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# Multivariate Sturm–Habicht sequences: real root counting on n-rectangles and triangles.

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**Abstract**

The main purpose of this note is to show how Sturm–Habicht Sequence can be generalized to the multivariate case and used to compute the number of real solutions of a polynomial system of equations with a finite number of complex solutions. Using the same techniques, some formulae counting the number of real solutions of such polynomial systems of equations inside $n$–dimensional rectangles or triangles in the plane are presented.

Sturm–Habicht Sequence is one of the tools that Computational Real Algebraic Geometry provides to deal with the problem of computing the number of real roots of an univariate polynomial in $\mathbb{Z}[x]$ with good specialization properties and controlled complexity (see [GLRR$_{1,2,3}$]). The purpose of this note is to show how Sturm–Habicht Sequence can be easily generalized to the multivariate case and used to compute the number of real solutions of a polynomial system of equations with a finite number of complex solutions. Using the same technics it will be showed how to count real solutions of such polynomials systems of equations inside $n$-dimensional rectangles or in triangles in the plane. These counting algorithms will work only when the considered polynomial system of equations has a finite number of complex solutions.

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The paper is divided in two sections. In the first one, the definitions and main properties of Sturm–Habicht Sequence are showed and the second one is devoted to present the notion of Multivariate Sturm–Habicht Sequence and to present how it can be used to deal with the Real Root Counting Problem. The main tool to achieve this goal is the generalization of the "Volume Function" introduced in [Milne] which is based in the early work of C. Hermite on this topic (see [Hermite] and [KN]). Similar formulae to the ones to be presented in the second section were also obtained in [Pedersen].

## Sturm–Habicht Sequence

Let $\mathbb{K}$ be an ordered field and $\mathbb{F}$ a real–closed field with $\mathbb{K} \subseteq \mathbb{F}$. This section is devoted to introduce the main properties of Sturm–Habicht Sequence to be used in what follows. The proof of the theorems quoted in this section, related to properties of Sturm–Habicht sequence, can be found in [GLRR$_{1,2,3}$].

**Definition.** *Let $P$ be a polynomial in $\mathbb{K}[x]$ with $p = \deg(P)$. If we write*

$$\delta_k = (-1)^{\frac{k(k+1)}{2}}$$

*for every integer $k$, the Sturm-Habicht sequence associated to $P$ is defined as the list of polynomials $\{\mathbf{StHa}_j(P)\}_{j=0,\dots,p}$ where $\mathbf{StHa}_p(P) = P$, $\mathbf{StHa}_{p-1}(P) = P'$ and for every $j \in \{0,\dots,p-2\}$:*

$$\mathbf{StHa}_j(P) = \delta_{p-j-1}\mathbf{Sres}_j(P, P')$$

*where $\mathbf{Sres}_j(P, P')$ denotes the subresultant of index $j$ for $P$ and $P'$. For every $j$ in $\{0,\dots,p\}$ the principal $j$-th Sturm–Habicht coëfficient, $\mathbf{stha}_j(P)$, is defined as the coefficient of $x^j$ in $\mathbf{StHa}_j(P)$.*

Next definitions introduce several sign counting functions that we shall use to relate the polynomials in the Sturm–Habicht Sequence of $P$ with the number of real roots of $P$ in an open interval.

**Definition.** *Let $\{a_0, a_1, \dots, a_n\}$ be a list of non zero elements in $\mathbb{F}$. We define:*

- $\mathbf{V}(\{a_0, a_1, \dots, a_n\})$ as the number of sign variations in the list $\{a_0, a_1, \dots, a_n\}$, that is the number of consecutive signs $\{+, -\}$ or $\{-, +\}$,

- $\mathbf{P}(\{a_0, a_1, \ldots, a_n\})$ as the number of sign permanences in the list $\{a_0, a_1, \ldots, a_n\}$, that is the number of consecutive signs$\{+,+\}$ or $\{-,-\}$.

**Definition.** *Let $P$ be a polynomial in $\mathbb{K}[x]$ and $\alpha \in \mathbb{F}$ with $P(\alpha) \neq 0$. We define the integer number $\mathbf{W}_{\mathrm{StHa}}(P;\alpha)$ in the following way:*

- we construct a list of polynomials $\{g_0, \ldots, g_s\}$ in $\mathbb{K}[x]$ obtained by deleting the polynomials identically 0 from $\{\mathbf{StHa}_j(P)\}_{j=0,\ldots,p}$,
- $\mathbf{W}_{\mathrm{StHa}}(P;\alpha)$ is the number of sign variations in the list $\{g_0(\alpha), \ldots, g_s(\alpha)\}$ using the following rules for the groups of 0's:
  - ⋆ we count 1 sign variation for the groups:

$$[-,0,+],\quad [+,0,-],\quad [+,0,0,-] \text{ and } [-,0,0,+]$$

  - ⋆ we count 2 sign variations for the groups:

$$[+,0,0,+] \text{ and } [-,0,0,-]$$

The *Sturm-Habicht Structure Theorem* (see [GLRR3])) implies that it is not possible to find more than two consecutive zeros in the sequence $\{g_0(\alpha), \ldots, g_s(\alpha)\}$ and that the sign sequences $[+,0,+]$, $[-,0,-]$ can not appear.

**Definition.** *Let $P$ be a polynomial in $\mathbb{K}[x]$ and $\alpha, \beta \in \mathbb{F}$ with $\alpha < \beta$. We define*

$$\mathbf{W}_{\mathrm{StHa}}(P;\alpha,\beta) = \mathbf{W}_{\mathrm{StHa}}(P;\alpha) - \mathbf{W}_{\mathrm{StHa}}(P;\beta)$$

Next theorem shows how to use the Sturm–Habicht sequence of $P$ and the function $\mathbf{W}_{\mathrm{StHa}}$ to compute the number of real roots of $P$ inside an open interval.

**Theorem.** *Let $P$ be a polynomial in $\mathbb{K}[x]$ and $\alpha, \beta \in \mathbb{F}$ with $\alpha < \beta$ and $P(\alpha)P(\beta) \neq 0$. Then:*

$$\mathbf{W}_{\mathrm{StHa}}(P;\alpha,\beta) = \#(\{\gamma \in (\alpha,\beta) : P(\gamma) = 0\})$$

This section is finished showing how to use Sturm–Habicht sequence to compute the total number of real roots (in $\mathbf{F}$) of a polynomial in $\mathbb{K}[x]$. First the definition of a new sign counting function is introduced.

**Definition.** *Let $a_0, a_1, \ldots, a_n$ be elements in $\mathbf{F}$ with $a_0 \neq 0$ and we suppose that we have the following distribution of zeros:*

$$\{a_0, a_1, \ldots, a_n\} =$$

$$= \{a_0, \ldots, a_{i_1}, \overbrace{0, \ldots, 0}^{k_1}, a_{i_1+k_1+1}, \ldots,$$

$$a_{i_2}, \overbrace{0, \ldots, 0}^{k_2}, a_{i_2+k_2+1}, \ldots, a_{i_3}, 0, \ldots\ldots, 0, a_{i_{t-1}+k_{t-1}+1}, \ldots, a_{i_t}, \overbrace{0, \ldots, 0}^{k_t}\}$$

*where all the $a_i$'s that have been written are not 0. We define $i_0+k_0+1 = 0$ and:*

$$\mathbf{C}(\{a_0, a_1, \ldots, a_n\}) = \sum_{s=1}^{t}(\mathbf{P}(\{a_{i_{s-1}+k_{s-1}+1}, \ldots, a_{i_s}\})$$

$$-\mathbf{V}(\{a_{i_{s-1}+k_{s-1}+1}, \ldots, a_{i_s}\})) + \sum_{s=1}^{t-1} \varepsilon_{i_s}$$

*where:*

$$\varepsilon_{i_s} = \begin{cases} 0 & \text{if } k_s \text{ is odd} \\ (-1)^{\frac{k_s}{2}} \operatorname{sign}(\frac{a_{i_s+k_s+1}}{a_{i_s}}) & \text{if } k_s \text{ is even} \end{cases}$$

**Theorem.** *If $P$ is a polynomial in $\mathbb{K}[x]$ with $p = \deg(P)$ then:*

$$\mathbf{C}(\{\mathbf{stha}_p(P), \ldots, \mathbf{stha}_0(P)\}) = \#(\{\gamma \in \mathbf{F} : P(\gamma) = 0\})$$

## Volume Functions and Real Root Counting

Let $\mathbb{K} \subseteq \mathbf{F} \subseteq \mathbf{L}$ be a field extension with $\mathbf{K}$ ordered, $\mathbb{F}$ real closed and $\mathbf{L}$ algebraically closed. If $J$ is a zero dimensional ideal in $\mathbb{K}[\underline{x}] = \mathbf{K}[x_1, \ldots, x_n]$ and $\mathcal{V}_{\mathbf{L}}(J) = \{\Delta_1, \ldots, \Delta_s\}$ is the set of zeroes in $\mathbf{L}^n$ of $J$, the main questions to be considered in this section are the computation of the number of $\Delta_i$'s in $\mathbf{F}^n$ and the number of $\Delta_i$'s inside a prescribed

$n$-dimensional rectangle in $\mathbf{F}^n$. The main tool to solve these two Real Root Counting Problems will be the Volume Function which is presented in the next definition. The term "Volume Function" was introduced by P. Milne in [Milne] in order to compute the number of real solutions of $J$ inside a prescribed $n$-dimensional rectangle.

**Definition.** *Let $\ell$ be a polynomial in $\mathbf{K}[\underline{x},\underline{y}] = \mathbb{K}[x_1,\ldots,x_n,y_1,\ldots,y_n]$. The volume function associated to $\ell$ and $J$ is the polynomial in $\mathbf{K}[U,\underline{y}]$ defined by the following equality:*

$$\mathbf{V}_{\ell,J}(U,\underline{y}) = \prod_{\Delta\in\mathcal{V}_{\mathbf{L}}(J)} (U - \ell(\Delta,\underline{y}))$$

*where, in the previous product, the multiplicities are taken into account.*

Clearly, if $D$ is the dimension of $\mathbf{K}[\underline{x}]/J$ as $\mathbf{K}$-vector space (ie the sum of the $\Delta_i$'s multiplicities) then the degree of $\mathbf{V}_{\ell,J}$, as polynomial in $U$, is equal to $D$. The Volume Function $\mathbf{V}_{\ell,J}$ can be determined by computing a lexicographic Grobner Bases of $\langle J, U - \ell(\underline{x},\underline{y})\rangle$ by considering any monomial ordering verifying $\underline{x} > \underline{y} > U$. Another more efficient way is based on the using of any Grobner Bases of $J$ to compute the traces of the powers of $\ell(\underline{x},\underline{y})$ (with respect the extension $\mathbb{K} \subseteq \mathbf{K}[\underline{x}]/J$),

$$\mathrm{Trace}((\ell(\underline{x},\underline{y})^k) = \sum_{\Delta\in\mathcal{V}_{\mathbf{L}}(J)} \ell(\Delta,\underline{y})^k$$

and, in the application of the Newton Identities to recover the coefficients of $\mathbf{V}_{\ell,J}$. Next theorem shows how the Volume Function is useful for the Real Root Counting Problem.

**Theorem.** *Let $\underline{a}$ be a point in $\mathbf{F}^n$ and $\ell$ a polynomial in $\mathbf{K}[\underline{x},\underline{y}]$ verifying that*

$$j \neq k \iff \ell(\Delta_j,\underline{a}) \neq \ell(\Delta_k,\underline{a})$$

*Then the number of real solutions of $J$ (solutions in $\mathbf{F}^n$) is equal to the number of real roots (roots in $\mathbb{F}$) of $\mathbf{V}_{\ell,J}(U,\underline{a})$:*

$$\#(\mathcal{V}_{\mathbf{F}}(J)) = \mathbf{C}(\{\mathbf{stha}_j(\mathbf{V}_{\ell,J}(U,\underline{a}))\}_{0\le j\le D})$$

**Proof.** The proof is very easy since the condition imposed to $\ell$ and $\underline{a}$ allows to assure that there are no solutions of $J$, $\Delta$, in $\mathbb{L}^n - \mathbb{F}^n$ making $\ell(\Delta, \underline{a})$ an element of $\mathbb{F}$:

$$\begin{aligned} &\mathbf{C}(\{\mathbf{stha}_j(\mathbf{V}_{\ell,J}(U,\underline{a}))\}_{0\le j\le D}) = \#(\{\beta \in \mathbb{F} : \mathbf{V}_{\ell,J}(\beta,\underline{a}) = 0\}) = \\ &= \#(\{\beta \in \mathbb{F} : \exists i \quad \beta = \ell(\Delta_i,\underline{a})\}) = \#(\mathcal{V}_{\mathbb{F}}(J)) + \\ &+\#(\{\Delta_i \in \mathbb{L}^n - \mathbb{F}^n : \ell(\Delta_i,\underline{a}) \in \mathbb{F}\}) = \#(\mathcal{V}_{\mathbb{F}}(J)) \end{aligned}$$

as we wanted to show.

■

The previous condition on $\ell$ and $\underline{a}$ can be replaced by the following two conditions described only in terms of the real solutions:

$$\Delta, \Delta' \in \mathcal{V}_{\mathbb{F}}(J) \qquad \Delta \neq \Delta' \qquad \Longleftrightarrow \qquad \ell(\Delta,\underline{a}) \neq \ell(\Delta',\underline{a})$$

$$\Delta \in \mathcal{V}_{\mathbb{F}}(J) \quad \Longleftrightarrow \quad \ell(\Delta,\underline{a}) \in \mathbb{F}$$

In general, the quantity $\mathbf{C}(\{\mathbf{stha}_j(\mathbf{V}_{\ell,J}(U,\underline{a}))\}_{0\le j\le D})$ provides an upper bound for the number of real solutions. Next proposition shows how the Sturm–Habicht principal coefficients of $\mathbf{V}_{\ell,J}(U,\underline{y})$ are related to $\ell$ and the zeros of $J$.

**Proposition.** *Let $\ell$ be a polynomial in $\mathbb{K}[\underline{x},\underline{y}]$. Then, for every $j$ in $\{0,\ldots,D\}$ the following identity holds:*

$$\mathbf{stha}_j(\mathbf{V}_{\ell,J}(U,\underline{y})) = \sum_{\Gamma\in\binom{[D-j]}{j}} \prod_{s<t\in\Gamma} (\ell(\Delta_s,\underline{y}) - \ell(\Delta_t,\underline{y}))$$

*where $[D] = \{1,\ldots,D\}$ and $\binom{[D]}{j}$ denotes the set of all the subsets in $[D]$ with $j$ elements.*

**Proof.** It is enough to apply the explicit description of Sturm–Habicht principal coefficients in terms of the roots of the polynomial $\mathbf{V}_{\ell,J}(U,\underline{y})$ (as presented in [GLRR3]).

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In the case of one variable, $J = \langle P(x) \rangle$, and using $\ell(x,y) = y - x$, the Sturm–Habicht principal coefficients of $\mathbf{V}_{\ell,J}(U,y)$ are exactly the Sturm–Habicht principal coefficients of $P(x)$. Thus, it seems natural to call "*$\ell$-Multivariate Sturm-Habicht Sequence*" the Sturm–Habicht Sequence of $\mathbf{V}_{\ell,J}(U,\underline{y})$ (with respect $U$).

The "Volume Function" used by P. Milne in [Milne] to compute the number of real solutions of $J$ inside a prescribed $n$-dimensional rectangle was:

$$\ell(\underline{x},\underline{y}) = \prod_{j=1}^{n}(y_j - x_j)$$

Next theorem shows how to use the Volume Function associated to this concrete $\ell$, to deal with such problem but avoiding any assumption on $\mathbf{V}_{\ell,J}$: in [Milne] the same result is proven but under the additional hypothesis that the Sturm sequence of $\mathbf{V}_{\ell,J}(U,\underline{y})$ and its derivative with respect $U$ must be normal (the degrees in the considered Sturm sequence decrease one by one).

**Theorem.** *Let*

$$\mathcal{R} = \prod_{i=1}^{n}[a_i, b_i], \qquad a_i < b_i$$

*be a $n$-dimensional rectangle in $\mathbb{F}^n$ and*

$$\ell(\underline{x},\underline{y}) = \prod_{j=1}^{n}(y_j - x_j)$$

*verifying the following conditions with respect the set $\mathcal{V}_{\mathbf{F}}(J)$:*

- *there are no points of $\mathcal{V}_{\mathbf{F}}(J)$ on any hyperplane in $\mathbb{F}^n$ containing a face of $\mathcal{R}$,*

- ⋆ *if $v_1, \ldots, v_N$ $(N = 2^n)$ are the vertices of $\mathcal{R}$ then for any $i \in \{1, \ldots, N\}$:*

$$j \neq k \implies \ell(\Delta_j, v_i) \neq \ell(\Delta_k, v_i)$$

*If we denote*

$$s[v_i] = \mathbf{W}_{\mathrm{StHa}}(\mathbf{V}_{\ell,J}(U, v_i); -\infty, 0)$$

*then the following equality holds:*

$$\#(\mathcal{V}_{\mathbb{F}}(J) \cap \mathcal{R}) = \frac{1}{2^{n-1}} \sum_{i=1}^{N} (-1)^{v_i} s[v_i]$$

*where:*

$$(-1)^{v_i} = (-1)^{\mu(v_i)} \qquad \begin{array}{cccc} \mu: & \mathcal{R} & \longrightarrow & \mathbb{F} \\ & (l_1, \ldots, l_n) & \longmapsto & \displaystyle\sum_{k=1}^{n} \frac{l_k - a_k}{b_k - a_k} \end{array}$$

**Proof.** For simplicity the proof is done for the case $n = 2$. The general case follows exactly the same strategy. Let $A_i$ $(1 \leq i \leq 9)$ be one of the nine open regions in $\mathbb{F}^2$ determined by the lines $x_1 = a_1$, $x_1 = b_1$, $x_2 = a_2$ and $x_2 = b_2$ (ordered from the most negative to the most positive),

| | | |
|---|---|---|
| $A_7$ | $A_8$ | $A_9$ |
| $A_4$ | $A_5$ | $A_6$ |
| $A_1$ | $A_2$ | $A_3$ |

$B_i = \#(\mathcal{V}_{\mathbb{F}}(J) \cap A_i)$ and $\Gamma[v_i] = \#(\{\Delta \in \mathcal{V}_{\mathbb{L}}(J) - \mathbb{F}^2 : \ell(\Delta, v_i) < 0\})$, for any vertex $v_i$. Then the value of $s[(a_1, a_2)]$ can be described in the following terms:

$$\begin{aligned} s[(a_1, a_2)] &= \mathbf{W}_{\text{StHa}}(\mathbf{V}_{\ell,J}(U, a_1, a_2); -\infty, 0) = \\ &= \#(\{\beta \in \mathbf{F} : \mathbf{V}_{\ell,J}(\beta, a_1, a_2) = 0, \beta < 0\}) = \\ &= \#(\{\beta \in \mathbf{F} : \exists \Delta \in \mathcal{V}_{\mathbb{L}}(J), \beta = \ell(\Delta, a_1, a_2), \beta < 0\}) = \\ &= \#(\{\Delta \in \mathcal{V}_{\mathbb{L}}(J) : \ell(\Delta, a_1, a_2) < 0\}) = \\ &= \#(\{\Delta \in \mathcal{V}_{\mathbf{F}}(J) : \ell(\Delta, a_1, a_2) < 0\}) + \\ &+ \#(\{\Delta \in \mathcal{V}_{\mathbb{L}}(J) - \mathbf{F}^2 : \ell(\Delta, a_1, a_2) < 0\}) = \\ &= B_2 + B_3 + B_4 + B_7 + \Gamma[(a_1, a_2)] \end{aligned}$$

In the same way, the following equalities are obtained:

$$\begin{aligned} s[(a_1, b_2)] &= B_2 + B_3 + B_5 + B_6 + B_7 + \Gamma[(a_1, b_2)] \\ s[(b_1, a_2)] &= B_3 + B_4 + B_5 + B_7 + B_8 + \Gamma[(b_1, a_2)] \\ s[(b_1, b_2)] &= B_3 + B_6 + B_7 + B_8 + \Gamma[(b_1, b_2)] \end{aligned}$$

The condition imposed to $\ell$ in $(\star)$ allows to assure that there are no solutions of $J$, $\Delta$, in $\mathbf{L}^n - \mathbf{F}^n$ making $\ell(\Delta, v_i)$ less than zero (ie an element in $\mathbf{F}$) and thus $\Gamma[v_i] = 0$ $(1 \leq i \leq 4)$. Thus solving the linear system of equations

$$\begin{aligned} s[(a_1, a_2)] &= B_2 + B_3 + B_4 + B_7 \\ s[(a_1, b_2)] &= B_2 + B_3 + B_5 + B_6 + B_7 \\ s[(b_1, a_2)] &= B_3 + B_4 + B_5 + B_7 + B_8 \\ s[(b_1, b_2)] &= B_3 + B_6 + B_7 + B_8 \end{aligned}$$

for $B_5$, it is obtained that:

$$B_5 = \frac{1}{2}(s[(a_1, b_2)] - s[(a_1, a_2)] + s[(b_1, a_2)] - s[(b_1, b_2)])$$

as we wanted to show.

∎

The section is finished by showing how the Multivariate Sturm–Habicht Sequence can be used to determine the number of real solutions of a polynomial system of equations inside a triangle in the plane. Similar formulae can be derived for simplices but for sake of simplicity only the case of the triangle in the plane is presented.

**Theorem.** *Let $\mathcal{T}$ be a $\mathbf{K}$–triangle in $\mathbf{F}^2$ with vertices $v_1$, $v_2$ and $v_3$ and $H_i(x_1, x_2)$ $(i \in \{1, 2, 3\})$ the equation of the line defined by $v_j$ and $v_k$ with $H_i(v_i) < 0$. Let $\ell$ be the polynomial in $\mathbf{K}[\underline{x}, \underline{y}]$*

$$\ell(\underline{x}, \underline{y}) = H_1(\underline{x})H_2(\underline{x})H_3(\underline{y}) + H_1(\underline{x})H_2(\underline{y})H_3(\underline{x}) + H_1(\underline{y})H_2(\underline{x})H_3(\underline{x})$$

*verifying the following conditions with respect the set $\mathcal{V}_{\mathbf{F}}(J)$:*

- *there is no points of $\mathcal{V}_{\mathbf{F}}(J)$ on any line in $\mathbf{F}^2$ containing an edge of $\mathcal{T}$,*

$\star$ *for any $i \in \{1, 2, 3\}$:*

$$j \neq k \implies \underline{\ell(\Delta_j, v_i) \neq \ell(\Delta_k, v_i)}$$

*If we note*

$$s[v_i] = \mathbf{W}_{\mathrm{StHa}}(\mathbf{V}_{\ell,J}(U, v_i); -\infty, 0)$$

*then the following equality holds:*

$$\#(\mathcal{V}_{\mathbb{F}}(J) \cap \mathcal{T}) = \frac{s[v_1] + s[v_2] + s[v_3] - S}{2}$$

*where $S$ is the cardinal of $\mathcal{V}_{\mathbb{F}}(J)$:*

$$S = \mathbf{C}(\{\mathbf{stha}_j(\mathbf{V}_{\ell,J}(U, v_1))\}_{0 \le j \le D})$$

**Proof.** Let $A_i$ $(1 \le i \le 7)$ be one of the seven open regions in $\mathbb{F}^2$ determined by the lines $H_1 = 0$, $H_2 = 0$ and $H_3 = 0$

$$\begin{array}{ll} A_1 = [H_1 > 0, H_2 < 0, H_3 > 0] & A_2 = [H_1 > 0, H_2 < 0, H_3 < 0] \\ A_3 = [H_1 > 0, H_2 > 0, H_3 < 0] & A_4 = [H_1 < 0, H_2 < 0, H_3 > 0] \\ A_5 = [H_1 < 0, H_2 < 0, H_3 < 0] & A_6 = [H_1 < 0, H_2 > 0, H_3 < 0] \\ & A_7 = [H_1 < 0, H_2 > 0, H_3 > 0] \end{array}$$

and $B_i = \#(\mathcal{V}_{\mathbb{F}}(J) \cap A_i)$. Then the assumptions $(\bullet)$ and $(\star)$ allow to describe the integers $s[v_i]$ in the following way:

$$\begin{aligned} s[v_1] &= B_2 + B_5 + B_7 \\ s[v_2] &= B_1 + B_5 + B_6 \\ s[v_3] &= B_3 + B_4 + B_5 \end{aligned}$$

Adding these three equations, the following equality is obtained:

$$s[v_1] + s[v_2] + s[v_3] = 2B_5 + S$$

as desired since $B_5 = \#(\mathcal{V}_{\mathbb{F}}(J) \cap \mathcal{T})$.

■

One important advantage of the previous two theorems is found in the fact that once the $\ell$-Multivariate Sturm–Habicht Sequence is determined then it can be used for solving the Real Root Counting Problem in any $n$-rectangle or triangle satisfying the regularity hypothesis in such theorems. In contrast to the formulae in [Pedersen], these two theorems do not require computing a characteristic polynomial and, once the "Volume Function" is determined, the complexity of the computations is quadratic in $D$.

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