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Inner metric properties of 2-dimensional semi-algebraic sets.

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Abstract

We consider 2- dimensional semialgebraic topological manifolds from the differentialgeometric point of view. Curvatures at singularities are defined and a Gauss-Bonnet formula holds. Moreover, Aleksandrov's axioms for an intrinsic geometry of surfaces are fullfilled.

1 Introduction

A metric d on a space X is called inner metric, if for all $x, y \in X$ one has $d(x,y) = \inf(l(\gamma))$. Here the infimum is taken over all rectificable curves γ connecting x with y and $l(\gamma)$ is the length of γ (we set $d(x,y) = \infty$ if x and y are in different path components). We want to study semialgebraic sets from the inner metric point of view, and to start with, we consider sets of pure dimension 2. We will consider curvatures along 1-dimensional singular subsets and at very singular points. Let us give some examples:

Example 1.1. Consider the surface Q of the standard cube, being imbedded in $\mathbb{R}^3$. Then Q is locally euclidean at all points except for the 8 vertices (even at the edges). The total angle at each vertex is $\frac{3}{2}\pi$. So we count a curvature $\frac{1}{2}\pi$, concentrated at each vertex. For the total curvature we get $8 \cdot \frac{1}{2}\pi = 4\pi = 2\pi\chi(Q)$ where χ is the Euler characteristic.

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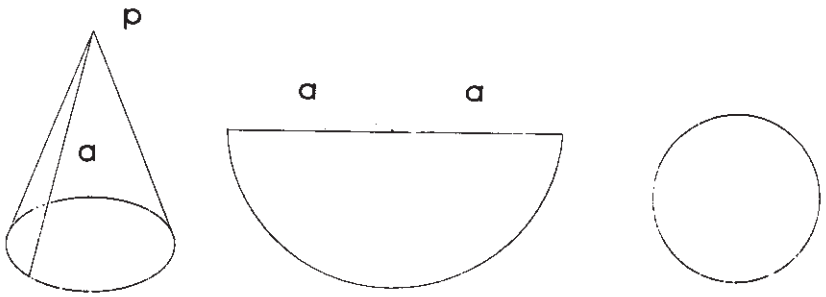
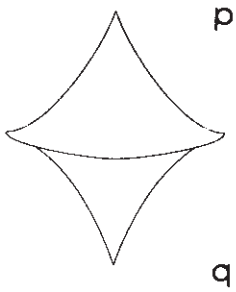
Example 1.2. Consider a 2-dimensional cone $C \subset \mathbb{R}^3$ with the bottom as in the figure below which one gets by gluing, say, a unit disc and a half disc of radius 2 along the circle and halfcircle and the edges a respectively. This time we get a total angle of π at p, hence the curvature π, and along the 1-dimensional singular edge we get the curvature $\int_0^{2\pi}(\frac{1}{2}+1)dt = 3\pi$. At the smooth points C is euclidean. So, as total curvature, we get again $4\pi = 2\pi\chi(C)$.



Example 1.3. Consider in $\mathbb{R}^3$ the surfaces

$$S = \{x \in \mathbb{R}^3 | x_3^2 = (1 - \sqrt{x_1^2 + x_2^2})^3, 0 \leq x_1^2 + x_2^2 \leq 1\}$$



Here the total angle at the points p and q is $\frac{4}{\sqrt{13}}\pi$, the total curvature along the singular edge S^1 cannot be computed by usual formulas for curvatures of curves in Riemannian 2-manifolds, since one cannot differentiate sufficiently often. However, it will turn out, that the curvature along S^1 is $\int_0^{2\pi} 2dt = 4\pi$. The curvature at the smooth points is negative.

In general, for any compact semi-algebraic 2-dimensional oriented manifold S we expect a formula like

$$\int_{S^0} K(x)dx + \int_{S^1} k(t)dt + \sum_{i=1}^{r} \kappa(p_i) = 2\pi\chi(S),$$

where S^0 is the non-singular part of S with Gauss-curvature K and canonical measure dx, k is the curvature along the 1-dimensional singular locus and κ is the curvature, which is concentrated at finitetly many very singular points $p_1, \ldots, p_r$. We are going to explain, how k and κ can be defined. We will also investigate, how this fits into Aleksandrovs surface theory [A-Z], and thus find a new class of models, for which Aleksandrovs axioms hold.

2 Semi-algebraic geometry

Let $S \subset \mathbf{R}^n$ be a smooth semi-algebraic 2-manifold, endoved with the induced Riemannian metric. As before, we denote by K the Gauß-curvature and by dx the canonical measure on S.

Proposition 2.1. $\int_S |K(x)|dx < \infty$

In fact, much more is known: If $S^d \subset \mathbf{R}^n$ is a smooth semi-algebraic d-manifold, then $\int_S |\omega|$ is bounded by a function, which only depends on the complexity of a description for S, that is on d, n and the number and degrees of the polynomials, which describe S [Eu]. Here ω is the Gauß-Bonnet form according to the induced Riemannian metric. Note that $|\omega|$ is also defined if S is non oriented. $\omega = 0$ if d is odd. Similary one has

Proposition 2.2. *Let $S \subset \mathbb{R}^n$ be a smooth semi-algebraic 1-manifold. Then*

$$\int_S |k(t)|dt < \infty$$

Here k denotes the usual curvature for regular curves and dt its length element. Again, the integral depends only on the complexity of describing equations for S. In fact, consider the semi-algebraic map

$$\gamma : S \to S^{n-1}; t \mapsto \frac{T_tS}{\|T_tS\|} \quad \text{(unit tangentvector)}.$$

Then $\int_S |k(t)|dt =$ length of $\gamma(S)$,
and the right hand side can be estimated by a Cauchy-Crofton formula [Sa, Chap. 18, 1.].

Now assume that S is a topological semi-algebraic 2-manifold with boundary ∂S. Let $S^0 = S \setminus \partial S$ be non singular. Then, except for a finite set $S^2 \subset \partial S$ at the points $y \in \partial S$ one has Nash-wings. That means the following:
One has a smooth neighbourhood Y of y in ∂S and a tubular neighbourhood $T \to Y \times [0,1[$ around Y together with a section $\sigma : Y \times [0,1[\to T; (y,t) \mapsto \sigma(y,t)$, so that the vanishing order of $\frac{\partial}{\partial t}\sigma|_0$ is constant along Y. It follows, that the tangent plane map

$$S^0 \to G_{2n} \quad ; \quad p \mapsto T_p S^0$$

where G_{2n} is the Grassmann bundle, can be extended continuously from S^0 to $S \setminus S^2$. [B-C-R, Th. 9.6.13 and Th. 9.6.8]. In this situation y is called a Nash-point. We also need the following

Lemma 2.3. *Let $S \subset \mathbf{R}^n$ be a semi-algebraic topological 2-manifold. For $p \in S$ let $S(\rho, p)$ be the $n-1$ sphere with center p and radius ρ. Let S be stratified: $S = S^0 \cup S^1 \cup S^2$, dim $(S^i) = 2 - i$ such that S^0 is smooth.*

Then for all $\epsilon > 0$ there exists $\delta > 0$ such that for all δ with $0 < \rho < \delta$ one has:

i) $S(\rho, p)$ intersects the strata $S^i, i = 0, 1, 2$, transversally.

ii) For $q \in S(\rho, p) \cap S^1$ one has $| < X, Y > -1| < \epsilon$ where X is unit tangent vector of S^1 at q and Y is unit normal vector on $S(\rho, p)$ at q (S^1 intersects $S(\rho, p)$ nearly orthogonally. X and Y are defined up to a sign.)

iii) For $q \in S(\rho, p) \cap S^0$ one has $| < Y, N > -1| < \epsilon$ where again Y is unit normal vector on $S(\rho, p)$ at q and N unit normal vector on $S(\rho, p) \cap S^0$ at q (in the tangent space of S^0 at q).

Since we do not know a reference for this, we give a sketch of the proof: For $\epsilon > 0$ it is easy to find a semi-algebraic stratification of S such that on each 2-stratum the variation of the tangent space in the Grassmannian is less than ϵ. Since statement ii) for 1-strata is rather obvious we get i) and iii) nearly automatically.

We are indebted to the referee for this argument.

3 Curvature and Gauß-Bonnet formula

In the sequel we will have to consider curves and angles between curves on 2-manifolds (possibly with boundary). We will do this in various situations, for instance, in semi-algebraic topological 2-manifolds and in Aleksandrov surfaces. So this is just the general setting: A curve is always a simple continuous map $\gamma : I \to S$ where I is an interval in $\mathbf{R}$ or I is the unit circle. In the latter case γ is called closed. The restriction of γ to I^0 will be denoted by $\overset{\circ}{\gamma}$.

Let $p \in S$ and let γ, δ be curves with common initial point p but no further common points. If $p \notin \partial S, \gamma$ and δ define two germs of regions, say $(\gamma, \delta)_l$ and $(\gamma, \delta)_r$ locally around p. If S, γ and δ fullfill certain regularity conditions, there will exist section angles, corresponding to the two regions, which we denote by $\alpha_l(\gamma, \delta)$ and $\alpha_r(\gamma, \delta)$. In general, it can happen that $\alpha_l(\gamma, \delta) + \alpha_r(\gamma, \delta) =: \alpha(p) \neq 2\pi$. This $\alpha(p)$ is called total angle around p. If $p \in \partial S$, there is only one section angle.

Similary, a curve γ defines two sides and to each side corresponds a curvature k_l or k_r respectively of γ. (If γ is closed, one has to assume that a neighbourhood of γ is oriented). Again, not always $k_l + k_r = 0$. On ∂S one has only one curvature, say, k_l. In that case we set $k_r = 0$. Finally, consider a point $\gamma(t), t \in I^0$ and the curves $\gamma_1(s) = \gamma(t - s), \gamma_2(s) = \gamma(t + s)$. If the section angles $\alpha_l(\gamma_1, \gamma_2)$ and $\alpha_r(\gamma_1, \gamma_2)$ are defined, we define exterior angles $\beta_l(\gamma(t)) := \pi - \alpha_l(\gamma_1, \gamma_2)$ and $\beta_r(\gamma(t)) := \pi - \beta_r(\gamma_1, \gamma_2)$.

Let S be a 2-dimensional semi-algebraic topological manifold with boundary ∂S.

Notations 3.1. We assume that $S^0 = S \setminus \partial S$ is smooth. We set $\partial S = S^1 \cup S^2$ where S^1 consists of the Nash-points of ∂S. This includes that

i) ∂S is smooth at each point $y \in S^1$.

ii) The tangentplane-map $p \mapsto T_p S$ extends continously to $S^0 \cup S^1$.

By the last section S^2 can assumed to be finite.

The finite set S^2 again splits into two parts: $S^2 = S^{21} \cup S^{22}$ where S^{21} consist of those points y to which the tangent plane-map extends continously, but ∂S is not smooth at y, and S^{22} consists of the remaining

points. A semi-algebraic topological 2-manifold S with boundary, for which S^0 is smooth, will be called a simple semi-algebraic surface (or briefly SSS).

Definition 3.2. *A curve $\gamma : I \to S$ is called regular, if γ is of class C^1 in the ambient space $\mathbb{R}^n \supset S, \dot{\gamma}(t) \neq 0$ for all $t \in I$, and if one of the following conditions holds*

i) $\gamma \subset \partial S$

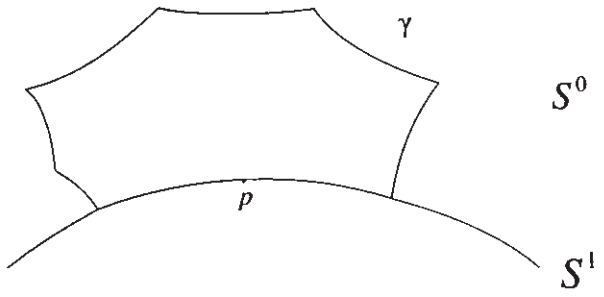
ii) $\overset{\circ}{\gamma} \subset S^0, \overset{\circ}{\gamma}$ *is of class* C^2 *and* $\int_\gamma |k(t)|dt < \infty$.

Here k is the inner Riemannian curvature of γ. Note that in case i) γ can assumed to be of class C^∞ inside S^1.

On S^1 we define the function k, called inner curvature, as follows: We choose a normal unit vector field N along S^1, tangent to S and pointing into the interior of S. The 1-manifold S^1 in $\mathbb{R}^n$ admits also a curvature vector field $\overline{k}$ along S^1. We set $k := < \overline{k}, N >$.

Proposition 3.3. *Let $p \in S^1$. Then in a neighbourhood U of p the local Gauß-Bonnet formula holds. Moreover, k only depends on the inner metric of S.*

More precisely, this means the following: In a neighbourhood U of p, which is homeomorphic to a half plane and for which $\partial S \cap U \subset S^1$ we consider a simple closed piecewise regular curve γ where the pieces may be parts of S^1.

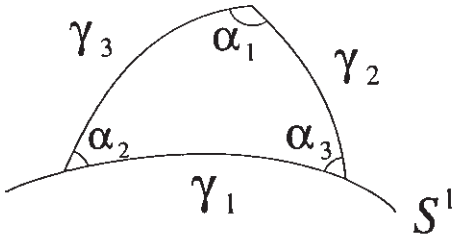


Then the inner curvature k_l with respect to the inner part $\mathrm{Int}(\gamma)$ of the region enclosed by γ is defined at all points of γ and also the exterior

angle β_i at the corners y_i of γ. Moreover, one has

$$\int_{\mathrm{Int}(\gamma)} K(x)dx + \int_{\gamma} k_l(s)ds + \sum \beta_i = 2\pi$$

Proof. (it might be better to say: sketch of the proof. For more details see [Ku]) We write U as a Nash wing as in the previous section. Also we may assume that γ is a triangle,



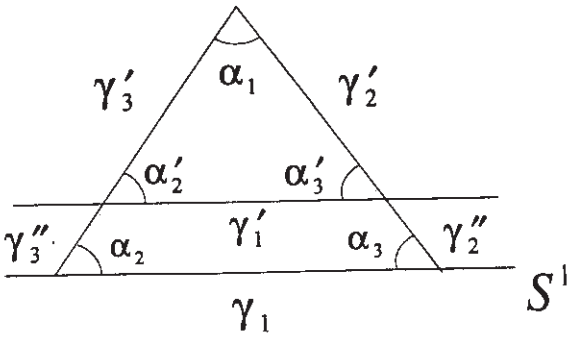
$\gamma = \gamma_1 \circ \gamma_2 \circ \gamma_3$ with $\gamma_1 \subset S^1, \gamma_2, \gamma_3 \subset S^0$ and corresponding interior angles α_i opposite to γ_i for $i = 1, 2, 3$. In the setting of Nash-wings we have

$$\gamma_1 \subset Y = \{(y, 0) | y \in Y\}.$$

We replace Y by the curve $\delta = \{(y, \epsilon) | y \in Y\}$ for a small y.

For sufficiently small ϵ we get the following: A decomposition of arcs: $\gamma_2 = \gamma_2'' \circ \gamma_2', \gamma_3 = \gamma_3' \circ \gamma_3''$. And a decomposition of the triangle $\gamma = \gamma_1 \circ \gamma_2 \circ \gamma_3$ into the triangle $\gamma' = \gamma_1' \circ \gamma_2' \circ \gamma_3'$ with $\gamma_1' \subset \delta$ and with interior angles $\alpha_3', \alpha_1' = \alpha_1, \alpha_2'$ respectively, and into the quadrangle $\gamma'' = \gamma_1 \circ \gamma_2'' \circ (\gamma_1')^{-1} \circ \gamma_3''$.

We may assume that γ' and γ'' are again simple closed curves.



Moreover, for sufficiently small ϵ, α_i' is arbitrarily close to α_i and $\int_{\gamma_i'} k(s)ds$ is arbitrarily close to $\int_{\gamma_i} k(s)ds$.

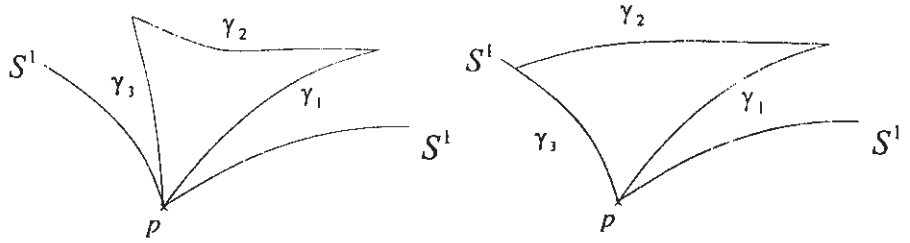
Since on the other hand $\int_{\mathrm{Int}(\gamma'')} K\,dx$ gets arbitrarily small and the Gauß-Bonnet formula holds for γ' it also holds for γ.

Clearly, the correctness of the Gauß-Bonnet formula defines the curvature function k on S^1 uniquely. Hence k only depends on the interior metric of S.

■

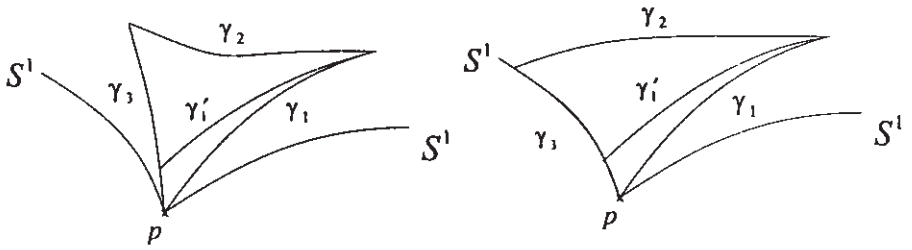
Proposition 3.4. *Let $p \in S^{21}$. Then in a neighbourhood U of p the local Gauß-Bonnet formula holds.*

Proof. Again it is enough to consider triangles. We have the two cases as indicated in the figure below:



We replace γ_1 by a close curve γ_1' respectively, as indicated in the next

figure,



and argue similary as for proposition 3.3, where in the second case we also use this proposition.

■

As a corollary of Propositions 3.3 and 3.4 we get

Proposition 3.5. *With the above setting for S assume that $S^{22} = \emptyset$. Let $T \subset S$ be a compact oriented 2-dimensional subvariety such that the boundary ∂T is a piecewise regular curve (for instance $T = S$). Then*

$$\int_T K(x)dx + \int_{\partial T} k_l(s)ds + \sum_{i=1}^{n} \beta_i = 2\pi\chi(T)$$

where β_i is the exterior angle at the corner p_i of ∂T and k_l the curvature with respect to Int (T).

It ramains to define angles at the points $p \in S^{22}$. More generally, let $p \in S$ and let γ, δ be regular curves emanating from p which do not intersect infinitely often in a neighbourhood of p. Then γ and δ define a section, say $(\gamma, \delta)_l$ around p.

Proposition and Definition 3.6. *With the above notations let $S(\rho, p)$ be the $n-1$ sphere in $\mathbf{R}^n$ around p with sufficiently small radius ρ. Set $\sigma_\rho := (\gamma, \delta)_l \cap S(\rho, p)$. Then the limit*

$$\alpha_l(\gamma, \delta) = \lim_{\rho \to 0} l(\sigma_\rho) \cdot \rho^{-1}$$

where $l(\sigma_\rho)$ is the length of σ_ρ exists and defines an angle only depending on the inner metric of S.

Proof. We parametize σ_ρ by arc length. For $q \in \sigma_\rho \cap S^0$ we choose Y and N as in Lemma 2.3. Let $\bar{k}$ be the curvature in $\mathbb{R}^n$ of σ_ρ at q. We get (up to a sign) $k_l = < \bar{k}, N >$. Hence, by Lemma 2.3,

$$k_l = < \bar{k}, Y > + \epsilon(\rho) = \frac{1}{\rho} + \epsilon(\varphi)$$

where $\epsilon(\rho) \to 0$ for $\rho \to 0$. Therefore

$$\int_{\sigma_\rho} dk_l = \frac{l(\sigma_\rho)}{\rho} + \epsilon(\rho) l(\sigma_\rho).$$

Now consider $\tau < \rho$ and the rectangle R enclosed by

$$\eta = \gamma|[\tau, \rho] \; \sigma_\rho(\delta|[\tau, \rho])^{-1} \; \sigma_\tau^{-1}$$

As we mentioned before, the inner angles at the four corners of R converge to $\pi/2$ for $\rho \to 0$. Also, since γ and δ have bounded curvature, we get

$$\int_{\gamma|[\tau,\rho]} k_l(\gamma(t))dt = \epsilon(\rho)$$

and correspondingly for δ. Hence by Proposition 3.5 we get

$$|l(\sigma_\rho)\rho^{-1} - l(\sigma_\tau)\tau^{-1}| = \int_{R^0} K(x)dx + \epsilon(\rho)$$

which shows the claim.

■

Corollary 3.7. *In Proposition 3.5 the assumption that $S^{22} = \emptyset$ can be dropped.*
Clearly, for $p \notin S^{22}$, the angle defined in 3.6 coincides with that we had before.

4 Piecewise semi-algebraic surfaces

We will extend our class of spaces.

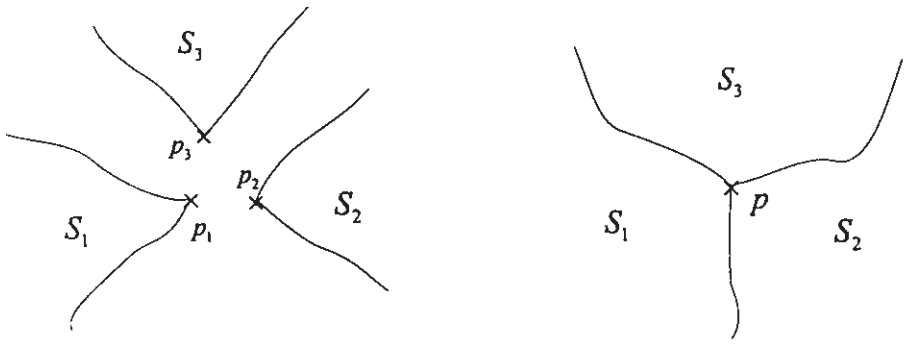
Construction and notations 4.1. Assume we are given two semi-algebraic 2-manifolds with boundary S_1, S_2 as above. Let Y_i be a closed

submanifold of ∂S_i of pure dimension 1, $i = 1, 2$ and $\varphi : Y_1 \to Y_2$ an isometry. Then we get a new space S by gluing S_1 and S_2 along Y_1 and Y_2 respectivly:

$$S = S_1 \cup S_2/\varphi$$

A 2-manifold (possibly with boundary) which one gets by gluing finitely many simple semi-algebraic surfaces S_i as above will be called a piecewise semi-algebraic surface (PSS). In fact, a semi-algebraic 2-manifold with boundary is a PSS, but the converse,in general, does not hold.

Stratification 4.2. Let S be a PSS. We set $S^0 = \{$smooth interior points of $S\}$. So for p of S^0 a neighbourhood of p is isometric to a smooth semi-algeraic 2-manifold. $S^1 = \{$Nash points$\}$. For $p \in S^1$ a neighbourhood of p is isometric to a Nash-wing (if $p \in \partial S$) or a gluing of two Nash-wings along their boundaries (if $p \notin \partial S$). Again $S^2 := S \setminus (S^0 \cup S^1)$ is finite. A neighbourhood $p \in S^2$ comes from gluing neighbourhoods of $p_i \in S_i^1 \cup S_i^2$ for finitely many simple semi-algebraic surfaces S_i such that the points $p_i \in S_i$ are identified.



We say, that $p \in S^{21}$ if all $p_i \in S_i^1 \cup S_i^{21}$ and $p \in S^{22}$ otherwise.

Let S be a PSS. Then S is a two-manifold with boundary. Also, if S comes from gluing together simple semi-algebraic surfaces $S_i, i = 1, \ldots, r$, then one has natural isometric imbeddings

$$\sigma_i : S_i \to S$$

Moreover, for $p \notin S^0$ in a neighbourhood of p, S^1 consists of finitely many branches. We have to enlarge the class of curves we consider.

Quasiregular curves 4.3. Let $\gamma : I \to S$ be a curve. Then there are countably many open intervals $U_i \subset I, i \in J$, such that $\gamma|U_i \subset S^0$. We set

$$C := I \setminus \bigcup_{i \in I} U_i.$$

One has $\gamma|C \subset S^1 \cup S^2$. We make the following assumptions:

a) For all $i \in I, \gamma|\overline{U}_i$ is piecewise regular (see Definition 3.2).

b) If $\gamma(t) = p \notin S^0$, then there exists $\epsilon > 0$ such that $\gamma|]t-\epsilon, t[$ (and $\gamma|]t, t+\epsilon]$ respectivly) intersects at most one branch of S^1 around p (in particular, it does not meet p.)

Condition b) allows to define exterior angles $\beta_l(\gamma(t))$ and $\beta_r(\gamma(t))$. In fact, consider again $\gamma_1(s) = \gamma(t-s)$ and $\gamma_2(s) = \gamma(t+s)$. If γ_i has infinitely many common points with a branch δ of S^1 at $\gamma(t)$, just replace γ_i by δ and consider the corresponding section angle and exterior angle between γ_i and δ.

c) $\int_{I \setminus C} |k_l(\gamma(t))|dt + \sum |\beta_l(\gamma(t))| < \infty$.

The summation runs over all $t \in \overset{\circ}{I}$= interior of I. A curve, which fullfills the assumptions a), b) and c) will be called quasiregular. This definition is symmetric with respect to left and right, since for almost all $t \in I$ one has $k_l(\gamma(t)) = -k_r(\gamma(t))$ and $\beta_l(\gamma(t)) = -\beta_r(\gamma(t))$.

Left and right curvature 4.4. Now we have to assume, that a neighbourhood of γ is oriented, if γ is closed. Moreover, let γ be quasiregular. We define left and right curvatures on γ as regular signed measures:

$$\int_\gamma dk_l := \int_{I \setminus C} k_l(\gamma(t)) + \int_{\gamma(C)} k_l(s)ds + \sum_{t \in \overset{\circ}{I}} \beta_l(t)$$

and correspondingly for the right hand side.

Curvature 4.5. Let S be an oriented PSS. We have to define a regular signed measure dK on $S \setminus \partial S$. So let $U \subset S \setminus \partial S$ be open. We set

$$\int_U dK = \int_{U \cap S_0} K(x)dx + \int_{U \cap S^1} dk_l + dk_r + \sum_{p \in U \cap S^2} (2\pi - \alpha(p))$$

Here $0 \leq \alpha(p)$ is the total angle around p. (For SSS's total angles are defined in 3.6. The extension to PSS's should be clear.) With these notations we have

Theorem 4.6. (Gauß-Bonnet-Formula) *Let S be a PSS and let $T \subset S$ be an oriented compact 2-dimensional subvariety such that ∂T is a union of finitely many simple closed quasiregular curves. Then*

$$\int_{T \setminus \partial T} dK + \int_{\partial T} dk_l = 2\pi\chi(T).$$

where dk_l is the curvature with respect to Int (T).

The proof of this is much easier then the description of all the settings above, although they are quite natural. So we leave the proof for the reader.

5 Aleksandrovs axiom

Let (S, d) be an inner metric space such that each point $p \in S$ admits a neighbourhood $U(p)$ which is homeomorphic to an open disc in $\mathbb{R}^2$, that is, S is a topological 2-manifold without boundary.

Geodesics and angles 5.1. A rectificable curve $\gamma : [a, b] \to S$ is called shortest geodesic, if $d(\gamma(a), \gamma(b)) = l(\gamma) :=$ length of (γ). Also, γ is called geodesic, if for all $t \in [a, b]$ there exists $\epsilon > 0$ such that for all $t_1 \leq t_2$ with $|t_i - t| < \epsilon$ one has : $\gamma|[t_1, t_2]$ is shortest geodesic. Next, consider two curves $\gamma_i : [0, e] \to S, i = 1, 2$, such that $\gamma_1(0) = \gamma_2(0)$. We define the upper angle $\overline{\alpha}(\gamma_1, \gamma_2)$ between γ_1 and γ_2 at $\gamma_i(0)$ as follows: For all $t_1, t_2 \in]0, e]$ consider the euclidean triangle with sides g_1, g_2, c of length $d(\gamma_1(0), \gamma_1(t_1)), d(\gamma_1(0), \gamma_2(t))$ and $d(\gamma_1(t), \gamma_2(t))$ respectivly. Let $\alpha(t_1, t_2)$ be the angle between g_1 and g_2. Then set

$$\overline{\alpha}(\gamma_1, \gamma_2) := \limsup\{\alpha(t_1, t_2) | (t_1, t_2) \to 0\}$$

This $\overline{\alpha}(\gamma_1, \gamma_2)$ is called the upper angle between γ_1 and γ_2 similary, we define the angle

$$\alpha(\gamma_1, \gamma_2) := \lim\{\alpha(t_1, t_2) | (t_1, t_2) \to 0\}$$

if it exists. One has $\pi \geq \overline{\alpha}(\gamma_1, \gamma_2), \alpha(\gamma_1, \gamma_2)$.

Triangles 5.2. Now let U be an neighbourhood, homeomorphic to an open disc in $\mathbb{R}^2$, of some point. We consider a triangle $\Delta \subset U$ with vertices A_1, A_2, A_3 and shortest geodesics γ_k connecting A_i and A_j. We assume that the γ_k fit together to a Jordan curve. Then Δ has an interior $\overset{\circ}{\Delta}$ and a boundary $\partial\Delta$ consisting of the three geodesic sections. Suppose that for any two points $p, q \in \partial\Delta$ and any simple arc γ connecting p and q such that γ up to the endpoints is entirely contained in $U \setminus \Delta$ one has $l(\gamma) \geq l(\delta)$ where $\delta \in \partial\Delta$ is the section which together with γ bounds a region outside of $\overset{\circ}{\Delta}$. Then Δ is called simple. Let $\overline{\alpha}_i$ be the upper angle at A_i. We define the upper defect of Δ by

$$\overline{\alpha}_2 + \overline{\alpha}_3 - \pi$$

Aleksandrovs axiom 5.3. For each $p \in S$ there exists a neighbourhood $U = U(p)$, homeomorphic to an open disc in $\mathbb{R}^2$, and a bound $K = K(p)$ such that for any collection $\Delta_1, \ldots, \Delta_r$ of simple triangles in U with disjoint interiors one has

$$\sum_{i=1}^{r} |\overline{\text{def}}\,(\Delta_i)| \leq K.$$

For this axiom we briefly say that S has locally bounded curvature. Also S is called an Aleksandrov surface. In [A-Z] it is shown, that for Aleksandrov surfaces one can develop a very strong synthetic 2-dimensional differential-geometry, which shares many properties with two dimensional Riemannian geometry. However, there are also serious differences. For instance, in the synthetic setting may exist ramifying geodesics.

We will show that Aleksandrovs axiom holds for piecewise semi-algebraic spaces. We are not able to do that directly, since exterior angles are very difficult to compute in the neighbourhood of singularities. Instead of that, we will approximate piecewise semi-algebraic spaces by well behaved ones.

Of course, Aleksandrovs axiom holds for Riemannian twofolds, if they are smooth enough. This follows immediately from the Gauß-Bonnet formula and the Proposition below

Proposition 5.4. *Let S be a Riemannian twofold of class C^2 and let γ_1, γ_2 be curves in S of class C^1 emanating from the same point p. Then the angle between γ_1, γ_2 at p exists and coincides with the Riemannian angle.*

This is well known. See also [Sch] for a detailed proof.

We need one of the main results on Aleksandrovs surfaces

Notation 5.5. Let X be a space with two metrics d_1, d_2. Then the Hausdorff distance $d(d_1, d_2)$ between these is defined by $d(d_1, d_2) := \sup\{|d(x,y) - d'(x,y)|(x,y) \in X \times X\}$

Theorem 5.6. *Let S be a topological twofold (without boundary) and let $(d_n) \to d$ be a Hausdorff convergent sequence of metrics on S. Assume that all d_n, d are inner and compatible with the topology of S. If (S, d_n) has uniformly locally bounded curvature, then (S, d) is an Aleksandrov surface.*

Proof. [A-Z, Chap. IV, Th. 15 in connection with III, Th. 11 f.f.]

■

6 Piecewise semi-algebraic surfaces are Aleksandrov surfaces

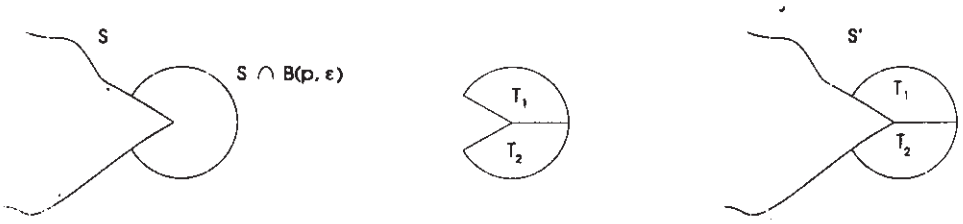
We are going to show the statement of the headline.

Theorem 6.1. *Let S be a piecewise semi-algebraic surface without boundary. Then Aleksandrovs axiom holds for S.*

The proof is done in two steps: First, by cutting and pasting we replace the metric d on S by a metric d' which is close to d and for which the axiom holds, and then we use approximation according to Theorem 5.6.

1. Recall that S is built up from simple semi-algebraic surfaces S_i by pasting pieces of the boundaries. We replace the metric on each S_i. Let $p \in S_i^2$. Let $B(p, \epsilon)$ be a small ball of radius ϵ around p. By Theorem 2.3, $B(p, \epsilon) \cap S_i$ is homeomorphic to a triangle Δ with sides a, b, c where c is opposite to p. Taking the original length's of the sides, we may have $l(c) > l(a) + l(b)$. We consider a spiral $\sigma(t), t \in [0, 1]$ in $\mathbf{R}^2$ around the origin, such that $\| \sigma(o) \| = l(a), \| \sigma(1) \| = l(b)$ and $l(\sigma) = l(c)$. We

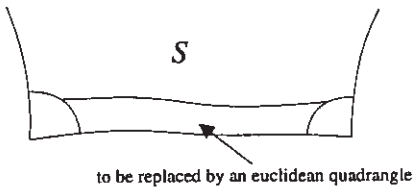
divide σ into k sections T_j for which the angle at the origin is smaller than π. Then we cut off $B(p,\epsilon) \cap S$ from S and replace it by pasting T_j along σ and $S(p,\epsilon) \cap S$ and the common edges of the T_j :



Here $S(p,\epsilon)$ is the sphere of radius ϵ around p. Note that the number k of sections T_j and the total curvature of σ in $\mathbb{R}^2$ can be bounded independently of ϵ. After identifying the new surface S' with S, we get a new metric d' on S which is close to d for small ϵ. By construction it is easly achieved that $d|\partial S = d'|\partial S$. We do this replacement for each $p \in S_i^2$. Let

$$R_i := S_i \setminus \cup_{p \in S_i^2} B(p,\epsilon).$$

Then at $R_i \cap S_i^1$ we have finitely many Nash-wings. Here we replace a tubular ϵ-neighbourhood T_ϵ of S_i^1 in such a Nash-wing by an euclidean quadrangle of corresponding side lengths.



If T_ϵ is unbounded at both sides (half bounded), we replace by a euclidean parallel strip (half parallel strip). For half parallel strips one has only to adapt the bounded side length. Again, it is easy to identify the new surface S' with S in a way such that the inner metric d' coming

from S' is close to d for small ϵ, and $d|\partial S = d'|\partial S$.

Finally we point out, that we have a description of the curves $S(p,\epsilon) \cap S_i$ and $\partial T_\epsilon \cap S_i$ the complexity of which only depends on the complexity of a description for S_i but not on ϵ. Hence the total absolute curvature of these curves is uniformly bounded, independently of ϵ (see Proposition 2.2). We do this for all S_i and patch them together as before. Thus we get an inner metric d_ϵ on S so that (S, d_ϵ) again is a PSS and d_ϵ tends to d for $\epsilon \to 0$.

2. Now Aleksandrovs axiom holds for (S, d_ϵ). This follows from [A-Z, Chap. IX, 3]. In our situation this can also be seen directly. In fact, a geodesic arc in (S, d_ϵ) is easily seen to be quasiregular (for a detailed proof see [Sch]). Moreover, each building block of (S, d_ϵ) is completely contained in a C^∞-Riemannian twofold. Therefore, by Proposition 5.4, upper angles between geodesics coincide with the Riemannian angles. Hence, by the Gauß-Bonnet formula, for a simple triangle Δ we have $\operatorname{def}(\Delta) = \int_\Delta^\circ dK$.

Now, by the definition of dK (see 4.5) and the first part of the proof we even get: There is a bound ρ, independent of ϵ, such that for (S, d_ϵ) one has $\int_S dK \leq \rho$. In particular, the family $\{(S, d_\epsilon) | \epsilon > 0, \epsilon \text{ sufficiently small }\}$ has uniformly locally bounded curvature. Since $d_\epsilon \to d$ for $\epsilon \to 0$, by Theorem 5.6 it follows that (S, d) is an Alekandrov surface too.

■

7 Angles, directions and curvature of curves

We want to apply the theory of Aleksandrov surface to piecewise semi-algebraic surfaces S. To this end we have to review some notations and results (see [A-Z] for a general reference).

Angles and directions 7.1. Let S be an Aleksandrov surface. Let $p \in S$ and γ a curve emanating from p. We say that γ has a direction at p if the angle α between γ and γ at p exists. Then necessarily $\alpha = 0$. If γ is geodesic, it has obviously a direction at p. If there is a second curve $\delta(s)$ emanating from p and having a direction at p, then also the angle $\beta =: \alpha(\gamma, \delta)$ between γ and δ exists [A-Z, Chap. VI, Th. 1]. If, moreover, γ and δ have infinitely many common points in any neighbourhood of

p, then $\beta = 0$. If $\beta = \pi$, possibly β has no geometrical meaning. For instance, this occurs if S is piecewise semi-algebraic, $p \in S^2$ and the total angle at p exceeds 2π. Therefore one defines sector angles as follows. Assume γ and δ have no common points $\neq p$ in some neighbourhood of p. Then γ and δ define two sectors $(\gamma, \delta)_l$ and $(\gamma, \delta)_r$.

Consider in such a sector, say $(\gamma, \delta)_l$ a sequence $\gamma = \gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_r = \delta$ of simple continuous curves emanating from p, having directions at p and being labeled in their natural order. We set

$$\alpha(\gamma_0, \ldots, \gamma_r) = \sum_{i=0}^{r-1} \alpha(\gamma_i, \gamma_{i+1})$$

and

$$\alpha(\gamma, \delta)_l = \sup\{\alpha(\gamma_0, \ldots, \gamma_r)\}$$

Moreover, we set

$$\alpha(p) = \alpha(\gamma, \delta)_l + \alpha(\gamma, \delta)_r \quad \text{(total angle at } p\text{)}$$

If any two geodesics emanating from p have common points arbitrarily close to p we set $\alpha(p) = 0$. Let γ, δ define sectors as above, and let $\alpha(\gamma, \delta)$ the angle between γ and δ at p. One has

a) $\alpha(\gamma, \delta) = \min\{\alpha(\gamma, \delta)_l, \alpha(\gamma, \delta)_r, \pi\}$

b) If $\eta \in (\gamma, \delta)_l$ has a direction at p, then $\alpha(\gamma, \delta)_l = \alpha(\gamma, \eta)_l + \alpha(\eta, \delta)_l$

c) If $(\gamma, \delta)_l$ is convex, then $\alpha(\gamma, \delta) = \min\{\alpha(\gamma, \delta)_l, \pi\}$

From b) it follows that the total angle $\alpha(p)$ is well defined. For more details see [A-Z, Chap. IV, 4].

Curvature of curves 7.2. Let $\gamma : [a, b] \to S$ be a curve with directions at the end points $p = \gamma(a)$ and $q = \gamma(b)$. We also assume that $\alpha(p) \neq 0, \alpha(q) \neq 0$. If γ consists piecewise of shortest geodesics, γ is called a polygone. In a suitable small neighbourhood γ has a left and a right hand side. In particular, if γ is a polygone and $p_i \neq p, q$ a vertex, we have left and right exterior angles $\beta_l(p_i)$ and $\beta_r(p_i)$ defined by γ at p_i. We set

$$\bar{k}_l(\gamma) = \sum_{p_i} \beta_l(p_i), \quad \bar{k}_r(\gamma) = \sum_{p_i} \beta_r(p_i)$$

where p_i runs over the vertices $\neq p, q$ of γ.
Now assume, that γ is not necessarily polygonal, and let γ_i be a sequence of polygones with endpoints p, q which converges to γ from the left hand side (so that $\gamma_i \cap \gamma$ consists of vertices only). Then, locally at p and q we have regions enclosed by γ_i and γ and corresponding sectorangles σ_i and τ_i. The sequence $(\overline{k}_l(\gamma_i) + \sigma_i + \tau_i)$ tends to a well defined value [A-Z, Chap. IV, Th. 2] which is called left curvature ([A-Z] uses the word "left rotation") and denoted by $k_l(\gamma)$. If γ is polygone, then $\overline{k}_l(\gamma)$ and $k_l(\gamma)$ in general do not coincide.

Next, let $\gamma : I \to S$ be a curve. We say that γ has bounded curvature (in [A-Z] one says more precisely "bounded variation of curvature") if the following conditions hold:

a) At all $p \in \gamma$ the total angle $\alpha(p) \neq 0$.

b) For all $p \in \overset{\circ}{\gamma}$, γ has left and right directions and hence left and right exterior angles $\beta_l(p)$ and $\beta_r(p)$.

c) There is a bound ρ such that for all ordered sequences of points $p_1, \ldots, p_r \in \overset{\circ}{\gamma}$ one has

$$\sum_{i=1}^{r+1} (|k_l(\gamma_i)| + |k_r(\gamma_i)|) + \sum_{i=1}^{r} (|\beta_l(p_i)| + |\beta_r(p_i)|) < \rho.$$

Here $\gamma_1, \ldots, \gamma_{r+1}$ are the sections of γ defined by the p_i.

If γ has bounded curvature, one has natural regular signed measures dk_l and dk_r on γ. [A-Z, Chap. IX, Th. 1]

Examples 7.3. Assume that $\alpha(p) \neq 0$ for $p \in \gamma$.

a) Let γ be a geodesic in S. Then γ has bounded curvature. However, it is not true in general, that $dk_l = dk_r = 0$. One only has $dk_l \leq 0, dk_r \leq 0$.

b) Let S be piecewise semi-algebraic, and let γ be a (quasi)regular curve in S. Then γ has bounded curvature and the set functions dk_r and dk_l coincide with those we defined in 4.4. We will apply this for the special case that γ is a semi-algebraic path.

Proposition 7.4. *Let $\gamma : [a,b] \to S$ be a curve of bounded curvature. Then γ has directions at the endpoints.*

Proof. [A-Z, Chap. IX, Th. 2].

■

Gauss-Bonnet formula 7.5. There is a canonical regular signed measure dK on S [A-Z, Chap. V]. Let $T \subset S$ be a compact oriented submanifold of dimension 2 with boundary ∂S. So ∂S consists of finitely many closed curves $\gamma_1, \ldots, \gamma_r$. Then

$$\sum_{T^\circ} dK + \sum_{i=1}^{r} k_l(\gamma_i) = 2\pi\chi(T)$$

Where the left hand side is the inner side of T.

Proof. [A-Z, Chap. VI, Th. 5].

■

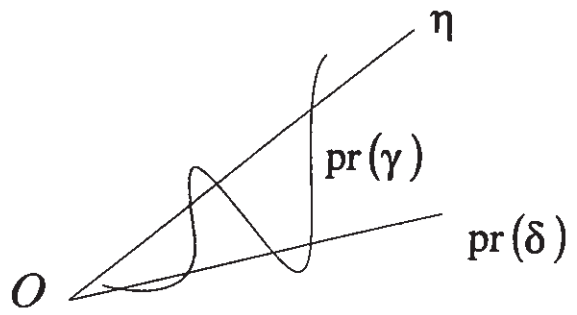
8 Piecewise semi-algebraic surfaces revisted

Now let S be a piecewise semi-algebraic surface. Since S is also an Aleksandrov surface, we have many things doubly defined: sector angles, curvature on S and left and right curvatures of curves. We have also two Gauß-Bonet formulas. These will help us to identify these quantities. Let us provide the notations we had for piecewise semi-algebraic surfaces with tildes. First we show

Lemma 8.1. *Let $p \in S^1, \delta$ a branch of S^1 at p and γ a geodesic emanating from p which intersects δ in any neighbourhood of p. Then γ is of class C^1 at p and $\tilde{\alpha}(\gamma, \delta) = 0$.*

Proof. The projection of S into the tangent space T_pS defines a semi-algebraic C^1 chard $pr : S \to \mathbf{R}^2$ in a neighbourhood of p and the origin respectively. It is enough to show that the claim holds for $pr(\gamma)$ and $pr(\delta)$ at 0. If the claim does not hold we have a linear ray η, starting at

0, which includes with $pr(\delta)$ a positive angle β, so that γ also intersects η in any neighbourhood of 0.



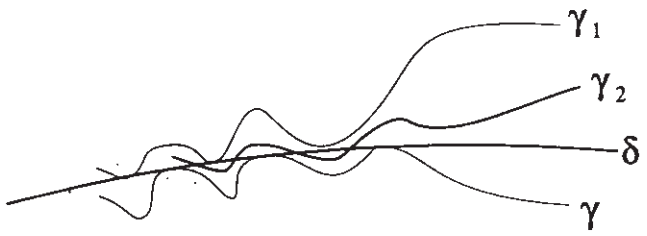
Then in any neighbourhood of 0 there are pieces of $pr(\gamma) \subset pr(S^0)$ the total curvature of which exceeds a certain value. Since the map $q \mapsto T_qS$ (in some ambient space) extends continuously to S^1, the variation of the curvature on γ converges to that on $pr(\gamma)$. On the other hand, γ being geodesic has no curvature on S^0. Contradiction. Similary it follows, that the right derivation on γ is continuous.

■

Lemma 8.2. *Under the assumptions of Lemma 8.1 assume we have a second geodesic γ_1 which intersects δ in any neighbourhood of p. Then $\alpha(\gamma, \gamma_1) = \tilde{\alpha}(\gamma, \gamma_1) = 0$.*

Proof. Clearly $\tilde{\alpha}(\gamma, \gamma_1) = 0$. Also $\alpha(\gamma, \gamma_1) = 0$ if γ and γ_1 intersect in any neighbourhood of p. Assume first, that in any neighbourhood of p there exist $q \in \gamma \cap S^0, q_1 \in \gamma_1 \cap S^0$ and a $C^2 -$ arc η in S^0 from p to q such that we get a triangle Δ with vertices p, q, q_1 enclosed by a simple closed curve consisting of a section of γ, γ_1 and η. For such a triangle the defect tends to zero (for dK and $d\tilde{K}$) and the angles at q, q_1 coincide in both settings. Hence both Gauß-Bonnet formulas yield, that also $\alpha(\gamma, \gamma_1) = \tilde{\alpha}(\gamma, \gamma_1)$.

In the remaining case, γ and γ_1 may have common tangents with δ, arbitrarily close to p. Then we choose a shortest geodesic γ_2 connecting a point "between" γ and γ_1 with p.



In the situation of the figure by what we have shown we get $\alpha(\gamma, \gamma_2) = 0 = \alpha(\gamma_2, \gamma_1) = \alpha(\gamma, \gamma_1)$. Still it might happen that γ_2 remains on the same side of δ than γ_1 but then one can achieve that γ_2 has a common point with γ, arbitrarily close to p. It follows $\alpha(\gamma, \gamma_1) < \epsilon$ and $\alpha(\gamma_1, \gamma_2) < \epsilon$.

■

Lemma 8.3. *Let p and δ be as in the previous Lemmata. Then δ has a direction at p.*

Proof. Assume that the claim does not hold. Then there are geodesics γ, γ_1 emanating from p such that $\alpha(\gamma, \gamma_1) > 0$ and δ intersects γ and γ_1 in any neighbourhood of p. Contradiction to Lemma 8.2.

■

Lemma 8.4. *Let γ be a regular curve in S emanating from $p \in S$ with $\alpha(p) \neq 0$ such that, apart from p, γ is contained in S^0. Then γ has a direction at p. Moreover, $dk = d\tilde{k}$ on γ.*

Proof. First we show that for γ in S^0 we have $dk = d\tilde{k}$. For this let us first assume that γ is geodesic, $\gamma \subset S^0$. Then $dk_l \leq 0$ and $dk_r \leq 0$. Of course we have $dK = d\tilde{K}$ on S^0. Hence $0 \geq \int_\gamma (dk_l + dk_r) = \int_\gamma dK = \int_\gamma d\tilde{K} = 0$. The first equality is by [A-Z, Chap. VI, Th. 6]. It follows

$dk_l = dk_r = 0$. Next let $\gamma \subset S^0$ be regular. For small curves γ we may consider a simple triangle in S^0 with sides γ, δ, η where δ and η are geodesics. We know, by Proposition 5.4, that for the angles we have $\alpha_i = \tilde{\alpha}_i, i = 1, 2, 3$. Comparing the Gauß-Bonnet formulas we get $\int_\gamma dk = \int_\gamma d\tilde{k}$.

Now we allow that the startpoint p of γ is possibly not in S^0. However, by Proposition 7.4, γ has a direction at p.

■

Proposition 8.5. *Let $p \in S^0 \cup S^1$ and let γ_1, γ_2 be regular curves emanating from p. Then $\alpha(\gamma_1, \gamma_2) = \tilde{\alpha}(\gamma_1, \gamma_2)$ and correspondingly for section angles at p. In particular, for the total angle at p one has $\tilde{\alpha}(p) = \alpha(p) = 2\pi$.*

Proof. We need only to show the first claim. Since $\alpha(\gamma_1, \gamma_2)$ exists, we can compute it by considering sequences $(\gamma_1(t_n)), (\gamma_2(s_n)), t_n \to 0, s_n \to 0$ for $\gamma_i : [0, \epsilon[\to S$, such that $d(p, \gamma_1(t_n)) = d(p, \gamma_2(s_n)) \neq 0$. In this situation the claim follows easily from the continuity of the Riemann metric at p.

■

Corollary 8.6. *(Reflection law) Let $p \in S^1, \delta_1, \delta_2$ the two branchs of S^1 at p and let γ be a geodesic passing through p with branchs γ_1, γ_2 emanating from p. Then $\tilde{\alpha}(\gamma_1, \delta_1) = \tilde{\alpha}(\gamma_2, \delta_2)$.*

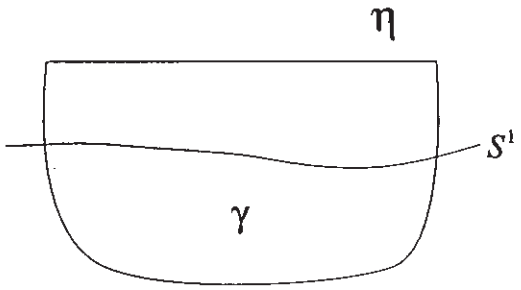
Proof. One has $\alpha(\gamma_1, \delta_1) + \alpha(\gamma_2, \delta_1) \geq \pi$ and $\alpha(\gamma_1, \delta_2) + \alpha(\gamma_2, \delta_2) \geq \pi$. [A-Z, Chap. VI, Th. 6]. On the other hand $\alpha(\gamma_1, \delta_1) + \alpha(\gamma_2, \delta_1) + \alpha(\gamma_1, \delta_2) + \alpha(\gamma_2, \delta_2) = 2\pi$.

■

Corollary 8.7. $dK = d\tilde{K}$.

Proof. This is clear on S^0. Next let γ be a small arc in S^1. In a small neighbourhood U of γ we choose a piecewise regular closed curve η which

intersects S^1 at the endpoints of γ.



By Proposition 5.4, Lemma 8.4 and Proposition 8.5 we know that $dk = d\tilde{k}$ on η, hence

$$\int_{\mathrm{Int}(\eta)} dK = \int_{\mathrm{Int}(\eta)} d\tilde{K}.$$

Since dK is regular, for small U the left hand side tends to

$$\int_\gamma dK = \int_\gamma dk_r + \int_\gamma dk_l = \int_\gamma d\bar{k}_l + \int_\gamma d\bar{k}_r.$$

The left equality holds by [A-Z, Chap. VI, Th. 3] and the right one by Theorem 4.6.
Similary, a point $p \in S^2$ can be enclosed by a small simple piecewise regular curve. Then, using the same argument, we get

$$K(p) = \tilde{\alpha}(p) - 2\pi = \tilde{K}(p)$$

■

From this proof we see furthermore

Corollary 8.8.

a) Let γ be a regular curve in S with $\alpha(p) \neq 0$ for all $p \in \gamma$. Then $dk = d\tilde{k}$ on γ.

b) For $p \in S$ with $\alpha(p) \neq 0$ and any two regular curves γ_1, γ_2 emanating from p one has $\alpha_l(\gamma_1, \gamma_2) = \tilde{\alpha}_l(\gamma_1, \gamma_2)$ and $\alpha_r(\gamma_1, \gamma_2) = \tilde{\alpha}_r(\gamma_1, \gamma_2)$.

c) For all $p \in S, \alpha(p) = \tilde{\alpha}(p)$.

Proof. For part b) only the case where $p \in S^2$ has not yet been considered. This can again been done by comparing both Gauß-Bonnet formulas. Part c) follows from the fact that also $K(p) = \alpha(p) - 2\pi$.

■

Part a) extends to quasiregular curves. We will not enter into this since quasiregularity is just an auxiliary notation. For the remaining part of this paper let $S \subset \mathbb{R}^n$ be a semi-algebraic twofold without boundary. This is the situation we are originally interested in. However, our intrisic approach shows that many properties of S do not depend on the special presentation of S. In particular, we get some insight into the behavior of geodesics.

Proposition 8.9. *Let* $S \subset \mathbf{R}^n$ *be semi-algebraic and let* γ *be a curve with bounded curvature in* S. *Then at all points* p *on* γ *the curve* γ *in* $\mathbb{R}^n$ *has left and right derivatives respectively and these are left (resp.) right continuous.*

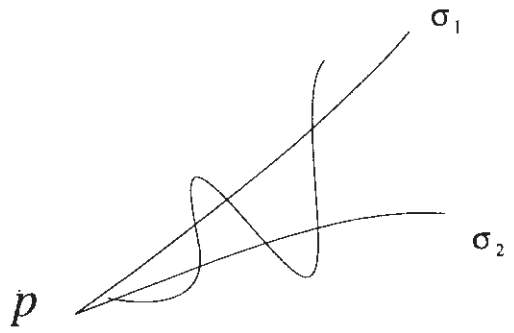
Proof. We may assume that $p \in S$ with $\alpha(p) \neq 0$ is the initial point of γ. Chose a semi-algebraic chart $\varphi : U \to V \subset \mathbf{R}^2$ where U is a neighbourhood of p, V is an open disc with center 0 and $\varphi(p) = 0$. We set $V = \{ty | t \in [0,1[, y \in C\}$ where $C \subset \mathbf{R}^2$ is the unit circle. Then we can choose φ in such a way that the following conditions hold:

i) For fixed $y \in C$ the curve $c_y : [0,1[\to U; t \mapsto \varphi^{-1}(ty)$ is of class C^1.

ii) The map $[0,1[\times C \to \mathbf{R}^n; (t,y) \mapsto c_y(t)$ is continuous.
Moreover, for $\epsilon > 0$ the chart can be chosen in such a way, that uniformly
$|\int_I k(t)dt| < \epsilon$ where $I \subset c_y, y \in C$ and $k(t)$ is the curvature on c_y.

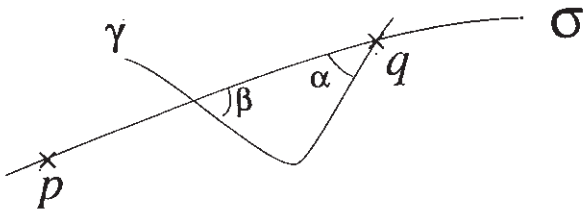
This can be seen by using resulution of singularities for the Zariski-closure of S. For the last statement see also the argument in Lemma 2.3. Let $\sum$ be the bundle of all curves $c_y(t)$ for $y \in C$. Since $\alpha(p) > 0, \gamma$ can only finitely often turn around p. [A-Z, Chap. IX, Th. 2]. Therefore, if γ has no right derivative at p, there are σ_1 and $\sigma_2 \in \sum$ having different

right derivatives in $\mathbb{R}^n$, such that γ runs infinitely often back and forth between σ_1 and σ_2.



It follows, that γ shares the direction at p with σ_1 and σ_2. Contradiction.

Similary, for any $\epsilon > 0$ there exists $\rho > 0$ such that in $B(\rho, p)$ one has: γ is contained in a section $(\sigma_1, \sigma_2)_l, \sigma_1, \sigma_2 \in \sum$, such that $\alpha_l(\sigma_1, \sigma_2) < \epsilon$. Therefore, if $q \in \gamma \cap B(\rho, p)$ and $\sigma \in \sum$ with $q \in \sigma$, then, at $p, \alpha_l(\sigma, \gamma) < \epsilon$. Also, for q close to p, the right derivative of γ at q becomes close to the left derivative. If the latter is the same as the left derivative of σ at q, we are done. Otherwise there is a first point r (possibly p) where γ and σ meet again.



By the Gauß-Bonnet formula, the angles α and β, as in the figure, get small if q is close to p. This shows, that the right derivatives of γ are continuous. ■

Corollary 8.10. *Let $S \subset \mathbf{R}^n$ be semi-algebraic, $p \in S$ with $\alpha(p) \neq 0$ and let γ be a curve, emanating from p, such that $\overset{\circ}{\gamma}$ is geodesic. Then γ has continuous right derivatives.*

Proof. It is known, that geodesics in Aleksandrov surfaces have bounded curvature. In our situation, one can easily compute it:

$$\int_\gamma dk_l = \int_{\gamma \cap S^1} dk_l + \sum_{p \in \gamma^0} \beta_l(p)$$

where $\beta_l(p)$ is the left exterior angle at p. Note that $\beta_l(p) \leq 0$ for all $p \in \gamma^0$ and for almost all of them one has equality.

■

It is not hard to show the preceding result directly, if $p \notin S^{22}$, but for $p \in S^{22}$ we were not able to do that. Next we will get rid of the assumption that $\alpha(p) \neq 0$.

Proposition 8.11. *For $p \in S$ the following conditions are equivalent:*

a) The total angle $\alpha(p) = 0$

b) All semi-algebraic arcs emanating from p have the same right derivative in $\mathbf{R}^n$.

Proof. We have $\alpha(p) = \tilde{\alpha}(p) = \lim_{\rho \to 0} l(\sigma_\rho)\rho^{-1}$ (see Definition 3.6).

■

Of course, this result is expected, but we were not able to get it by direct computation of angles in Aleksandrov surfaces. If $\alpha(p) = 0$, automatically all curves γ emanating from p have directions and also right derivatives in $\mathbf{R}^n$. The angle between two of them is zero. So quite generally it holds that $\tilde{\alpha}_l(\gamma_1, \gamma_2) = \alpha(\gamma_1, \gamma_2)$ if both angles are defined.

However, there may exist curves γ emanating from p such that $\overset{\circ}{\gamma}$ is geodesic but the right derivative is not continuous. Then γ turns infinitely often around p with respect to each semi-algebraic chart.

Natural stratification 8.12. Let S be semi-algebraic. Then S admits a natural semi-algebraic stratification which only depends on the inner metric of S : We set T^n = set of points which admit a Riemannian

neighbourhood, where the fundamental form is of class C^n. Then $S^0 \subset T^\infty$. Moreover, $T^0 \setminus S^0$ consists of those points p of S^1, where $dk_l = -dk_r$ in a neighbourhood of p and possibly certain points $p \in S^2$ where necessarily $\alpha(p) = 2\pi$. Hence $T^0 \setminus S^0$ is semi-algebraic. From this one can show that $T_0 \setminus T^i$ is a semi-algebraic subset of $S^1 \cup S^2$ for $i < \infty$. Also $R^1 := (S^1 \cup S^2) \setminus T^0$ is semi-algebraic. It consists of those points p, which admit neighbourhoods U such that $\mathrm{vol}(U)$ is arbitrarily small but $|\int_U dK|$ is bigger than some constant. In R^1 in turn we have a finite set of very singular points: Ramification points and points p, for which $\alpha(p) \neq 2\pi$. To the latter belong the isolated points of R^1.

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