REVISTA MATEMÁTICA de la
Universidad Complutense de Madrid
Volumen 10, número 2: 1997
http://dx.doi.org/10.5209/rev_REMA.1997.v10.n2.17465

Note on ∞-superharmonic functions.

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Abstract

The purpose of this note is to show that all viscosity supersolutions of

$$\Delta_\infty v \equiv \sum \frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}\frac{\partial^2 v}{\partial x_i \partial x_j} \leq 0$$

are variational. That is, they are limits of p-superharmonic functions, induced by the operator

$$\Delta_p v = \operatorname{div}(|\nabla v|^{p-2}\nabla v) \ ,$$

as p approaches ∞. In addition, it is shown that each viscosity supersolution of $\Delta_\infty v \leq 0$ is Lipschitz continuous.

1 Introduction

The solutions of the differential equation

$$\Delta_\infty h \equiv \sum_{i,j=1}^{n} \frac{\partial h}{\partial x_i}\frac{\partial h}{\partial x_j}\frac{\partial^2 h}{\partial x_i \partial x_j} = 0 \qquad (1.1)$$

are called *∞-harmonic functions*. They play an essential rôle as the best Lipschitz extensions of their boundary values, cf. [A] and [J]. Their regularity properties are poorly understood, but at least it is known that

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Mathematics Subject Classification: 35J70, 26A16.
*Research partially supported by NSF grant DMS-9501561.
Servicio Publicaciones Univ. Complutense. Madrid, 1997.

they belong to $C(\Omega) \cap W^{1,\infty}_{\text{loc}}(\Omega)$. The mere concept of solution is difficult, because the equation does not have an ordinary weak formulation containing only the first partial derivatives, while the second ones needed to evaluate (1.1) are not even known to exist. There are two options to overcome this difficulty.

First, one uses the concept of *viscosity solutions*. This has the advantage that $\Delta_\infty \varphi$ has to be calculated only for smooth test-functions. Second, one approximates the equation by equations like

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \quad , \quad \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \pm\varepsilon^{p-1} \ ,$$

as p approaches ∞. Both approaches are needed, so far. A strange mixture of viscosity and variational methods prevails.

For the details of the variational method see [DBM]. The viscosity method is developed in [J] where a remarkable uniqueness result is obtained. It is proven in [J] that, given continuous boundary values in an arbitrary bounded domain in the n-dimensional Euclidean space, there is a unique viscosity solution attaining the given boundary values at every boundary point. As a matter of fact, this viscosity solution is the uniform limit of the corresponding p-harmonic functions, as p approaches ∞. (The solutions of the equation $\operatorname{div}(|\nabla h|^{p-2}\nabla h) = 0$ are called p-harmonic.) Although the framework of viscosity solutions is needed to prove uniqueness, it does not produce any "new" solutions.

The objective of our note is to prove that even the *viscosity supersolutions* of the equation are variational, i.e., they are locally uniform limits of p-superharmonic functions, as p approaches ∞. We use an obstacle problem in the Calculus of Variations, a tool that is of independent interest. A noteworthy consequence of the variational characterization is that certain estimates now are automatically extended to the full class of viscosity supersolutions. As an example we mention Harnack's inequality (Corollary 4.5) and Liouville's theorem (Corollary 4.7).

2 Some Definitions

The viscosity supersolutions of $\Delta_\infty v \leq 0$ are equivalent to the ∞-*superharmonic functions* defined via a comparison principle. To be on the safe side, we mention the definitions. Let Ω denote a domain in $\mathbb{R}^n$.

2.1 Definition. *The function $v : \Omega \to (-\infty, \infty]$ is a viscosity supersolution, if*

(i) $v \not\equiv \infty$

(ii) *v is lower semicontinuous, and*

(iii) *at any given point x we have $\Delta_\infty \varphi(x) \leq 0$, if $\varphi \in C_0^\infty(\Omega)$, $\varphi \leq v$ in Ω, and $\varphi(x) = v(x)$.*

Notice that Δ_∞ has to be calculated only for the test-function φ, not for v itself. Analogously, a viscosity subsolution is defined. Finally, a function that is both a viscosity super- and a viscosity subsolution is called a viscosity solution. Thus viscosity solutions are continuous by definition. By the result of R. Jensen the Dirichlet boundary value problem has a unique viscosity solution, cf. [J]. To be more precise, suppose that Ω is bounded and that $f : \partial\Omega \to \mathbb{R}$ is a given continuous function. Then the equation $\Delta_\infty h = 0$ has a *unique* viscosity solution h in Ω with boundary values

$$\lim_{x \to \xi} h(x) = f(\xi)$$

at each $\xi \in \partial\Omega$. As a matter of fact, $\lim_{p\to\infty} h_p = h$ uniformly in Ω, where h_p is the solution to the equation $\Delta_p h_p = 0$ with boundary values f in Ω. (It is known that h_p is unique and that h_p attains the prescribed boundary values, if $p > n =$ the dimension of the space.) To begin with, it is not clear that different sequences of p's approaching ∞, would yield the same function h. It is here that Jensen's uniqueness result is indispensable. Accordingly, the full sequence converges to h.

2.2 Definition. *The function $v : \Omega \to (-\infty, \infty]$ is ∞-superharmonic, if*

(i) $v \not\equiv \infty$,

(ii) *v is lower semicontinuous, and*

(iii) *v obeys the comparison principle in any subdomain D with $\bar{D} \subset\subset \Omega$: if $h \in C(\bar{D})$ is ∞-harmonic in D and $h \leq v$ on ∂D, then $h \leq v$ in D.*

Notice that this definition requires that the concept of ∞-harmonic function has been defined in advance. Here we take the ∞-harmonic functions as the viscosity solutions. This mixture of two concepts in one definition is not esthetic.[1]

2.3 Proposition. *The viscosity supersolutions and the ∞-superharmonic functions are the same functions.*

Proof. The viscosity supersolutions satisfy the comparison principle by [J, Theorem 2.1] and hence they are ∞-superharmonic.

Suppose now that v is ∞-superharmonic. Given $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ such that $\varphi(x) \le v(x)$, when $x \in \Omega$, and $\varphi(x_0) = v(x_0)$, we have to show that $\Delta_\infty \psi(x_0) \le 0$. Suppose, on the contrary, that $\Delta_\infty\varphi(x_0) > 0$ for some φ. By continuity $\Delta_\infty\varphi(x) > 0$, when $|x - x_0| \le r$. Denote $B = B(x_0, r)$. Consider the auxiliary function

$$w(x) = \varphi(x) - \varepsilon|x - x_0|^2 .$$

A direct calculation yields

$$\begin{aligned}\Delta_{\infty w}(x) &= \Delta_\infty\varphi(x) - 2\varepsilon|\nabla\varphi(x) - 2\varepsilon(x - x_0)|^2 \\ &\quad - 2\varepsilon(x - x_0)\cdot\nabla|\nabla\varphi(x)|^2 + 4\varepsilon^2\sum_{i,j}(x_i - x_{0i})\frac{\partial^2\varphi(x)}{\partial x_i\partial x_j}(x_j - x_{0j}) \\ &= \Delta_\infty\varphi(x) + O(\varepsilon).\end{aligned}$$

Hence $\Delta_\infty w(x) > 0$ in B, when $\varepsilon > 0$ is small enough. This means that w is a classical subsolution to the equation and as such it satisfies a comparison principle: $w \le h$ in B, h denoting the ∞-harmonic function having the same boundary values on ∂B as w. In particular, $v(x_0) = \varphi(x_0) = w(x_0) \le h(x_0)$.

On the other hand,

$$h|_{\partial B} = v|_{\partial B} = \varphi|_{\partial B} - \varepsilon r^2 \le v|_{\partial B} - \varepsilon r^2 .$$

By the assumption $v(x) \ge h(x) + \varepsilon r^2$ in B (the translation by the constant εr^2 does not matter). Thus $v(x_0) \ge h(x_0) + \varepsilon r^2$, which contradicts the inequality $v(x_0) \le h(x_0)$ above. Hence the assumption $\Delta_\infty\varphi(x_0) > 0$ was false. This proves that v is a viscosity supersolution.

■

[1] We do not know, whether one may further restrict the h's in (iii)$'$ to those having second partial derivatives, so that the condition $\Delta_\infty h = 0$ could be directly verified, at least at almost every point.

3 The Obstacle Problem

We will prove that the solution to an obstacle problem in the Calculus of Variations is ∞-superharmonic. Suppose that $\psi : \overline{\Omega} \to \mathbb{R}$ is a given Lipschitz continuous function and that $\psi \in W^{1,\infty}(\Omega)$. For simplicity, assume that Ω is a bounded domain. The function ψ will act as an obstacle: all admissible functions are forced to lie above ψ. We aim at constructing a function $v_\infty \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$ such that $v_\infty \geq \psi$, $v|\partial\Omega = \psi|\partial\Omega$, and for each subdomain $D \subset \Omega$

$$\|\nabla v_\infty\|_{\infty,D} \leq \|\nabla v\|_{\infty,D} \tag{3.1}$$

whenever $v \in C(\overline{D}) \cap W^{1,\infty}(D)$, $v \geq \psi$ in D, and $v|\partial D = v_\infty|\partial D$. In other words, one can characterize v_∞ as the best Lipschitz extension to Ω of the boundary values of ψ, under the constraint that the admissible functions are forced to lie above the obstacle.

The solution to the obstacle problem is unique. Fortunately, we need not deduce that from (3.1). For our purpose it is enough to construct *one* solution as the limit of p-superharmonic functions, which solve the same obstacle problem for the integral $\int_\Omega |\nabla v|^p\,dx$. To this end, we minimize the variational integral $\int_\Omega |\nabla v|^p\,dx$ in the class

$$\mathcal{F}_p = \left\{ v \in C(\overline{\Omega}) \cap W^{1,p}(\Omega) \mid v \geq \psi \text{ in } \Omega\,,\ v = \psi \text{ on } \partial\Omega \right\}. \tag{3.2}$$

There is a unique minimizer in this class, say v_p. Thus

$$\int_\Omega |\nabla v_p|^p\,dx \leq \int_\Omega |\nabla v|^p\,dx \tag{3.3}$$

for each $v \in \mathcal{F}_p$. We refer to [L] about this obstacle problem. Notice that the class of admissible functions is not empty, since $\psi \in \mathcal{F}_p$. (We tacitly assume that $p > n$, so that the boundary values certainly are attained in the classical sense.)

Using the familiar inequalities

$$\|\nabla v_p\|_{p,\Omega} \leq |\Omega|^{1/p}\|\nabla\psi\|_{\infty,\Omega}\,,$$

$$|v_p(x) - v_p(y)| \leq 2n|x-y|^{1-\frac{n}{p}}\|\nabla v_p\|_{p,\Omega}$$

and some compactness arguments, we deduce that a subsequence of v_p's converges uniformly to a function v_∞ and that $v_\infty \in \mathcal{F}_\infty$. Actually, the full sequence converges, see Remark 3.5 below.

3.4 Theorem. *The constructed solution v_∞ to the obstacle problem is ∞-superharmonic in Ω. It is ∞-harmonic in the set $\{v_\infty > \psi\}$.*

Proof. We claim that v_∞ is ∞-superharmonic. Choose a subdomain $D \subset\subset \Omega$ and suppose that $h_\infty \in C(\bar{D})$ is an ∞-harmonic function such that $h_\infty \le v_\infty$ on the boundary ∂D. By Jensen's uniqueness theorem h_∞ is variational, i.e., it is the uniform limit of p-harmonic functions with the same boundary values as h_∞ on ∂D. Given $\varepsilon > 0$, we have $v_p > v_\infty - \varepsilon$ for (a subsequence of) large p's. On the boundary ∂D we have $h_p \le v_\infty < v_p + \varepsilon$ for large p's. By the comparison principle for p-superharmonic functions, the inequality $h_p \le v_p + \varepsilon$ holds in D. At the limit we get $h_\infty \le v_\infty + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we have proved that $v_\infty \ge h_\infty$ in D. Thus v_∞ satisfies the comparison principle. This proves that v_∞ is ∞-superharmonic.

To prove that v_∞ is ∞-harmonic in the set where the obstacle does not hinder, we proceed as follows. Given $\varepsilon > 0$, consider the open set

$$D_\varepsilon = \{x \in \Omega \mid v_\infty(x) > \psi(x) + \varepsilon\},$$

provided that it is not empty. When $p > p_\varepsilon$, $v_p(x) > v_\infty(x) - \varepsilon$ and $v_p(x) > \psi(x)$ in D_ε. Strictly speaking, this holds for a subsequence of p's. It is known that v_p is p-harmonic in the set $\{v_p > \psi\}$, cf. [L]. Especially, v_p is p-harmonic in D_ε, when p is large. This means that v_∞ is the uniform limit in D_ε of p-harmonic functions. It is easily seen that the uniform limit of p-harmonic functions, as p approaches ∞, always is ∞-harmonic. Thus we have established that v_∞ is ∞-harmonic in each D_ε, when $\varepsilon > 0$. This is the desired result.

■

3.5 Remark. A subsequence of p's was used in the construction of v_∞. Indeed, the full sequence converges. To see this, suppose that we have two functions v_∞^1 and v_∞^2 in the previous theorem, perhaps resulting from different subsequences. If the set $\{v_\infty^1 > v_\infty^2\}$ is not empty, v_∞^1 is ∞-harmonic in this set, because $v_\infty^1 > v_\infty^2 \ge \psi$ so that the obstacle does not hinder. But on the boundary of the same set, $v_\infty^1 = v_\infty^2$. By comparison, $v_\infty^2 \geq$ the ∞-harmonic function v_∞^1, a fact that contradicts the definition of the aforementioned set. This proves that $v_\infty^1 \leq v_\infty^2$ in Ω. By symmetry, $v_\infty^1 \geq v_\infty^2$.

Finally, it is worth our while mentioning that the minimization property (3.1) follows rather directly from the construction. (We will not need it.)

4 Viscosity Supersolutions Are Variational

We will show that every viscosity supersolution is variational. In other words, it is locally the uniform limit of p-superharmonic functions. To prove this we will solve the obstacle problem with the supersolution itself acting as obstacle! Therefore we had better first prove that we encounter a Lipschitz continuous obstacle.

4.1 Lemma. *The ∞-superharmonic functions are Lipschitz continuous on compact subsets. In particular, they are locally bounded and belong to $W^{1,\infty}_{\text{loc}}$.*

Proof. Although a direct proof is not difficult, we will deduce the result from Corollary 3.10 in [J], according to which bounded viscosity supersolutions are Lipschitz continuous. Thus we have only to show that the ∞-superharmonic function v is locally bounded. Since this is a local question we may as well assume that $v \geq 0$ in Ω, v being lower semicontinuous by definition.

If $v(x_0) = \infty$ at some point $x_0 \in \Omega$, then we would have that $v \equiv \infty$ in Ω, a situation excluded by definition. Indeed, choose a ball $B(x_0, r) \subset \Omega$. Then the inequalities

$$v(x) \geq k(r - |x - x_0|)\,, \qquad k = 1, 2, 3, \ldots \tag{4.2}$$

hold, when $x = x_0$ and when $|x - x_0| = r$. The function $h(x) = k(r - |x - x_0|)$ is ∞-harmonic in the domain $0 < |x - x_0| < r$, so that (4.2) holds in $B(x_0, r)$ by the comparison principle. This means that $v \equiv \infty$ in $B(x_0, r)$. Continuing like this, with a chain of balls, we get the contradiction.

If v is locally unbounded, we can always select a sequence of points $x_1, x_2, x_3, \ldots$ such that $v(x_k) \geq k$ and $x_0 = \lim x_k$ is an interior point.

For a sufficiently small r all $B(x_k, r) \subset \Omega$, and

$$v(x) \geq k(r - |x - x_k|) , \qquad k = 1, 2, 3, \ldots, \tag{4.3}$$

when $x = x_k$ and when $|x - x_k| = r$. Again the inequality holds in $B(x_k, r)$ by comparison. Thus $v(x) \geq \frac{kr}{2}$, when $|x - x_k| < \frac{r}{2}$. This certainly yields that $v(x_0) = \infty$ and so we are back to the first case.

■

4.4 Theorem.[1] *Any ∞-superharmonic function is variational, i.e., it is a locally uniform limit of p-superharmonic functions.*

Proof. Suppose that w_∞ is an arbitrary ∞-superharmonic function in Ω. By Lemma 4.1 it is locally Lipschitz continuous. Let $D \subset\subset \Omega$ denote a subdomain. By Rademacher's theorem ∇w_∞ exists a.e. in Ω and $w_\infty \in W^{1,\infty}(D)$.

We solve the obstacle problem in the domain D with w_∞ as obstacle. The solution v_∞ is obtained as the uniform limit in D of p-superharmonic functions. By the construction $v_\infty \geq w_\infty$. We refer to Section 3. In the (components of the) open set where the obstacle does not hinder v_∞ is ∞-harmonic, that is, v_∞ is ∞-harmonic in $\{v_\infty > w_\infty\}$. But on the boundary of this set $v_\infty = w_\infty$ (recall that both functions coincide on ∂D by the construction), whence the comparison principle yields that $w_\infty \geq v_\infty$ in the set where $w_\infty < v_\infty$. This is a clear contradiction, except if the aforementioned set is empty. We have proved that $v_\infty = w_\infty$ in D.

Using an exhaustion of Ω with bounded subdomains D_j, $D_1 \subset D_2 \subset \cdots$, and a diagonalization procedure, we obtain that $w_\infty = \lim v_p$ in Ω. The convergence is uniform on each subset D_j, but it may happen that v_p is defined in D_j only when $p \geq$ a certain index depending on j. For the diagonalization one has to observe that $v_p^{D_{j+1}} \geq v_p^{D_j}$ in D_j, where an obvious notation has been used.

This proves the theorem.

■

4.5 Corollary. *The Harnack inequality holds for all non-negative ∞-superharmonic functions:*

$$v(x) \leq e^{\frac{|x-y|}{R-r}} v(y) , \tag{4.6}$$

when $x, y \in B(x_0, r)$ and $B(x_0, R) \subset \Omega$, $0 < r < R$.

Proof. This was proved for variational ∞-superharmonic functions in [LM] and by Theorem 4.4 they are all of this kind.

■

4.7 Corollary (Liouville) *The only ∞-superharmonic functions bounded from below in the whole $\mathbb{R}^n$ are the constants.*

Proof. Adding a constant to the function, we may assume that it is non-negative in $\mathbb{R}^n$. Let $R \to \infty$ in (4.6).

■

Acknowledgement. This note was written while the first author was visiting The University of Texas at Austin and the University of Pittsburgh.

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