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Around real Enriques surfaces.

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Abstract

We present a brief overview of the classification of real Enriques surfaces completed recently and make an attempt to systemize the known classification results for other special types of surfaces. Emphasis is also given to the particular tools used and to the general phenomena discovered; in particular, we prove two new congruence type prohibitions on the Euler characteristic of the real part of a real algebraic surface.

ПЕТЯ. Я ворона убил.
ОЛЯ. Зачем, зачем? Кому же надо?
ПЕТЯ. Он каркал надо мной.
Велимир Хлебников. *Мирско́нца.*[1]

There was a young fellow from Clyde,
Who was once at a funeral spied,
When asked who was dead,
He smilingly said,
"I don't know. I just came for the ride."
Limerick.

1 Questions and their history

From the naïve point of view, a nonsingular real algebraic variety is just a set given in a real projective space by a nonsingular system of

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[1]Allegory in Russian; difficult to translate. Close to the limerick.

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polynomial equations with real coefficients. However, at a certain stage it becomes natural, and even necessary, to enhance and extend this notion. First, as polynomial equations also make sense over $\mathbb{C}$, one can consider the complexification. The resulting complex variety, given by the same equations in the corresponding complex projective space, is invariant under the complex conjugation involution, and the original real variety is its fixed point set. Then one can take the complexification out of the ambient space, considering it as an abstract complex analytic variety (in general, it may be singular), and thus arrive to the notion of complex analytic variety equipped with a *real structure*; the latter, by definition, is just an antiholomorphic involution, and it is this involution (and, in particular, its fixed point set) that becomes subject of the study.

In this paper we confine ourselves to dimension two and consider *nonsingular compact* (without boundary) *complex analytic surfaces with real structure.*

Note that instead of complex analytic surfaces one could as well consider algebraic surfaces over $\mathbb{C}$ endowed with a Galois involution. In both the cases a real structure on a surface X is an involutive isomorphism $c: X \to \bar{X}$. However, we prefer to deal with complex analytic manifolds as, on one hand, the analytic category is wider, and on the other hand, the problems we are interested in and the tools we are using are topological. Above all, in all the cases considered below the topology of real structures does not depend on the category chosen: all the prohibitions are of purely topological nature (and thus hold for analytic surfaces), while all the examples used to prove the completeness of these prohibitions are algebraic (and, moreover, can often be chosen even within a smaller class, like, say, algebraic surfaces of a given degree).

Apart of the main question, to study the real structure (involution) up to homeomorphism or diffeomorphism, there are several other, more visual, levels of investigation. In particular, the so-called purely real approach concerns only the topology of the real point set of the variety (i.e., the fixed point set of the real structure). The level of study being fixed, the question still can be posed in various ways. The first desire is to classify the real structures (or fixed point sets, or whatever is chosen for the subject). Then one has to confine oneself to a certain class of complex surfaces, say, one or several related deformation families.

Chronologically, the first family considered from this point of view were cubic surfaces in $\mathbb{R}p^3$, which were subjected to different classifications. Probably, the first classification taking into account the real structure was given by Schläfli [S1], who in 1858 introduced his famous 5 kinds of generic (i.e., nonsingular) cubic surfaces. It is rather difficult to believe that he had no idea about the shape of the real part of these surfaces; however, it was not until 1872 (see [S2], [S3]) that we could find in his papers any related remarks. Probably, in spite of Riemann's input, the topological setting in that time was still neither current nor respected.

Apparently, it was Klein who first explicitly posed and solved all the basic questions concerning topology of real cubic surfaces. In 1873 (see [Kl]) he showed that Schläfli's classification coincides with the topological classification of the real parts of cubic surfaces. Furthermore, he showed that the moduli space of cubic surfaces with a given topological type of the real part is connected, which in fact gives the complete topological classification of Galois involutions on cubic surfaces: two such involutions are equivalent if and only if their fixed point sets (i.e., the real parts of the surfaces) are homeomorphic.

Cubic surfaces occupy a special position among other surfaces: from the complex point of view they form one of the infinitely many components of the moduli space (or, in other words, belong to a particular deformation type) of rational surfaces. First results on the classification of general real rational surfaces were obtained by Enriques [Enr] in 1897. The classification was completed in 1912 by Comessatti (see [Co1], [Co2]), who extended Klein's results to arbitrary real rational surfaces and described the topology of the real parts for each (complex) deformation type (see Theorem 2.1 in Section 2).[2] In the late sixties Manin [M1], [M2] and Iskowskikh [Isk1]–[Isk3] put these results into the modern framework, completed some statements, gave new proofs, and generalized the results to 2-extensions of fields other than $\mathbb{R}$.

It is worth mentioning that it is due to his solution of this classification problem that Comessatti found a nontrivial bound for the number of components of a real rational surface, which he later generalized to his

[2]The description of the connected components of the moduli spaces is also contained, but, as far as we know, not explicitly stated in Comessatti's works [Co1]–[Co3]: with one exception, within one complex deformation type the moduli space of minimal real rational surfaces with a given topological type of the real part is connected.

famous estimate on the Euler characteristic: in the modern terminology this result states that the Euler characteristic of the real part of a real algebraic surface is bounded by the Hodge number $h^{1,1}$ of its complex part.

Another natural direction of developing the subject is the study of real quartics in $\mathbb{R}p^3$, which was started by Rohn and Hilbert. (Hilbert even included the corresponding questions in his famous list of problems.) After a period of relative oblivion, in the late 1960's they were made a subject of study by Utkin, who followed the approach of Rohn and Hilbert (which relates quartic surfaces to plane sextics) and used the classification of sextics just obtained by Gudkov. The topological classification of the real parts of real quartics was completed in 1976 by Kharlamov [Kh]. Once again the solution of a classification problem stipulated the discovery of new general phenomena: a series of congruences on the Euler characteristic of the real part of a real surface (Gudkov, Arnold, and Rokhlin congruences and their generalizations).

Quartic surfaces also belong to a special class: they are all so-called $K3$-surfaces. Complex analytic $K3$-surfaces form a connected moduli space, where quartics constitute a connected subspace. From the differential point of view all the complex $K3$-surfaces are diffeomorphic to one another. In fact, the topological classification of the real parts of general $K3$-surfaces coincides with and follows from the topological classification of the real parts of real quartics; moreover, the final answer is the same for all $K3$-surfaces, algebraic $K3$-surfaces, quartic surfaces in P^3, and hyperelliptic $K3$-surfaces, i.e., double plains branched over curves of degree 6 (and, in fact, for $K3$-surfaces embedded to P^N with a given degree). Note that unlike the two other classes considered in this paper (i.e., rational and Enriques surfaces) a $K3$-surface may be nonalgebraic, although all $K3$-surfaces are Kähler.

More advanced classification of real $K3$-surfaces was done in 1979 by Nikulin [N1], who found and rather explicitly described the connected components of the moduli space. According to Nikulin, two real $K3$-surfaces belong to one component if and only if their Galois involutions are topologically equivalent, and the action of the Galois involution is determined up to diffeomorphism by some simple numerical topological invariants.

Following the Enriques classification of complex algebraic surfaces, there remains only five special classes of surfaces: abelian surfaces, surfaces with a pencil of rational curves, hyperelliptic surfaces, surfaces with a pencil of elliptic curves of canonical (Kodaira) dimension 1, and Enriques surfaces.

Abelian surfaces were classified by Comessatti (see [Co3] for the precise statements and further references). Some results on the topology of hyperelliptic surfaces and real surfaces with a real pencil of rational curves and the classification of singular fibres of real pencils of elliptic curves were obtained by Silhol [Si].

The topological classification of the real parts of real Enriques surfaces, as well as of some canonical structures that they inherit from the complexification, was started by V. Nikulin [N2] and recently completed by the authors [DK1], [DK2]. Similar to what happened during the investigation of other special classes of surfaces, as a by-product of this study we discovered some new topological properties of the Galois involution. The purpose of this paper is to present these results, with an emphasis on the relatively new tools applied and the veritable information which they give about surfaces more general than the Enriques surfaces.

The paper is organized as follows: In §2 we cite some results which answer some of the questions posed above. In §3 we present a specific tool which we used to classify real Enriques surfaces and state some of its properties (see [DK2]). This tool, so called *Kalinin's spectral sequence*, which we know mainly due to O. Viro and I. Kalinin, unfortunately is not widely known to the specialists in real algebraic geometry and, in view of its general nature, rather belongs to topology of periodic transformation groups. In §4 we prove two new results on topology of real algebraic surfaces, which, on one hand, were originated by the classification of real Enriques surfaces and, on the other hand, illustrate applications of Kalinin's spectral sequence.

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2 Some answers

Below, when describing the topological type of the real part of a surface, we denote by S_g the orientable surface of genus g and by $V_q = \#_q\mathbb{R}p^2$, the nonorientable surface of genus q. We use any of $S = S_0 = V_0$ for the 2-sphere.

We start with reproducing Comessatti's result on the classification of minimal real rational surfaces.

2.1. Theorem (Comessatti [Co1–Co3]). *Each minimal real rational surface is one of the following:*

(1) *real projective plane* P^2*:* $\mathbb{R}X = V_1$;

(2) *real quadric* $P^1 \times P^1$*: there are four types:* S, S_1, *and two nonequivalent empty surfaces;*

(3) *ruled rational surfaces* Σ_m, $m \geq 2$*:*
m even: $\mathbb{R}X = \varnothing$ *or* S_1, *m odd:* $\mathbb{R}X = V_2$*;*

(4) *real conic bundles over* P^1 *whose reducible fibers are all real and consist of pairs of complex conjugated exceptional curves:* $\mathbb{R}X = mS$, *where* $2m \geq 4$ *is the number of reducible fibers;*

(5) *Del Pezzo surfaces of degree* $d = K^2 = 1$ *or* 2*:*
$d = 1$*:* $\mathbb{R}X = V_1 \sqcup 4S$, $d = 2$*:* $\mathbb{R}X = 3S$ *or* $4S$.

Remark. The two nonisotopic real structures on $X = P^1 \times P^1$ with $\mathbb{R}X = \varnothing$ is the exception mentioned in the introduction.

Remark. The Del Pezzo surface of degree 2 with $\mathbb{R}X = 3S$ can also be represented as a conic bundle over P^1 with six reducible fibers.

In order to state other results, we need the following notion:

2.2. Definition. *A Morse simplification is a Morse surgery which decreases the total Betti number, i.e., either removes a spherical component* ($S \to \emptyset$) *or contracts a handle* ($S_{g+1} \to S_g$ *or* $V_{p+2} \to V_p$)*. A particular complex deformation family being fixed, a topological type (i.e., a class of surfaces with homeomorphic real parts) is called extremal if it cannot be obtained from another topological type by a Morse simplification.*

Remark. Note that a Morse simplification may not correspond to a Morse simplification in a continuous family of complex surfaces. As a result, the notions of extremal topological type and extremal (in the obvious sense) surface may be different. E.g., according to Viro and Kharlamov [Vi], any surface whose real part is mod 2 homologous to zero in the complexification is extremal, though it may have nonextremal topological type.

In order to illustrate this notion we list all (not only minimal) topological types of Del Pezzo surfaces of degree 1 and 2. (Certainly, this result follows immediately from Comessatti's classification).

2.3. Theorem. *The topological types of the real parts of Del Pezzo surfaces of degree $d = 1$ and 2 are those (and only those) which may be obtained by a series of Morse simplifications from the following extremal types:*

$$d = 1: \quad V_9,\ V_3 \sqcup S,\ V_2 \sqcup V_1, \text{ and } V_1 \sqcup 4S;$$
$$d = 2: \quad V_8,\ V_2 \sqcup S,\ V_1 \sqcup V_1, 4S, \text{ and } S_1.$$

Finally, we list the topological types of the real parts of real $K3$- and Enriques surfaces.

2.4. Theorem (Kharlamov [Kh]). *The topological types of the real parts of $K3$-surfaces are those and only those which may be obtained by a series of Morse simplifications from the following extremal types:*

(1) *M-surfaces:* $S_{10} \sqcup S$, $S_6 \sqcup 5S$, $S_2 \sqcup 9S$;

(2) *$(M-2)$-surfaces:* $S_7 \sqcup 2S$, $S_3 \sqcup 6S$;

(3) *Pair of tori:* $2S_1$.

2.5. Theorem (see [DK1]). *There are 87 topological types of real Enriques surfaces. Each of them can be obtained by a sequence of Morse simplifications from one of the 22 extremal types listed below. Conversely, with the exception of the two types $6S$ and $S_1 \sqcup 5S$, any topological type obtained in this way is realized by a real Enriques surface.*

The 22 extremal types are:

(1) *M-surfaces:*

(a) $\chi(E_{\mathbb{R}}) = 8$	(b) $\chi(E_{\mathbb{R}}) = -8$
$4V_1 \sqcup 2S$,	$V_{11} \sqcup V_1$,
$V_2 \sqcup 2V_1 \sqcup 3S$,	$V_{10} \sqcup V_2$,
$V_3 \sqcup V_1 \sqcup 4S$,	$V_9 \sqcup V_3$,
$2V_2 \sqcup 4S$,	$V_8 \sqcup V_4$,
$V_4 \sqcup 5S$,	$V_7 \sqcup V_5$,
$V_2 \sqcup S_1 \sqcup 4S$,	$2V_6$,
	$V_{10} \sqcup S_1$;

(2) *$(M-2)$-surfaces with $\chi(E_{\mathbb{R}}) = 0$:*

$V_4 \sqcup 2V_1$,	$V_5 \sqcup V_1 \sqcup S$,
$V_3 \sqcup V_2 \sqcup V_1$,	$V_4 \sqcup V_2 \sqcup S$,
$V_6 \sqcup 2S$,	$2V_3 \sqcup S$,
$V_4 \sqcup S_1 \sqcup S$,	$2V_2 \sqcup S_1$;

(3) *Pair of tori:* $2S_1$.

Remark. Nikulin's classification of real $K3$-surfaces contains the following result: a real $K3$-surface X is determined up to equivariant diffeomorphism by the topological type of its real part $\mathbb{R}X$ and its *type* (i.e., whether the fundamental class $[\mathbb{R}X]$ is or is not homologous to zero in $H_2(\mathbb{C}X; \mathbb{Z}/2)$).

Since the fundamental group of a complex Enriques surface $\mathbb{C}E$ is $\mathbb{Z}/2$, its real part inherits an interesting additional structure: the set of its connected components naturally splits into two halves, $\mathbb{R}E = \mathbb{R}E^{(1)} \cup \mathbb{R}E^{(2)}$. Each half is covered by the real part of one of the two real structures on the covering $K3$-surface. The study of this decomposition was started by V. Nikulin [N2] as part of his attempt to classify real Enriques surfaces. The complete classification of triads $(\mathbb{R}E; \mathbb{R}E^{(1)}, \mathbb{R}E^{(2)})$ up to homeomorphism is given in [DK2]:

2.6. Theorem. *Each half of a real Enriques surface may be either S_1, or $2V_2$, or $\alpha V_g \sqcup aV_1 \sqcup bS$, $g > 1$, $a \geq 0$, $b \geq 0$, $\alpha = 0, 1$. With the exception of the types kS and $V_{2r} \sqcup kS$ any decomposition into halves satisfying the above condition is realizable. The exceptional topological* *types admit only the distributions listed in Figure 1.*

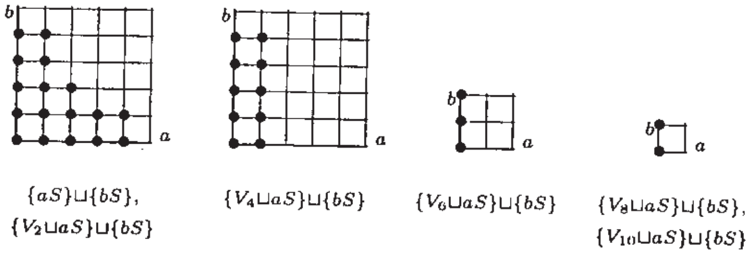

Figure 1. Exceptional topological types

Remark *(added in proof).* At present we have completed the classification of real Enriques surfaces up to deformation equivalence (see [DK3] for a preliminary report; two real Enriques surfaces are called deformation equivalent if they can be included into a one-parameter family of real Enriques surfaces). The principal result states that *the deformation type of a real Enriques surface E is determined by the topological type of its real structure.* In addition to such classical invariants as the topological type of the real part $\mathbb{R}E$, the above decomposition into halves, and the *type* (i.e., whether the fundamental classes $[\mathbb{R}E]$, $[\mathbb{R}E^{(1)}]$, and $[\mathbb{R}E^{(2)}]$ are homologous to zero, the Stiefel-Whitney class $w_2(\mathbb{C}E)$, or none of these in $H_2(\mathbb{C}E;\mathbb{Z}/2)$ or in the homology of certain auxiliary manifolds), a new invariant, so called *Pontrjagin-Viro form,* is necessary to distinguish between certain M-surfaces. As a by-product, the solution of this problem brings together such classical objects as cubic surfaces in P^3, intersections of quadrics in P^4, quartic curves in P^2, and order 6 curves in a quadric cone.

3 Tools

In this section we introduce our primary tool—so called *Kalinin's spectral sequence*, which was originally constructed by I. Kalinin [Ka] as a stabilized version of the Borel-Serre spectral sequence for equivariant cohomology. This sequence starts at the homology $H_*(Y)$ of a topological space Y with involution c and converges to the total homology $H_*(\operatorname{Fix} c)$ of the fixed point set of c. (Unless stated explicitly, *all the homology and cohomology groups have coefficients* $\mathbb{Z}/2$.) The resulting filtration $\mathcal{F}^*$ on $H_*(\operatorname{Fix} c)$ and the isomorphisms bv_p between the limit term of the spectral sequence and $\operatorname{Gr}_{\mathcal{F}} H_*(\operatorname{Fix} c)$ were discovered by O. Viro geometrically before Kalinin's work and were primarily related to the Smith exact sequence. As it is shown in [DK2], Kalinin's spectral sequence can be derived from the Smith exact sequence as well.

Below we give a geometrical description of Kalinin's spectral sequence and Viro homomorphisms bv_* and state their main properties. Proofs of these results and/or further references can be found in [DK2]. Since homology groups are more transparent and easier to deal with, we decided to use the homology language (though we cannot certainly help mentioning cohomology when speaking about multiplications and Poincaré duality). As our approach is geometrical, we have to appeal to the notion of chain. Depending on the nature of Y, one may work with singular, simplicial, smooth, or any other kind of chains considered in algebraic topology. To assure convergence of the spectral sequence, Y must satisfy certain conditions, which, strictly speaking, depend on the homology theory chosen (e.g., sheaf theories and locally compact finite dimensional spaces). However, in this paper all the results are applied to the best possible topological spaces—smooth manifolds, so they do not depend on this choice.

Thus, let us fix a good (see above) topological space Y with involution c and denote by $\operatorname{Fix} c$ the fixed point set of c. Consider the partial homomorphisms $\operatorname{bv}_r : H_*(\operatorname{Fix} c) \dashrightarrow H_r(Y)$ and the $\mathbb{Z}$-graded spectral sequence $({}^rH_*, {}^rd_*)$ defined as follows:

(1) bv_0 is zero on $H_{\geq 1}(\operatorname{Fix} c)$ and its restriction to $H_0(\operatorname{Fix} c)$ coincides with the inclusion homomorphism;

(2) bv_p is defined on a (nonhomogeneous) element $x \in H_*(\operatorname{Fix} c)$ represented by a cycle $\sum x_i$ (where x_i is the i-dimensional component

of x) if and only if there exist some chains y_i in Y, $1 \le i \le p$, so that $\partial y_1 = x_0$ and $\partial y_{i+1} = x_i + (1+c_*)y_i$ for $i \ge 1$. In this case $\mathrm{bv}_p x$ is represented by the class of $x_p + (1+c_*)y_p$ in $H_p(Y)$;

(3) $H^1_* = H_*(Y)$ and ${}^1d_* = 1 + c_*$;

(4) ${}^r d_p$, considered as a partial homomorphism $H_p(Y) \dashrightarrow H_{p+r-1}(Y)$, is defined on a cycle x_p in Y if and only if there are some chains $y_p = x_p, y_{p+1}, \dots, y_{p+r-1}$ so that $\partial y_{i+1} = (1+c_*)y_i$. In this case ${}^r d_p x_p = (1+c_*)y_{p+r-1}$.

3.1. Theorem. *The homomorphisms* bv_* *and spectral sequence* $({}^rH_*, {}^rd_*)$ *are natural with respect to equivariant maps. Furthermore,* ${}^rH_*$ *and* ${}^rd_*$ *do form a spectral sequence (i.e.,* ${}^rd_p$ *are well defined homomorphisms* ${}^rH_p \to {}^rH_{p+r-1}$ *and* ${}^{r+1}H_p = \operatorname{Ker} {}^rd_p / \operatorname{Im} {}^rd_{p-r+1}$*), and this sequence converges to* $H_*(\operatorname{Fix} c)$ *via* bv_**, i.e.,* bv_p *induces an (honest) isomorphism* $\mathcal{F}^p / \mathcal{F}^{p+1} \to {}^\infty H_p$*, where* $\mathcal{F}^p = \operatorname{Domain} \mathrm{bv}_p = \operatorname{Ker} \mathrm{bv}_{p-1}$.

There is an obvious cohomology version ${}^rH^* \Rightarrow H^*(\operatorname{Fix} c)$ of the spectral sequence, which is dual to the homology one. The cup- and cap-products in Y naturally extend to, respectively, a $\mathbb{Z}/2$-algebra structure in ${}^rH^*$ and ${}^rH^*$-module structure in ${}^rH_*$. If Y is a connected N-manifold and $\operatorname{Fix} c \ne \emptyset$, then the fundamental class $[Y]$ survives to ${}^\infty H_N$ and the multiplication $\cap[Y] : {}^rH^p \to {}^rH_{N-p}$ is an isomorphism (Poincaré duality), which, in the usual way, defines homology intersection pairing $\circ : {}^rH_p \otimes {}^rH_q \to {}^rH_{p+q-N}$. The relation between this pairing and the ordinary intersection pairing in $\operatorname{Fix} c$ is given by the following statement:

3.2. Theorem. *Let Y be a smooth closed N-dimensional manifold with a smooth involution $c : Y \to Y$, and let $F = \operatorname{Fix} c$ be the fixed point set of c. Denote by $w(\nu)$ the total Stiefel-Whitney class of the normal bundle ν of F in Y. Then for $a \in \mathcal{F}^p$ and $b \in \mathcal{F}^q$ one has $w(\nu) \cap (a \circ b) \in \mathcal{F}^{p+q-N}$, and*

$$\mathrm{bv}_p a \circ \mathrm{bv}_q b = \mathrm{bv}_{p+q-N}[w(\nu) \cap (a \circ b)].$$

The (homology) Steenrod operations $\mathrm{Sq}_t : H_*(Y) \to H_{*-t}(Y)$ also extend to ${}^rH_*$. In order to describe their relation to the ordinary Steenrod operations in $H_*(\operatorname{Fix} c)$, let us introduce the *weighted Steenrod operations* $\widehat{\operatorname{Sq}}_t x = \sum_{0\le j\le t} \binom{P-p-t+j}{t-j} \operatorname{Sq}_j x$, where $x \in H_p(Y)$ and $P > p+2t$ is a power of 2. (The binomial coefficients do not depend on P, see, e.g., Lemma I.2.6 in [SE].) Then one has:

3.3. Theorem. *If $x \in \mathcal{F}^p$ and $t \ge 0$, then $\widehat{\operatorname{Sq}}_t x \in \mathcal{F}^{p-t}$ and*

$$\operatorname{Sq}_t \operatorname{bv}_p x = \operatorname{bv}_{p-t} \widehat{\operatorname{Sq}}_t x.$$

We conclude this section with the description of Viro homomorphisms (in dimensions up to 2) in the case when Y is a real algebraic surface and c is the real structure. Let $C_1, C_2, \dots, C_k$ be the components of $\mathbb{R}Y$. Denote by $\langle C_i\rangle$ and $[C_i]$ their classes in $H_0(\mathbb{R}Y)$ and fundamental classes in $H_2(\mathbb{R}Y)$ respectively. Then the values $\operatorname{bv}_0\langle C_i\rangle$, $\operatorname{bv}_1 \alpha$ (with $\alpha \in H_1(\mathbb{R}Y)$), and $\operatorname{bv}_2[C_i]$ are always well defined and coincide, essentially, with the inclusion homomorphisms. The value $\operatorname{bv}_1\langle C_i - C_j\rangle$ is also well defined and is represented by the equivariant circle $(1+c_*)y_1$, where y_1 is an arc in $\mathbb{C}Y$ connecting a point in C_i with a point in C_j. If, under some appropriate choice of y_1, this circle is homologous to zero, i.e., $(1+c_*)y_1 = \partial y_2$, then $\operatorname{bv}_1\langle C_i - C_j\rangle = 0$ and $(1+c_*)y_2$ represents $\operatorname{bv}_2\langle C_i - C_j\rangle$. Similarly, if $\operatorname{bv}_1 \alpha = 0$, i.e., $\alpha = (1+c_*)y_1$ for some cycle y_1 in $\mathbb{C}Y$, then there exists a chain y_2 in $\mathbb{C}Y$ such that $\partial y_2 = \alpha + (1+c_*)y_1$, and $(1+c_*)y_2$ represents $\operatorname{bv}_2 \alpha$. Finally, if $\operatorname{bv}_1 \alpha = \operatorname{bv}_1\langle C_i - C_j\rangle$, then $\operatorname{bv}_2(\alpha + \langle C_i - C_j\rangle)$ is defined and is represented by $(1+c_*)y_2$, where $\partial y_2 = \alpha + (1+c_*)y_1$ and y_1 is an appropriately chosen arc connecting two points in C_i and C_j.

Elements of the form $\operatorname{bv}_2[C_i]$, $\operatorname{bv}_2\langle C_i - C_j\rangle$, $\operatorname{bv}_2 \alpha$, and $\operatorname{bv}_2(\alpha + \langle C_i - C_j\rangle)$ span ${}^\infty H_2(\mathbb{C}Y)$ (i.e., ${}^\infty H_2(\mathbb{C}Y)$ consists of their linear combinations $\sum \operatorname{bv}_2 x_i$; with an abuse of language we let $\sum \operatorname{bv}_2 x_j = \operatorname{bv}_2 \sum x_j$ provided that the latter is well defined, even if the summands are not well defined). According to Theorem 3.2, Kalinin's intersection form on ${}^\infty H_2(\mathbb{C}Y)$ is the one given by Table 1. In the table, the intersection $\alpha \circ \beta$ is regarded as an element of $H_0(Y_{\mathbb{R}})$, and $(\alpha\circ\beta)[\mathbb{R}Y]$ and $(\alpha\circ\beta)[C_i]$ are, respectively, the total intersection number and its portion falling into C_i. δ_{ij} stands for the Kronecker symbol: $\delta_{ii} = 1$ and $\delta_{ij} = 0$ if $i \ne j$. The intersection form extends linearly to the classes $\operatorname{bv}_2(\alpha + \operatorname{bv}_2\langle C_i - C_j\rangle)$, as if $\operatorname{bv}_2 \alpha$ and $\operatorname{bv}_2\langle C_i - C_j\rangle$ were well defined.

	$\mathrm{bv}_2\langle C_i - C_j\rangle$	$\mathrm{bv}_2\,\alpha$	$\mathrm{bv}_2[C_i]$
$\mathrm{bv}_2\langle C_k - C_l\rangle$	0	0	$\delta_{ik} + \delta_{il}$
$\mathrm{bv}_2\,\beta$	0	$(\alpha \circ \beta)[\mathbb{R}Y]$	$(\beta \circ \beta)[C_i]$
$\mathrm{bv}_2[C_k]$	$\delta_{ik} + \delta_{jk}$	$(\alpha \circ \alpha)[C_k]$	$\delta[ik]\chi(C_i)$

Table 1

The Bockstein homomorphism $H_2(\mathbb{C}Y) \to H_1(\mathbb{C}Y)$ descends to the term ${}^{\infty}H$ and coincides with $\mathrm{Sq}_1 : {}^{\infty}H_2(\mathbb{C}Y) \to {}^{\infty}H_1(\mathbb{C}Y)$; it is given by Theorem 3.3: $\mathrm{Sq}_1\,\mathrm{bv}_2[C_i] = \mathrm{bv}_1\,w_1(C_i)$ and $\mathrm{Sq}_1\,\mathrm{bv}_2(\alpha + \langle C_i - C_j\rangle) = \mathrm{bv}_1\,\alpha$. (Here and below $w_i(X)$ stands for the homology class dual to the i-th Stiefel-Whitney class of a manifold X.)

4 Other results

The classification results cited in Section 2 can be considered as an advanced experimental part of the study of the topology of real algebraic surfaces. As any experiment, it serves not only to confirm the applicability of a certain general theory but also to help to discover new phenomena. Several examples have already been mentioned in Section 1: the classification of real rational surfaces (Theorem 2.1) led Comessatti to his inequality on the Euler characteristic, and the Arnold-Gudkov-Rokhlin congruence [Ro] was first observed for real $K3$-surfaces (Statement (1) of Theorem 2.4).[3] The classification of real Enriques surfaces also gives considerable material for observations. Even a glance at the list of extremal types given by Theorem 2.5 and at the complete list generated from it reveals several regularities. One can notice, for example, that all the M-surfaces are nonorientable and that the orientable $(M-2)$-surfaces (appearing in the derived list) satisfy the same congruences as M-surfaces. In [DK2] we established (and made use of) several general results of this kind. (References to other related results known in the literature are also found in [DK2].) Below we suggest a slightly different

[3] More precisely, the congruence was conjectured by D. Gudkov based on his classification of plane sextics, which are closely related to $K3$-surfaces. Note that the experimental material known to Ragsdale and Comessatti could already reveal some congruences, but they both did not notice them and put attention only to the inequalities.

interpretation and generalization of this phenomenon. We hope that for a good observer the classification of real Enriques surfaces may provide materiel for other discoveries.

4.1. Theorem. *Let X be a compact (without boundary) complex analytic surface with real structure, and let $w_2(\mathbb{C}X) = 0$. Then:*

(1) *if $H_1(\mathbb{C}X) = 0$, then $\mathbb{R}X$ is orientable;*

(2) *if $H_1(\mathbb{C}X;\mathbb{Q}) = 0$ and $\mathbb{R}X$ is nonorientable, then X is an $(M-d)$-surface, $d \geq 2$, and*

   (a) *if $d = 2$, then $\chi(\mathbb{R}X) \equiv \sigma(\mathbb{C}X) \pmod{16}$;*

   (b) *if $d = 3$, then $\chi(\mathbb{R}X) \equiv \sigma(\mathbb{C}X) \pm 2 \pmod{16}$;*

   (c) *if $d = 4$ and $\chi(\mathbb{R}X) \equiv \sigma(\mathbb{C}X)+8 \pmod{16}$, then the image of $[\mathbb{R}X]$ belongs to $\operatorname{Tors} H_2(\mathbb{C}X;\mathbb{Z}) \otimes \mathbb{Z}/2 \subset H_2(\mathbb{C}X)$.*

4.2. Theorem. *Let X be a compact (without boundary) complex analytic surface with real structure, and let $w_2(\mathbb{C}X) \neq 0$. Then:*

(1) *if X is an M-surface, then $\mathbb{R}X$ is nonorientable;*

(2) *if, besides, $w_2(\mathbb{C}X) \in \operatorname{Tors} H_2(\mathbb{C}X;\mathbb{Z}) \otimes \mathbb{Z}/2$ and $\mathbb{R}X$ is orientable, then X is an $(M-d)$-surface, $d \geq 2$, and*

   (a) *if $d = 2$, then $\chi(\mathbb{R}X) \equiv \sigma(\mathbb{C}X) \pmod{16}$;*

   (b) *if $d = 3$, then $\chi(\mathbb{R}X) \equiv \sigma(\mathbb{C}X) \pm 2 \pmod{16}$;*

   (c) *if $d = 4$ and $\chi(\mathbb{R}X) \equiv \sigma(\mathbb{C}X)+8 \pmod{16}$, then the image of $[\mathbb{R}X]$ belongs to $\operatorname{Tors} H_2(\mathbb{C}X;\mathbb{Z}) \otimes \mathbb{Z}/2 \subset H_2(\mathbb{C}X)$.*

The proof of the congruence part (Statement (2)) of Theorems 4.1 and 4.2 is similar to that of the well known Arnold-Gudkov-Rokhlin type congruences. Let $\operatorname{Discr} H^+$ be the discriminant form of the lattice H^+ of the c_*-invariant elements of $H_2(\mathbb{C}X;\mathbb{Z})/\operatorname{Tors}$. Then in both the theorems it suffices to prove that, under the hypotheses, in (a), (b), and (c) one has $\dim \operatorname{Discr} H^+ = 0$, 1, and ≤ 2 respectively. This, in turn, would follow from the fact that either $\dim {}^\infty H_1(\mathbb{C}X) < \dim H_1(\mathbb{C}X)$ or portion of $\operatorname{Tors} H_2(\mathbb{C}X;\mathbb{Z}) \otimes \mathbb{Z}/2 \subset H_2(\mathbb{C}X)$ dies in ${}^\infty H_2(\mathbb{C}X)$. (We address an interested reader to [DK2, §6].) It is this assertion that is actually proved below.

Proof of Theorem 4.1. (1) If there is an element $\alpha \in H_1(\mathbb{R}X)$ with $\alpha^2 = 1$, then $(\mathrm{bv}_2\,\alpha)^2 = 1$. ($\mathrm{bv}_2\,\alpha$ is well defined since $H_1(\mathbb{C}X) = 0$.) This contradicts to the assumption that the intersection form in $H_2(\mathbb{C}X)$ and, hence, in ${}^\infty H_2(\mathbb{C}X)$ is even.

(2) By assumption, there is an element $\alpha \in H_1(\mathbb{R}X)$ with $\alpha^2 = 1$. Similar to (1) one concludes that for any such element $\mathrm{bv}_1\,\alpha \neq \mathrm{bv}_1\,C$ for any $C \in H_0(\mathbb{R}X)$ with $\mathrm{bv}_0\,C = 0$. (In particular, $\mathrm{bv}_1\,\alpha \neq 0$.) Furthermore, any nonorientable component C_i of $\mathbb{R}X$ is of even genus, i.e., $w_1^2(C_i) = 0$. Now it is easy to see that $\mathrm{bv}_1\,\alpha$ does not belong to the image of $\mathrm{Sq}_1 \circ \mathrm{bv}_2$. On the other hand, $\mathrm{Sq}_1 : H_2(CX)/H_2(\mathbb{C}X;\mathbb{Z}) \otimes \mathbb{Z}/2 \to \mathrm{Tors}_2\,H_1(\mathbb{C}Y;\mathbb{Z}) = H_1(\mathbb{C}Y)$ is an isomorphism; hence, $\mathrm{Im}\,\mathrm{bv}_2$ does not cover $H_2(\mathbb{C}X)/H_2(\mathbb{C}X;\mathbb{Z}) \otimes \mathbb{Z}/2$.

■

Proof of Theorem 4.2. (1) If $\mathbb{R}X$ is orientable, there is no element $\alpha \in H_*(\mathbb{R}X)$ with $(\mathrm{bv}_*\,\alpha)^2 = 1$; hence, $w_2(\mathbb{C}X)$ dies in ${}^\infty H_2(\mathbb{C}X)$ and the surface is not maximal.

(2) According to (1), portion of Tors $H_2(\mathbb{C}X;\mathbb{Z})\otimes\mathbb{Z}/2$ (at least $w_2(\mathbb{C}X)$) dies in ${}^\infty H_2(\mathbb{C}X)$.

■

References

[Co1] A. Comessatti, *Fondamenti per la geometria sopra le superficie razionali dal punto di vista reale*, Math. Annalen **43** (1912), 1-72.

[Co2] ——, *Sur la connessione delle superficie razionali reale*, Annali di Matematica **23** (1914), 215–285.

[Co3] ——, *Reele Fragen in der algebraischen Geometrie*, Jahresbericht d. Deut. Math. Vereinigung **41** (1932), 107–134.

[DK1] A. Degtyarev, V. Kharlamov, *Topological classification of real Enriques surfaces*, Topology **35** (1996), no. 3, 711–729.

[DK2] ——, *Halves of a Real Enriques Surface*, appearing in Comm. Math. Helv.; Current (extended) version: *Distribution of the components of a real Enriques surface*, Preprint of the Max-Planck Institute, MPI/95-58, 1995 (also available from AMS server as amspps # 199507-14-005).

[DK3] ——, *On the moduli space of real Enriques surfaces*, appearing in C.R. Acad. Sci. Paris.

[Enr] F. Enriques, *Sulle irrazionalita da cui puo farsi dipendere la resoluzione d'un equazione algebrica $f(x, y, z) = 0$ con funzione die due parametri*, Math. Ann. **49** (1897), 1–23.

[Is1] V. A. Iskovskikh, *On birational models of rational surfaces*, Izv. Acad. Nauk SSSR **29** (1965), 1417–1433.

[Is2] ——, *Rational surfaces with a pencil of rational curves*, Mat. Sb. **74** (1967), 608–638; English transl. in Math. USSR - Sb **3** (1967), 563–587.

[Is3] ——, *Minimal models of rational surfaces over arbitrary fields*, Izv. Acad. Nauk SSSR **43** (1979), 19–43; English transl. in Math. USSR - Izv **14** (1980), 17–39.

[Ka] I. Kalinin, *Cohomology characteristics of real algebraic hypersurfaces*, Algebra i Analiz **3** (1991), no. 2, 91–110 (Russian); English transl. in St. Petersburg Math. J. **3** (1992), no. 2, 313–332.

[Kh] V. M. Kharlamov, *The topological type of nonsingular surfaces in $\mathbb{R}p^3$ of degree 4*, Funktsional. Anal. i Prilozhen. **10** (1976), no. 4, 55–68 (Russian); English transl. in Functional Anal. Appl. **10** (1976), no. 4, 295–305.

[Kl] F. Klein, *Ueber Flächen dritter Ordnung*, Math. Ann. **6** (1873), 551–581.

[M1] Yu. I. Manin, *Rational surfaces over perfect fields*, Publ. Math. de l'I.H.E.S. **30** (1966), 55–113.

[M2] ——, *Rational surfaces over perfect fields*. II, Mat. Sb. **72** (1967), 161–192; English transl. in Math. USSR - Sb 1 (1967), 141–168.

[N1] V. V. Nikulin, *Integer quadratic forms and some of their geometrical applications*, Izv. Akad. Nauk SSSR, Ser. Mat. **43** (1979), 111–177 (Russian); English transl. in Math. USSR - Izv. **43** (1979), 103–167.

[N2] ——, *On the topological classification of real Enriques surfaces.* I, Amer. Math. Transl. (2) **173** (1996), 187–202.

[Ro] V. A. Rokhlin, *Congruences modulo 16 in the sixteenth Hilbert problem*, Funktsional. Anal. i Prilozhen. **6** (1972), no. 4, 58–64 (Russian); English transl. in Functional Anal. Appl. **6** (1972), no. 4, 301–306.

[S1] L. Schläfli, *On the distribution of surfaces of the third order into species, in reference to the absence or presence of singular points, and the reality of their lines*, Phil. Trans. Roy. Soc. London **153** (1863), 193–241.

[S2] ——, *Quand'è che dalla superficie generale di terzo ordine si stacca una parte che non sia realmente segata da ogni piano reale?*, Ann. Mat. Pura Appl., Serie II **5** (1871/73), 289–295.

[S3] ——, *Correzione alla Memoria intitolata: Quand'è che dalla superficie generale di terzo ordine si stacca un pezzo rientrante?*, Ann. Mat. Pura Appl., Serie II **7** (1875/76), 193–196.

[Si] R. Silhol, *Real Algebraic Surfaces*, Lecture Notes in Math. **1392** (1989).

[SE] N. E. Steenrod, D. B. A. Epstein, *Cohomology operations*, Princeton University Press, Princeton, 1962.

[Vi] O. Ya. Viro, *Progress in the topology of real algebraic varieties over the last six years*, Uspekhi Mat. Nauk **41** (1986),45–67 (Russian); English transl. in Russian Math. Surveys **41** (1986), 55–82.

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