [I1] (see, also, [I2], [I-V]) show that Viro's conjecture is not true. The counter-examples to Ragsdale conjecture are constructed as T-curves. So, it is natural to try to use the combinatorial patchwork construction in order to construct counter-examples to Viro's conjecture which are real algebraic surfaces in $\mathbf{R}P^3$.

We show in sections 3 and 4 that under some conditions of "maximality" of the triangulation participating in the combinatorial patchwork construction, Viro's conjecture is true for the resulting T-surfaces. However, using a "nonmaximal" triangulation (see exact definitions in section 2), we can obtain a T-surface X in $\mathbf{R}P^3$ with $dim\ H_1(\mathbf{R}X;\mathbf{Z}/2) > h^{1,1}(\mathbf{C}X)$ (see section 6).

We also construct T-surfaces of any degree in $\mathbf{R}P^3$ which are M-surfaces (it means that the total $\mathbf{Z}/2$-homology group of the real point set has the same rank as that of the complexification; see section 5).

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2 T-construction

Let m be a positive integer number (it would be the degree of the surface under construction) and T be the tetrahedron in $\mathbf{R}^3$ with vertices $(0,0,0)$, $(0,0,m)$, $(0,m,0)$, $(m,0,0)$. Let us take a triangulation τ of T with vertices having integer coordinates. Suppose that a distribution of signs at the vertices of τ is given. The sign (plus or minus) at the vertex with coordinates (i,j,l) is denoted by $\delta_{i,j,l}$.

Take the copies

$$T_x = s_x(T), \quad T_y = s_y(T), \quad T_z = s_z(T)$$
$$T_{xy} = s_x \circ s_y(T), \quad T_{xz} = s_x \circ s_z(T), \quad T_{yz} = s_y \circ s_z(T) \quad, T_{xyz} = s_x \circ s_y \circ s_z(T)$$

of T, where s_x, s_y, s_z are reflections with respect to the coordinate planes. Denote by T_* the octahedron

$$T \cup T_x \cup T_y \cup T_z \cup T_{xy} \cup T_{xz} \cup T_{yz} \cup T_{xyz}.$$

Extend the triangulation τ to a symmetric triangulation of T_*, and the distribution of signs $\delta_{i,j,l}$ to a distribution at the vertices of the extended triangulation by the following rule: passing from a vertex to its mirror image with respect to a coordinate plane we preserve its sign if the distance from the vertex to the plane is even, and change the sign if the distance is odd.

If a tetrahedron of the triangulation of T_* has vertices of different signs, select a piece of the plane (triangle or quadrangle) being the convex hull of the middle points of the edges having endpoints of opposite signs. Denote by S the union of the selected pieces. It is a piecewise-linear surface contained in T_*. Glue by $s_x \circ s_y \circ s_z$ the facets of T_*. The resulting space $\tilde{T}$ is homeomorphic to the real projective space $\mathbf{R}P^3$. Denote by $\tilde{S}$ the image of S in $\tilde{T}$.

Let us introduce an additional assumption: the triangulation τ of T is *convex*. This means that there exists a convex piecewise-linear function $\nu : T \longrightarrow \mathbf{R}$ whose domains of linearity coincide with the tetrahedra of τ. Sometimes, such triangulations are also called coherent (see [GKZ]) or regular (see [Zi]).

Theorem 2.1 (O. Viro). *Under the assumptions made above on the triangulation τ of T, there exist a nonsingular real algebraic surface X*

of degree m in $\mathbf{R}P^3$ and a homeomorphism $\mathbf{R}P^3 \to \tilde{T}$ mapping the set of real points $\mathbf{R}X$ of X onto $\tilde{S}$.

Moreover, a polynomial defining the surface X can be written down explicitly: if t is positive and sufficiently small, the polynomial

$$\sum_{(i,j,l)\in V} \delta_{i,j,l} x_0^i x_1^j x_2^l x_3^{m-i-j-l} t^{\nu(i,j,l)}$$

(where V is the set of vertices of τ) defines a surface with the properties described in Theorem 2.1.

We consider two special types of triangulations of T. A triangulation τ of T is called *primitive* if all the tetrahedra of τ are of volume 1/6. A T-surface constructed using a primitive triangulation is called *primitive.*

A triangulation τ' of T is called *maximal* if all the integer points of T are vertices of τ'. Clearly, any primitive triangulation is maximal. The notions of primitive and maximal triangulations coincide in dimension 2. The situation is different in dimension 3 : there exist maximal triangulations of T which are not primitive.

3 Euler characteristic of T-surface

Let us consider a k-dimensional simplex Q having vertices with integer coordinates and belonging to the orthant $\{x_i \geq 0\}$ of $\mathbf{R}^n$. We call the simplex Q *elementary* if the reductions modulo 2 of the vertices of Q are independent (generate an affine space of dimension k over $\mathbf{Z}/2$).

Suppose that a distribution of signs at the vertices of the simplex Q is given. Let us take the distributions of signs at the vertices of the symmetric copies of Q using the following generalization of the rule formulated in section 2 :

> the symmetric copy of a vertex a in an orthant b gets the sign $(-1)^{\vec{a}\cdot\vec{b}} sign(a)$, where $\vec{a}$ is the reduction modulo 2 of the vertex a ; the i-th coordinate of the vector $\vec{b}$ in $(\mathbf{Z}/2)^n$ is equal to 0 (resp. to 1) if $x_i > 0$ (resp. $x_i < 0$) for a point $(x_1, \ldots, x_n)$ in the interior of the orthant b; and $\vec{a}\cdot\vec{b}$ denotes the standard scalar product of two vectors in $(\mathbf{Z}/2)^n$.

We call a symmetric copy of Q *nonempty* if it has vertices of different signs.

Proposition 3.1. *If the simplex Q is elementary and does not belong to a coordinate hyperplane, then Q has exactly $2^n - 2^{n-k}$ nonempty symmetric copies.*

Proof. Let us, first, remark that the map $\vec{a} \mapsto \vec{a} \cdot \vec{b}$ is linear over $\mathbf{Z}/2$. The following operations do not change the property of any symmetric copy of Q to be nonempty:

(1) parallel translation of Q,

(2) changing of signs at all the vertices of Q.

Thus, we can suppose that the reduction $\vec{v}_0$ modulo 2 of a vertex v_0 of Q is 0 in $(\mathbf{Z}/2)^n$, and that the vertex v_0 has the sign "+". Denote the other vertices of Q and their reductions modulo 2 by $v_1, \dots, v_k$ and $\vec{v}_1, \dots, \vec{v}_k$, respectively. The condition that the copy of Q in an orthant b is empty (i. e. is not nonempty) can be expressed by a system of linear equations

$$\vec{v}_1 \cdot \vec{b} = \varepsilon_1, \ \dots \ , \vec{v}_k \cdot \vec{b} = \varepsilon_k,$$

where $\varepsilon_i = 0$ if the sign of the vertex v_i is positive, and $\varepsilon_i = 1$ if the sign of v_i is negative. The unknowns of the system are the coordinates of $\vec{b}$. A solution to the system does exist because the rank of the system is equal to k (the simplex Q is elementary). Moreover, the dimension of the space of solutions is equal to $n - k$. It means that the number of solutions is equal to 2^{n-k}, in other words, the simplex Q has exactly $2^n - 2^{n-k}$ nonempty copies.

∎

Proposition 3.1 is similar to Lemma 1 in [I-R].

Now we are able to calculate the Euler characteristic of a primitive T-surface.

Theorem 3.2. *If X is a primitive T-surface in $\mathbf{R}P^3$, then the Euler characteristic $\chi(\mathbf{R}X)$ of the real point set of X is equal to the signature $\sigma(\mathbf{C}X)$ of the complex point set of X. In other words, if X is a primitive T-surface of degree m in $\mathbf{R}P^3$, then*

$$\chi(\mathbf{R}X) = -\frac{m^3}{3} + \frac{4m}{3}.$$

Proof. Let us take an arbitrary primitive triangulation τ of the tetrahedron T and an arbitrary distribution of signs at the integer points of T. The piecewise-linear surface $\tilde{S}$ has a natural cell subdivision: each cell is the intersection of $\tilde{S}$ with a simplex of the triangulation of $\tilde{T}$.

All the simplices of τ are elementary. The number of simplices of τ of any dimension is fixed (the number of simplices of any dimension contained in each face of T is also fixed). Thus, we can calculate the Euler characteristic of $\tilde{S}$ according to Proposition 3.1.

The triangulation τ contains

m^3 tetrahedra,
$2m^3 + 2m^2$ triangles, and $4m^2$ of them are contained in the facets of T,
$7m^3/6 + 3m^2 + 11m/6$ edges, $6m^2$ of them are contained in the facets of T, and $6m$ of them are contained in the edges of T,
$(m + 1)(m + 2)(m + 3)/6$ vertices.

We obtain that the described cell subdivision of $\tilde{S}$ contains $7m^3$ two-dimensional cells, $12m^3$ edges and $14m^3/3 + 4m/3$ vertices. Thus,

$$\chi(\mathbf{R}X) = -\frac{m^3}{3} + \frac{4m}{3} = \sigma(\mathbf{C}X).$$

■

Theorem 3.3. *If X is a T-surface constructed using a maximal triangulation of the tetrahedron T, then $\chi(\mathbf{R}X) \geq \sigma(\mathbf{C}X)$.*

Proof. Let us, first, remark that all simplices of dimension ≤ 2 of a maximal triangulation τ' of T are elementary. Denote by q the number of tetrahedra of τ'. If any tetrahedron of τ' is elementary than, repeating the calculation of the proof of Theorem 3.2, we obtain $\chi(\tilde{S}) = 2m^3/3 - q + 4m/3$.

Each nonelementary tetrahedron of τ' has at least 6 nonempty copies, because the rank of the corresponding system of linear equations (see the proof of Proposition 3.1) is equal to 2. Thus,

$$\chi(\mathbf{R}X) = \chi(\tilde{S}) \geq \frac{2m^3}{3} - q + \frac{4m}{3} - q',$$

where q' is the number of nonelementary tetrahedra of τ'. It remains to remark that $q + q' \leq m^3$, and we obtain

$$\chi(\mathbf{R}X) \geq -\frac{m^3}{3} + \frac{4m}{3} = \sigma(\mathbf{C}X).$$

■

4 Case of primitive or maximal triangulation

As we saw in section 3, the Euler characteristic of a primitive T-surface in $\mathbf{R}P^3$ is determined by the degree and is equal to the signature $\sigma(\mathbf{C}X)$ of the complex point set of the surface.

For a real algebraic surface X (or, more generally, for a real algebraic variety of any dimension), we have Smith inequality (see, for example, [Wi]) :

$$b_*(\mathbf{R}X) \leq b_*(\mathbf{C}X)$$

between the ranks of total $\mathbf{Z}/2$-homology groups of the real and of the complex point sets of X. If $b_*(\mathbf{R}X) = b_*(\mathbf{C}X)$, the surface X is called *an M-surface*. We denote by $b_i(Y)$ the rank of i-th homology group of Y with $\mathbf{Z}/2$-coefficients.

Let us mention two congruences (see [Wi]).

Rokhlin congruence. *If X is an M-surface, then*

$$\chi(\mathbf{R}X) \equiv \sigma(\mathbf{C}X) \; mod \; 16.$$

Kharlamov-Gudkov-Krahnov congruence. *If X is an (M-1)-surface (in other words, if $b_*(\mathbf{R}X) = b_*(\mathbf{C}X) - 2$), then*

$$\chi(\mathbf{R}X) \equiv \sigma(\mathbf{C}X) \pm 2 \; mod \; 16.$$

Rokhlin congruence and Theorem 3.2 show that we can expect to construct primitive T-surfaces which are M-surfaces. We will see in section 5 that such surfaces do really exist in any degree. On the other hand, there are no (M-1)-surfaces among primitive T-surfaces in $\mathbf{R}P^3$ according to Kharlamov-Gudkov-Krahnov congruence and Theorem 3.2.

Theorem 4.1. *If X is a primitive T-surface in $\mathbf{R}P^3$ then*

$$b_1(\mathbf{R}X) \le h^{1,1}(\mathbf{C}X), \quad b_0(\mathbf{R}X) \le h^{2,0}(\mathbf{C}X) + 1.$$

Remarks. Theorem 4.1 states that Viro's conjecture holds in the case of primitive T-surfaces.

The inequality $b_0(\mathbf{R}X) \le h^{2,0}(\mathbf{C}X) + 1$ for primitive T-surfaces was proved by E. Shustin in [Sh].

Proof of Theorem 4.1. Using the Smith inequality

$$b_*(\mathbf{R}X) \le b_*(\mathbf{C}X) = m^3 - 4m^2 + 6m$$

(where m is the degree of X) and the equality

$$\chi(\mathbf{R}X) = \sigma(\mathbf{C}X) = -\frac{m^3}{3} + \frac{4m}{3}$$

proved in Theorem 3.2, we immediately obtain

$$b_1(\mathbf{R}X) \le h^{1,1}(\mathbf{C}X) = \frac{2m^3}{3} - 2m^2 + \frac{7m}{3}$$

and

$$b_0(\mathbf{R}X) \le h^{2,0}(\mathbf{C}X) + 1 = \frac{m^3}{6} - m^2 + \frac{11m}{6}.$$

■

Viro's conjecture also holds in the case of T-surfaces constructed using maximal triangulations.

Theorem 4.2. *If X is a T-surface constructed using a maximal triangulation of the tetrahedron T, then*

$$b_1(\mathbf{R}X) \le h^{1,1}(\mathbf{C}X).$$

Proof. The Smith inequality and the inequality $\chi(\mathbf{R}X) \ge \sigma(\mathbf{C}X)$ proved in Theorem 3.3, give again the desired inequality

$$b_1(\mathbf{R}X) \le h^{1,1}(\mathbf{C}X).$$

■

5 M-surfaces

We describe, first, a special primitive triangulation ρ of T suggested by O. Viro. We show that the T-construction using the triangulation ρ and an appropriate distribution of signs at the integer points of T gives an M-surface of degree m in $\mathbf{R}P^3$. In fact, the surfaces given by the procedure described below are homeomorphic to ones constructed (not as T-surfaces) by O. Viro in [V1].

Let us divide the tetrahedron T by the planes $z = l$, and denote by P_l the polytope

$$\{(x, y, z) \in T \ : \ l \leq z \leq l + 1, \quad l = 0, \ldots, m - 1\}.$$

Choose an arbitrary primitive convex triangulation of each triangle

$$T_l = T \cap \{z = l\}, \quad l = 0, \ldots, m - 1$$

(a triangulation of the triangle T_l is called *primitive* if all its triangles are of area 1/2, or, equivalently, if all the integer points of T_l are vertices of the triangulation).

Each polytope P_l is triangulated as follows. If l is even, take the join J_l of the side of T_l lying in the xz-coordinate plane and of the side of T_{l+1} lying in the plane $x + y + z = m$. If l is odd, take as J_l the join of the side of T_l lying in the plane $x + y + z = m$ and of the side of T_{l+1} lying in the xz-coordinate plane. The join J_l is naturally triangulated into the joins of segments

$$[(i, 0, l), (i+1, 0, l)], \ [(m-(l+1)-j, j, l+1), (m-(l+1)-(j+1), j+1, l+1)],$$
$$i = 0, \ldots, m - l - 1, \quad j = 0, \ldots, m - l - 2$$

if l is even, and J_l is triangulated into the joins of segments

$$[(m - l - j, j, l), (m - l - (j + 1), j + 1, l)], \ [(i, 0, l + 1), (i + 1, 0, l + 1)],$$
$$i = 0, \ldots, m - l - 2, \quad j = 0, \ldots, m - l - 1$$

if l is odd.

The polytope P_l is the union of J_l and of two tetrahedra $P_l^{(1)}$ and $P_l^{(2)}$. These tetrahedra can be triangulated into the cones over the triangles of the chosen triangulations of T_l and of T_{l+1}.

Clearly, the described triangulation ρ of T is primitive. To explain that ρ is convex, consider a triangulation of T formed by the tetrahedra

$$J_l,\ P_l^{(1)},\ P_l^{(2)}\quad (l = 0, \dots, m-1).$$

The later triangulation is convex. Let $\nu' : T \to \mathbf{R}$ be a convex function certifying the convexity of this triangulation, and let $\nu_l : T_l \to \mathbf{R}$ $(l = 0, \dots, m-1)$ be a convex function certifying that the chosen triangulation of T_l is convex. Consider a piecewise-linear function $\nu : T \to \mathbf{R}$ which is linear on each tetrahedron of ρ and takes the value $\nu'(r_l)+\varepsilon\nu_l(r_l)$ at an integer point r_l of T_l. It is easy to see that the function ν for a positive sufficiently small ε certifies the convexity of ρ.

Choose the following distribution of signs at the integer points of T:

> a point (i, j, l) gets the sign "+" if $i \equiv j \equiv l \equiv 0 \bmod 2$ or $l \equiv 1 \bmod 2$ and $ij \equiv 0 \bmod 2$;
> and it gets the sign "-" otherwise.

Proposition 5.1. *A T-surface X constructed using the triangulation ρ and the distribution of signs described is an M-surface. The real point set $\mathbf{R}X$ of X is homeomorphic to the disjoint union of $\frac{m^3}{6} - m^2 + \frac{11m}{6} - 1$ spheres and a sphere with $\frac{m^3}{3} - m^2 + \frac{7m}{6}$ handles if m is even or a projective plane with $\frac{m^3}{3} - m^2 + \frac{7m-3}{6}$ handles if m is odd.*

Proof. It is easy to verify that any integer point r lying strongly inside T has a symmetric copy $s(r)$ with the following property : all the neighbouring vertices of $s(r)$ (i. e. vertices connected with $s(r)$ by an edge of the triangulation) have the same sign, and this sign is opposite to the sign of $s(r)$. It means that the surface $\tilde{S}$ has a connected component homeomorphic to a sphere contained in the star of $s(r)$.

We found $\frac{m^3}{6} - m^2 + \frac{11m}{6} - 1 = h^{2,0}(\mathbf{C}X)$ components of $\tilde{S}$. There is at least one component of $\tilde{S}$ more, because the surface $\tilde{S}$ intersects the coordinate planes. On the other hand, according to Theorem 4.1, the number of connected components of $\mathbf{R}X$ does not exceed $h^{2,0}(\mathbf{C}X) + 1$. Thus, the real point set $\mathbf{R}X$ has exactly $h^{2,0}(\mathbf{C}X) + 1$ connected components.

Using the equalities

$$\chi(\mathbf{R}X) = \sigma(\mathbf{C}X), \quad b_0(\mathbf{R}X) = h^{2,0}(\mathbf{C}X) + 1,$$

we get $b_*(\mathbf{R}X) = b_*(\mathbf{C}X)$, i. e. X is an M-surface. Furthermore,

$$b_1(\mathbf{R}X) = h^{1,1}(\mathbf{C}X),$$

and, thus, the topological type of $\mathbf{R}X$ coincides with one described in the statement of Proposition.

■

6 Counter-examples to Viro's conjecture

We saw in section 4 that Viro's conjecture is true for T-surfaces constructed using a maximal triangulation. Surprisingly enough, a non-maximal triangulation of T can produce a T-surface X in $\mathbf{R}P^3$ with $b_1(\mathbf{R}X) > h^{1,1}(\mathbf{C}X)$.

Let us describe, first, the construction of *an extension* of a triangulation of the triangle $T_0 = T \cap \{z = 0\}$.

Suppose that m is even and that a primitive triangulation τ_0 of T_0 with the vertices having integer coordinates is given. Divide the tetrahedron T into two parts $T \cap \{z \geq 2\}$ and $T \cap \{z \leq 2\}$ by the plane $z = 2$. Take in the first part the triangulation coinciding with the triangulation ρ described in the construction of M-surfaces.

Divide now the second part $T \cap \{z \leq 2\}$ by the plane $x + y + kz = m$ (where $m = 2k$) into the tetrahedron $\bar{T}$ with vertices $(0,0,0)$, $(m,0,0)$, $(0,m,0)$, $(0,0,2)$ and the cone C with the vertex $(0,0,2)$ over

$$\{(x,y,z) \in T \ : \ x + y + z = m, \ 0 \leq z \leq 2\}$$

(see Figure 1).

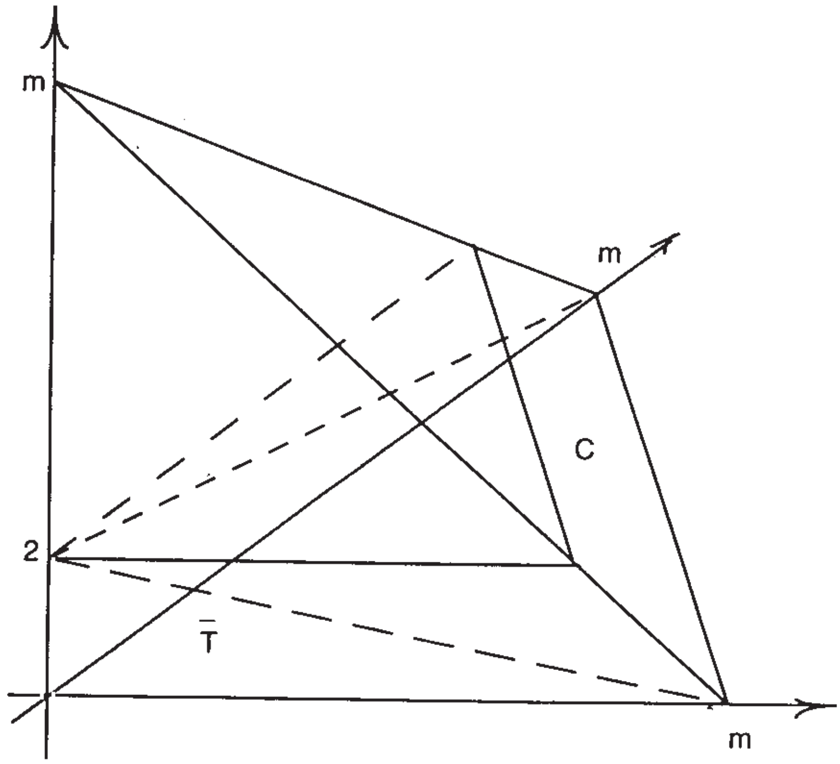

Figure 1

To triangulate the tetrahedron $\bar{T}$, we take the cones over all the triangles of τ_0, and subdivide (in the unique possible way) the cones containing integer points of the plane $z = 1$ in order to obtain a maximal triangulation of $\bar{T}$.

To describe the triangulation of the cone C, let us consider the cone $\hat{C}$ with the vertex $(k+1, 0, 1)$ over the triangle $T \cap \{x + y + kz = m\}$. The rest of the cone C is divided into two parts by the plane $z = 1$ (see Figure 2). Denote the lower part (contained in $C \cap \{0 \leq z \leq 1\}$) by C_0, and denote the upper part (contained in $C \cap \{1 \leq z \leq 2\}$) by C_1.

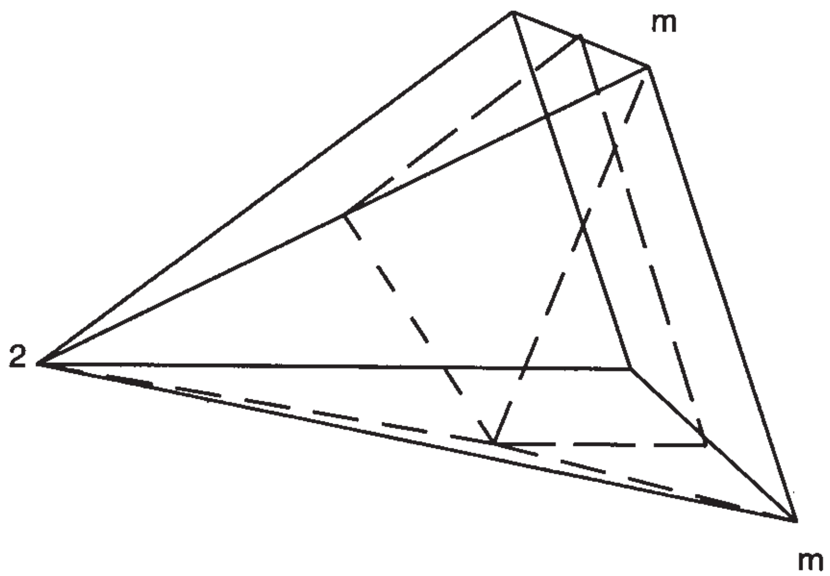

Figure 2

The triangulation of the triangle $T \cap \{x + y + kz = m\}$ is already fixed (it comes from the triangulation of $\bar{T}$). Thus, we can triangulate the cone $\hat{C}$ by the cones with the vertex $(k+1, 0, 1)$ over the triangles of the triangulation of $T \cap \{x + y + kz = m\}$.

Subdivide C_0 taking the cone C' with the vertex $(0, m, 0)$ over the facet of C_0 belonging to the plane $z = 1$, and the join J' of segments $[(m, 0, 0), (0, m, 0)]$ and $[(k+1, 0, 1), (m-1, 0, 1)]$. Let us choose an arbitrary primitive convex triangulation of the quadrangle $C_0 \cap \{z = 1\}$. It gives a natural primitive triangulation of C' (taking the cones over the triangles of the chosen triangulation of $C_0 \cap \{z = 1\}$). The join J' is triangulated by the joins of segments $[(m-j, j, 0), (m-j-1, j+1, 0)]$ and $[(i, 0, 1), (i+1, 0, 1)]$ (where $i = k+1, \ldots, m-2$; $j = 0, \ldots, m-1$).

It remains to triangulate the part C_1. Subdivide C_1 into the join of segments $[(m-1, 0, 1), (0, m-1, 1)]$ and $[(0, 0, 2), (m-2, 0, 2)]$ (triangulated by the joins of segments $[(m-j-1, j, 1), (m-j-2, j+1, 1)]$

and $[(i,0,2),(i+1,0,2)]$, where $i=0,\dots,m-3$; $j=0,\dots,m-2)$ and the naturally triangulated cones : with the vertex $(0,0,2)$ (resp. $(0,m-1,1)$) over the quadrangle $C_1\cap\{z=1\}$ (resp. over the triangle $T_2=T\cap\{z=2\}$).

The described maximal triangulation of T is called *the extension* of the triangulation τ_0 and is denoted by $ext(\tau_0)$.

Arguments, similar to ones used in the previous section to show that the triangulation ρ is convex, prove that if τ_0 is convex then $ext(\tau_0)$ is also convex. Almost all tetrahedra of $ext(\tau_0)$ are of volume 1/6. The only tetrahedra of a greater volume (more precisely, of volume 1/3) are the cones with the vertex $(0,0,2)$ over the odd triangles of τ_0 (we call a triangle of τ_0 *odd* if it does not have a vertex with the both even coordinates).

Suppose now that a distribution δ_0 of signs at the integer points of T_0 is given. Let us describe a distribution $ext(\delta_0)$ of signs at the integer points of T which we call *an extension* of δ_0. In the part $T\cap\{z\geq 2\}$ we take the distribution of signs described in the construction of M-surfaces. It remains, thus, to fix a distribution of signs at the integer points of $T\cap\{z=1\}$. We do it as follows:

> take an arbitrary distribution in $T\cap\{z=1\}\cap\{x+y<k\}$,
> all the integer points of the segment $[(k,0,1),(0,k,1)]$ but the point $(0,k,1)$ get the sign "-",
> for the other points of T_1 we apply the rule : a point $(i,j,1)$ gets the sign "-" if i and j are both odd, and the sign "+" otherwise.

Let us take a triangulation τ_0^1 and a distribution δ_0^1 of signs at the integer points of T_0 producing a counter-example to Ragsdale conjecture with $p=\frac{3k^2-3k+2}{2}+1$ (see [I1], [I2], [I-V]). The triangulation τ_0^1 can be obtained placing the hexagon H shown in Figure 3 inside of T_0 (on suppose that $m\geq 10$) in such a way that the center of H has both the nonzero coordinates odd, and extending, then, the triangulation of H to a primitive convex triangulation of T_0. To obtain a distribution of signs at the integer points of T_0, we complete the distribution presented in Figure 3 by the rule :

> a point $(i,j,0)$ gets the sign "-" if i and j are even, and $i+j<m$,

a point $(i, j, 0)$ gets the sign "+" otherwise.

Remark that this distribution of signs at the integer points of T_0 is slightly different form the distribution described in [I1], [I2], [I-V].

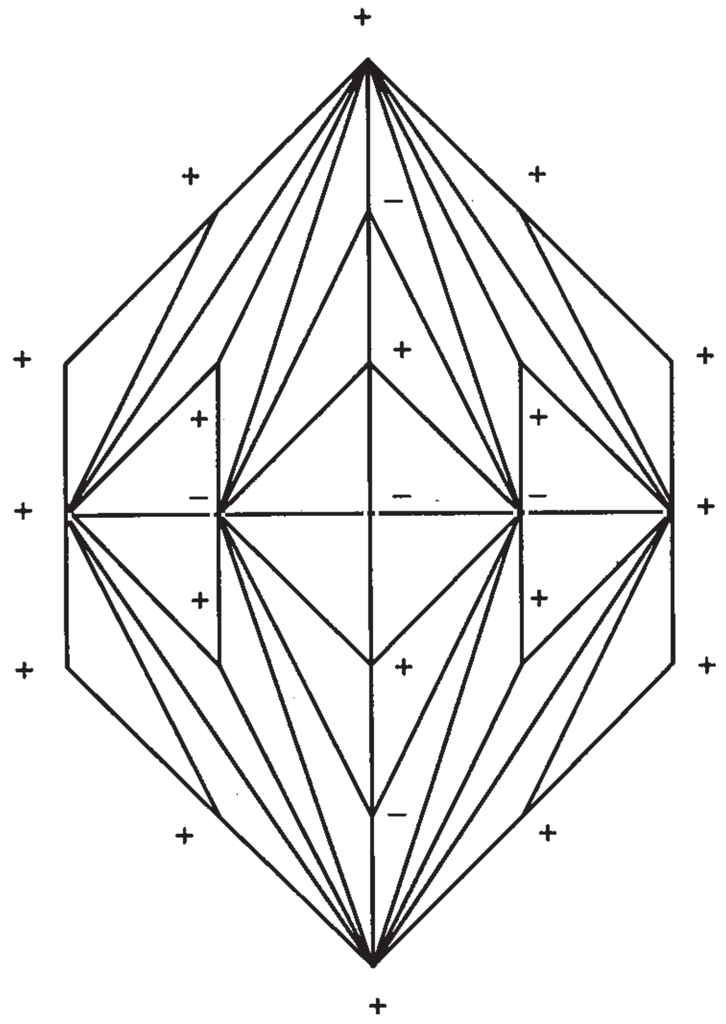

Figure 3

Proposition 6.1 *The maximal triangulation* $ext(\tau_0^1)$ *and a distribution* *of signs $ext(\delta_0^1)$ produce a T-surface X of degree m in $\mathbf{R}P^3$ with*

$$\chi(\mathbf{R}X) = -\frac{m^3}{3} + \frac{4m}{3}, \quad b_0(\mathbf{R}X) = h^{2,0}(\mathbf{C}X) - 2.$$

The real point set $\mathbf{R}X$ of X is homeomorphic to the disjoint union

$$\left(\frac{m^3}{6} - m^2 + \frac{11m}{6} - 5\right) S^2 \coprod S_2 \coprod S_{\frac{m^3}{3} - m^2 + \frac{7m}{6} - 5}$$

of $\frac{m^3}{6} - m^2 + \frac{11m}{6} - 5$ spheres, a sphere with 2 handles and a sphere with $\frac{m^3}{3} - m^2 + \frac{7m}{6} - 5$ handles.

Proof. Let us, first, calculate $\chi(\mathbf{R}X)$. It was already remarked that almost all tetrahedra of $ext(\tau_0^1)$ are of volume 1/6. The only tetrahedra of greater volume (of volume 1/3) are the cones over the odd triangles of τ_0^1. Each of these tetrahedra of volume 1/3 has 6 nonempty symmetric copies (a tetrahedron of volume 1/3 of a maximal triangulation has 6 nonempty copies if the product of signs at its vertices is positive, and it has 8 nonempty copies if the product of signs is negative). Thus, the arguments of the proof of Theorems 3.2 and 3.3 show that $\chi(\mathbf{R}X) = \sigma(\mathbf{C}X)$.

Calculate now the number of connected components of $\tilde{S}$. Exactly as in the proof of Theorem 5.1, any integer point lying strongly inside $(T \cap \{z \geq 2\}) \cup C$ has a symmetric copy with the star containing a component of $\tilde{S}$ homeomorphic to a sphere. It is easy to see that the stars of integer points lying strongly inside T and belonging to the segment $[(k, 0, 1), (0, k, 1)]$ also contain the components of $\tilde{S}$ homeomorphic to a sphere. Consider the integer points lying strongly inside the tetrahedron $\bar{T}$. Let us call *even interior points of* T_0 the integer points $(i, j, 0)$ such that $i > 0$, $j > 0$, $i + j < m$, i and j are both even. There is a correspondence between the even interior points of T_0 and the points of $Int(\bar{T}) \cap \mathbf{Z}^3$: any integer point lying strongly inside $\bar{T}$ is a middle point of a segment joining the point $(0, 0, 2)$ and an even interior point of T_0. We denote the middle point of a segment $[(0, 0, 2), r]$ (where r is an even interior point of T_0) by $f(r)$.

Suppose that an even interior point r does not belong to the hexagon H. Then r has the sign "-". If $f(r)$ has also the sign "-", then the union of stars of r and of $f(r)$ (in the triangulation of T_*) contains a component of $\tilde{S}$ homeomorphic to a sphere. If $f(r)$ has the sign "+", then the union of stars of r and of $s_z(f(r))$ contains a component of $\tilde{S}$ homeomorphic to a sphere.

We have found $h^{2,0}(\mathbf{C}X)-4$ spheres of $\tilde{S}$ (a sphere was associated to any integer point lying strongly inside of T except 4 points of the form $f(r)$, where r is an even interior point of T_0 belonging to the hexagon H). There are two connected components of $\tilde{S}$ more. One component is homeomorphic to a sphere with two handles and lies inside of $\bar{H} \cup s_z(\bar{H})$, where $\bar{H}$ is a cone with the vertex $(0,0,2)$ over H. The remaining part of $\tilde{S}$ is connected. The number $b_1(\tilde{S})$ can be calculated via the Euler characteristic.

■

Theorem 6.2. *If m is an even integer number not less than 10, then there exists an (M-2)-surface X of degree m in $\mathbf{R}P^3$ such that $b_1(\mathbf{R}X) = h^{1,1}(\mathbf{C}X)+2$.*

Proof. Let us take the triangulation $ext(\tau_0^1)$ of T and the distribution of signs $ext(\delta_0^1)$ at the integer points of T. According to Proposition 6.1 the resulting surface $\tilde{S}$ is homeomorphic to

$$\left(\frac{m^3}{6} - m^2 + \frac{11m}{6} - 5\right) S^2 \coprod S_2 \coprod S_{\frac{m^3}{3}-m^2+\frac{7m}{6}-5}.$$

Remove now 4 vertices of the form $f(r)$, where r is an even interior point of T_0 belonging to H (see the proof of Proposition 6.1), with all the adjacent edges. Denote the new triangulation (which is nonmaximal) by $ext'(\tau_0^1)$ and consider the surface $\tilde{S}'$ constructed using $ext'(\tau_0^1)$ and the restriction $ext'(\delta_0^1)$ of the distribution $ext(\delta_0^1)$ to the set of vertices of $ext'(\tau_0^1)$. Clearly, the surface $\tilde{S}'$ is homeomorphic to

$$\left(\frac{m^3}{6} - m^2 + \frac{11m}{6} - 5\right) S^2 \coprod S_2 \coprod S_{\frac{m^3}{3}-m^2+\frac{7m}{6}-1}$$

because we added 4 handles to the component homeomorphic to

$S_{\frac{m^3}{3}-m^2+\frac{7m}{6}-5}$. Thus, the number of $b_1(\tilde{S}')$ is equal to

$$\frac{2m^3}{3} - 2m^2 + \frac{7m}{3} + 2.$$

∎

Using counter-examples of degree $2k$ to Ragsdale conjecture with more than $\frac{3k^2-3k+2}{2}+1$ even ovals (see [I1], [I2], [I-V]), one can construct surfaces X of degree $2k$ in $\mathbf{R}P^3$ with $b_1(\mathbf{R}X) > h^{1,1}(\mathbf{C}X)+2$.

Theorem 6.3. *If $m = 2k$ is an even integer not less than 10, then there exists a surface X of degree m in $\mathbf{R}P^3$ such that*

$$b_1(\mathbf{R}X) = h^{1,1}(\mathbf{C}X) + 2\left[\frac{(k-3)^2+4}{8}\right]$$

(where $[u]$ denotes the greatest integer which does not exceed u).

Proof. We start from a triangulation τ_0^a and a distribution δ_0^a of signs at the integer points of T_0 giving a counter-example to Ragsdale conjecture with

$$p = \frac{3k^2-3k+2}{2} + a,$$

where $a = \left[\frac{(k-3)^2+4}{8}\right]$ (see [I1], [I2], [I-V]). The triangulation τ_0^a can be obtained in the following way. Consider the partition of the triangle T_0 shown in Figure 4. Let us take in each shadowed hexagon the triangulation (and the signs) of the hexagon H. The triangulation of the union of the shadowed hexagons can be extended to a primitive convex triangulation τ_0^a of T_0. To obtain the distribution δ_0^a of signs at the integer points of T_0, we choose the signs outside of the union of the shadowed hexagons again using the rule :

> a point $(i,j,0)$ gets the sign "-" if i and j are even, and $i+j<m$,
> a point $(i,j,0)$ gets the sign "+" otherwise.

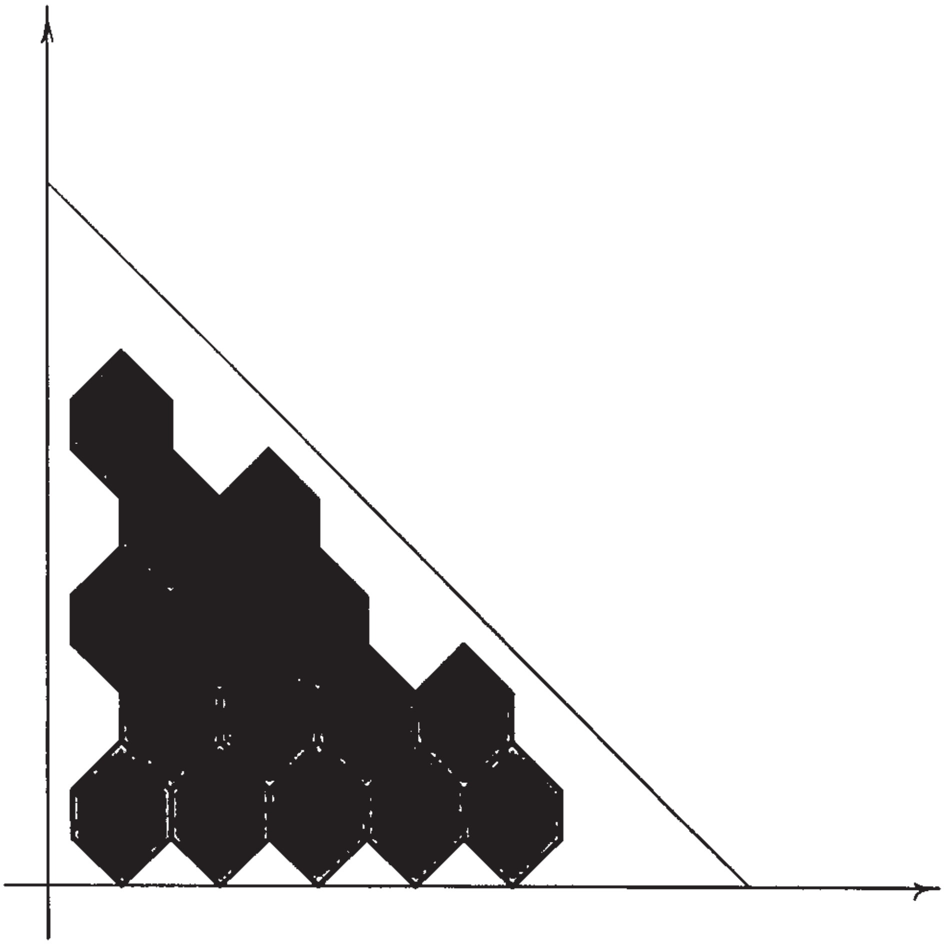

Figure 4

Consider the triangulation $ext(\tau_0^a)$ of T and the distribution $ext(\delta_0^a)$

of signs at the integer points of T. The resulting surface $\tilde{S}$ is homeomorphic to

$$\left(\frac{m^3}{6} - m^2 + \frac{11m}{6} - 1 - 4a\right) S^2 \coprod aS_2 \coprod S_{\frac{m^3}{3} - m^2 + \frac{7m}{6} - 5a}.$$

Remove now the vertices of the triangulation $ext(\tau_0^a)$ (with adjacent edges) of the form $f(r)$, where r is an even interior point of T_0 belonging to one of the shadowed hexagons, and take the restriction $ext'(\delta_0^a)$ of the distribution $ext(\delta_0^a)$ to the vertices of the new triangulation $ext'(\tau_0^a)$. We obtain a surface $\tilde{S}'$ homeomorphic to

$$\left(\frac{m^3}{6} - m^2 + \frac{11m}{6} - 1 - 4a\right) S^2 \coprod aS_2 \coprod S_{\frac{m^3}{3} - m^2 + \frac{7m}{6} - a}$$

with

$$b_1(\tilde{S}') = \frac{2m^3}{3} - 2m^2 + \frac{7m}{3} + 2a.$$

■

Remarks.

1. Removing, if necessary, some of the shadowed hexagons in the construction of Theorem 6.3, we get counter-examples to Viro's conjecture with the real point set homeomorphic to

$$\left(\frac{m^3}{6} - m^2 + \frac{11m}{6} - 1 - 4a\right) S^2 \coprod aS_2 \coprod S_{\frac{m^3}{3} - m^2 + \frac{7m}{6} - a},$$

where $a = 1, \ldots, \left[\frac{(k-3)^2+4}{8}\right]$.

2. The counter-example of the smallest degree in $\mathbf{R}P^3$ given by Theorems 6.2 and 6.3 is a surface of degree 10. The real point set of this surface is homeomorphic to

$$80S^2 \coprod S_2 \coprod S_{244}.$$

It is unknown if there exist counter-examples of degree less than 10. The smallest degree we can expect for a counter-example to Viro's conjecture is degree 5.

3. Repeating the procedure described above for the new counter-examples to the Ragsdale conjecture constructed by B. Haas [Ha], one can construct surfaces X of degree $2k$ in $\mathbf{R}P^3$ with

$$b_1(\mathbf{R}X) = h^{1,1}(\mathbf{C}X) + 2a',$$

where $a' = \left[\frac{k^2-7k+16}{6}\right]$.

4. We can obtain counter-examples to Viro's conjecture which are asymptotically better than the examples described above: there exist T-surfaces X of degree $2k$ in $\mathbf{R}P^3$ with $b_1(\mathbf{R}X) = h^{1,1}(\mathbf{C}X) + 2A$, where $A = k^3/24+$ terms of smaller degrees. To construct such surfaces, we divide the tetrahedron T by the planes $z = 2l$ (where $l = 1, \ldots k-1$), and define a triangulation and a distribution of signs in each part of the subdivision using the procedure described in the proof of Theorem 6.3 for $T \cap \{0 \leq z \leq 2\}$.

References

[GKZ] I. Gelfand, M. Kapranov and A. Zelevinski, *Discriminants, resultants and multidimensional determinants*, Birkhäuser, Boston, 1994.

[Ha] B. Haas, *Les multilucarnes: Nouveaux contre-exemples à la conjecture de Ragsdale*, C. R. Acad. Sci. Paris (to appear).

[I1] I. Itenberg, *Contre-exemples à la conjecture de Ragsdale*, C. R. Acad. Sci. Paris, Ser. 1. **317** (1993), 277-282.

[I2] I. Itenberg, *Counter-examples to Ragsdale Conjecture and T-curves*, Contemporary Mathematics **182** (1995), 55-72.

[I-R] I. Itenberg, M.-F. Roy, *Multivariate Descartes' rule*, Beiträge zur Algebra und Geometrie, (to appear).

[I-V] I. Itenberg, O. Viro, *Patchworking algebraic curves disproves the Ragsdale conjecture*, Preprint, Uppsala University, 1995.

[Kh] V. Kharlamov, *Real algebraic surfaces*, Proc. Internat. Congress Math., vol. 1, Helsinki, 1978, pp. 421-428. (Russian).

[Ra] V. Ragsdale, *On the arrangement of the real branches of plane algebaric curves*, Amer. J. Math. **28** (1906), 377-404.

[Ri] J.-J. Risler, *Construction d'hypersurfaces réelles [d'après Viro]*, Séminaire N.Bourbaki, no. 763, vol. 1992-93, Novembre 1992.

[Sh] E. Shustin, *Critical points of real polynomials, subdivisions of Newton polyhedra and topology of real algebraic hypersurfaces*, Amer. Math., Soc. Transl. (2) **173** (1996), 203-223.

[Si] R. Silhol, *Real Algebraic Surfaces*, Lect. Notes in Math., vol. 1392, Springer-Verlag, Berlin Heidelberg, 1989.

[V1] O. Viro, *Construction of M-surfaces*, Functional Anal. Appl. **13** (1979).

[V2] O. Viro, *Gluing of algebraic hypersurfaces, smoothing of singularities and construction of curves*, Proc. Leningrad Int. Topological Conf. (Leningrad, Aug. 1983), Nauka, Leningrad, 1983, pp. 149-197. (Russian).

[V3] O. Viro, *Gluing of plane real algebraic curves and construction of curves of degrees 6 and 7*, Lect. Notes in Math., 1060, Springer-Verlag, Berlin Heidelberg, 1984, pp. 187-200.

[V4] O. Viro, *Progress in the topology of real algebraic varieties over the last six years*, Rus. Math. Surv. **41** (1986), no. 3, 55-82.

[V5] O. Viro, *Real algebraic plane curves: constructions with controlled topology*, Leningrad Math. J. **1** (1990), 1059-1134.

[V6] O. Viro, *Patchworking real algebraic varieties*, Preprint, Uppsala University, 1994.

[Wi] G. Wilson, *Hilbert's sixteenth problem*, Topology **17** (1978), no. 1, 53-73.

[Zi] G. Ziegler, *Lectures on Polytopes*, Springer-Verlag, New York 1994.

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