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The number of conics tangent to five given conics : the real case.

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Abstract

It is classical result, first established by de Jonquières (1859), that generically the number of conics tangent to 5 given conics in the complex projective plane is 3264. We show here the existence of configurations of 5 real conics such that the number of real conics tangent to them is 3264.

0 Introduction

The following is a classical problem in enumerative geometry :

Given 5 *generic conics, find the number of conics tangent to them.*

In 1848 J. Steiner believed to have found that there are 6^5. In 1859, E. de Jonquières found the correct answer : 3264; however, he did not publish his result because it was in contradiction with Steiner's, and because M. Chasles didn't trust him. Finally, Chasles established the correct answer in 1864, and Th. Berner again in (1865) (cf. [6], page 268).

The problem has been reworked more recently by Fulton-McPherson [2] and Procesi-De Concini [1] (see also [3], example 9.1.9 on page 158).

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We shall prove the existence of configurations of 5 real conics that admit exactly 3264 real conics tangent to them at real points. By a real conic we mean a conic whose equation has real coefficients and by the exact number we mean that there are no multiplicities to take into account : each solution to the problem is a smooth conic which is simply tangent at exactly 1 real point of each of the 5 given conics.

The configuration of 5 conics will be found as a small deformation of the 5 degenerate conics constituted by suitable pairs of lines crossing at the vertices of a regular pentagon in an affine plane. By taking different pairs of lines, it is possible to find configurations of 5 conics with a number of conics tangent to them smaller then 3264, but we do not investigate this any further here.

The main ingredient that we shall use to control the deformation in the real case is theorem 8, which might have some interest by itself. It says that if the derivatives of a C^∞ map F at some point x_0 coincide up to order 2 with those of the map $(x_1, \ldots, x_k) \mapsto (x_1^2, \ldots, x_k^2)$, then there exist regular values near $F(x_0)$ with 2^k preimages near x_0.

We have been informed that W. Fulton had realized already in 1986-87, using a similar approach, that there exist 5 real conics with 3264 real conics tangent to them.

1 First contacts

Most statements of this paragraph will be made over $\mathbb{R}$, but they remain valid, as well as their proofs, over $\mathbb{C}$.

Let us denote by $\mathcal{Q}$ (respectively $\mathcal{Q}_r$) the space of all bilinear symmetric forms (respectively the bilinear symmetric forms of rank r) on $\mathbb{R}^3$. Denote by $\mathbb{P}\mathcal{Q}$ the projective space of $\mathcal{Q}$ and by $\mathbb{P}\mathcal{Q}_r$ the locally closed subvariety of $\mathbb{P}\mathcal{Q}$ corresponding to $\mathcal{Q}_r$. Let $\mathbb{P}^2 = \mathbb{P}\mathbb{R}^2$ be the real projective plane.

Geometrically, $\mathbb{P}\mathcal{Q}$ is the space of all (possibly empty) real conics of $\mathbb{P}^2$, $\mathbb{P}\mathcal{Q}_3$ is the set of all smooth conics, $\mathbb{P}\mathcal{Q}_2$ is the set of all singular conics consisting of 2 distinct lines, and $\mathbb{P}\mathcal{Q}_1$ is the set of all double lines.

For $q \in \mathcal{Q}\setminus\{0\}$ (resp. $x \in \mathbb{R}^3 \setminus \{0\}$) we denote by $[q]$ (resp. $[x]$) its image in $\mathbb{P}\mathcal{Q}$ (resp. $\mathbb{P}^2$); for simplicity, we will sometimes drop the brackets [ ]. Consider the subvariety W of $(\mathbb{P}\mathcal{Q})^5 \times \left(\mathbb{P}^2\right)^5 \times \mathbb{P}\mathcal{Q}_3$

defined by:
$W = \{([q_1],[q_2],\cdots,[q_5],[x_1],\cdots,[x_5],[q]) \in (\mathbb{P}Q)^5 \times (\mathbb{P}^2)^5 \times \mathbb{P}Q_3 \mid$
$[x_i] \neq [x_j], i \neq j$ and the following equations hold, $i = 1,\cdots,5$:
$(\mathrm{I}) q_i(x_i,x_i) = 0\,, (\mathrm{II}) q(x_i,x_i) = 0\,, (\mathrm{III}) q_i(x_i,\cdot) \wedge q(x_i,\cdot) = 0\}.$
Note that in fact the equations $q(x_i,x_i) = 0$ and $q_i(x_i,x_i) = 0$ imply already that $q_i(x_i,\cdot) \wedge q(x_i,\cdot)$ vanishes on $\{x_i\} \wedge \mathbb{R}^3$ and therefore equation (III) can be viewed in $\left(\mathbb{R}^3 \wedge \mathbb{R}^3 / \{x_i\} \wedge \mathbb{R}^3\right)^* \simeq \mathbb{R}$. Alternatively, if we choose x_i', $x_i'' \in \mathbb{R}^3$ such that their images in $\mathbb{R}^3/[x_i^0]$ are linearly independent, then in a neighbourhood of $([q_1],\ldots,[q_5],[x_1^0],\ldots,[x_5^0],[q]) \in W$ equations (III) can be written :

$$(q_i(x_i,\cdot) \wedge q(x_i,\cdot))\,(x_i',x_i'') = q_i(x_i,x_i')q(x_i,x_i'') - q_i(x_i,x_i'')q(x_i,x_i') = 0 \quad .$$

The conditions defining W mean that the 2 conics defined by $q_i(x) = 0$ and $q(x) = 0$ are tangent at $[x_i]$; if $[x_i]$ is singular on q_i, it means simply that $x_i \in q \cap q_i$. In order to simplify the notation, we shall say that x_i belongs to q and q_i, or $x_i \in q \cap q_i$, and that q and q_i are tangent at x_i. We shall denote by $(q)_{\mathrm{sing}}$ and $(q)_{\mathrm{reg}}$ respectively the singular and the regular part of q.

Denote by

$$F: W \to (\mathbb{P}Q)^5$$

the restriction to W of the natural projection $(\mathbb{P}Q)^5 \times \left(\mathbb{P}^2\right)^5 \times \mathbb{P}Q_3 \to (\mathbb{P}Q)^5$. The problem is to find the maximal number of elements of $F^{-1}(u)$, for $u \in (\mathbb{P}Q)^5$ belonging to a suitable open, dense subset $\mathcal{U} \subset (\mathbb{P}Q)^5$ that we will define in this paragraph.

Remark. The image of W by the projection

$$p: (\mathbb{P}Q)^5 \times \left(\mathbb{P}^2\right)^5 \times \mathbb{P}Q_3 \to (\mathbb{P}Q)^5 \times \mathbb{P}Q_3$$

is the set of $([q_1],\ldots,[q_5],q)$ such that q is tangent to q_i, $i = 1,\ldots,5$ at some unspecified point. Denote by W_0 the locally closed subvariety of $(\mathbb{P}Q)^5 \times \mathbb{P}Q_3$ of the $([q_1],\ldots,[q_5],[q])$ that are such that the intersection of q and q_i, $i = 1,\ldots,5$, consists of 3 distinct points, at 2 of which q and q_i are tranversal, and the third (necessarily a real point) at which q and q_i are tangent. Denote by $F_0 : W_0 \to (\mathbb{P}Q)^5$ the natural projection; our genuine problem is to compute the cardinality of the fibers $F_0^{-1}(u)$ for u in a suitable open subset of $(\mathbb{P}Q)^5$.

Clearly, W_0 is open and dense in $p(W)$ and p induces a bijection from $p^{-1}(W_0) \cap W$ to W_0. It follows from Proposition 1 below that W and $(\mathbb{P}Q)^5$ have the same dimension, and so there the open subset $\mathcal{U} = (\mathbb{P}Q)^5 \setminus \overline{F(W \setminus p^{-1}(W_0)}$ is non-empty and p induces a bijection: $F^{-1}(u) \to F_0^{-1}(u)$ for $u \in \mathcal{U}$.

This justifies that we concentrate on the study of the generic fibers of F rather than F_0.

In fact we shall denote by $\mathcal{U}$ an open set in $(\mathbb{P}Q)^5$ that will shrink during this paragraph, as we add more and more genericity conditions.

Recall that for $[x] \in \mathbb{P}^n = \mathbb{P}(\mathbb{R}^{n+1})$ the tangent space $T_{[x]}\mathbb{P}^n \simeq \mathbb{R}^{n+1}/[x]$; we shall write $\overline{x}$ for an element of $T_{[x]}\mathbb{P}^n$, or for some of its representatives in $\mathbb{R}^{n+1}$.

Proposition 1. *The variety W is smooth, of dimension 25. For $w = ([q_i], [x_i], [q]) \in W$, the tangent space T_wW is the set of $(\overline{q}_1, \ldots, \overline{q}_5, \overline{x}_1, \ldots, \overline{x}_5, \overline{q})$ such that:*

$$\begin{cases} \text{(I)} & 2q_i(x_i, \overline{x}_i) + \overline{q}_i(x_i, x_i) = 0 \\ \text{(II)} & 2q(x_i, \overline{x}_i) + \overline{q}(x_i, x_i) = 0 \qquad \textit{for } i = 1, \ldots, 5\,. \\ \text{(III)} & (\overline{q}_i(x_i, \cdot) + q_i(\overline{x}_i, \cdot)) \wedge q(x_i, \cdot) + q_i(x_i, \cdot) \wedge (\overline{q}(x_i, \cdot) + q(\overline{x}_i, \cdot)) = 0 \end{cases}$$

Proof. Let $w = ([q_1], [q_2], \ldots, [q_5], [x_1], \ldots, [x_5], [q]) \in W$ and let us take the following derivates of the equations defining W at the point w :

$$\frac{\partial \text{I}}{\partial q_i}(\overline{q}_i) = \overline{q}_i(x_i, x_i) \quad , \quad \frac{\partial \text{II}}{\partial x_i}(\overline{x}_i) = 2q(x_i, \overline{x}_i) \quad , \quad \frac{\partial \text{III}}{\partial q_i}(\overline{q}_i) = \overline{q}_i(x_i, \cdot) \wedge q(x_i, \cdot)\,.$$

Choose x'_i, $x''_i \in \mathbb{R}^3$ linearly independent in $\mathbb{R}^3/[x_i] \simeq T_{[x_i]}\mathbb{P}^2$, $i = 1, \ldots, 5$. It is readily checked that the linear map

$$(\overline{q}_i, \overline{x}_i) \mapsto \Big(\overline{q}_i(x_i, x_i), 2q(x_i, \overline{x}_i), (\overline{q}_i(x_i, \cdot) \wedge q(x_i, \cdot))(x'_i, x''_i)\Big)_{i=1,\ldots,5}$$

is surjective, which shows that W is smooth of dimension 25, and that the projection $W \to \mathbb{P}Q_3$ is a fibration.

The second assertion follows by taking the total derivatives of the equations I, II and III defining W

∎

We now introduce a first series of genericity conditions on $([q_1], \dots, [q_5]) \in (\boldsymbol{P}Q)^5$. Although the q_i's are real conics, the lines and points mentioned below are taken into account even if they are not in $\boldsymbol{P}^2(\boldsymbol{R})$:

(G_1) : $\forall$ distinct i, j, k, $q_i \cap q_j \cap q_k = \emptyset$ (in $\boldsymbol{P}^2(\boldsymbol{C})$).

$(\check{G}_1)$: $\forall$ distinct i, j, k, q_i, $q_j$and q_k have no common tangent (in $\boldsymbol{P}^2(\boldsymbol{C})$).

(G_2) : $\forall$ distinct i, j, k, ℓ, any common tangent to q_i and q_j does not contain points in $q_k \cap q_\ell$ (in $\boldsymbol{P}^2(\boldsymbol{C})$).

(G_3) : $\forall$ distinct i, j, k, ℓ, m, if $d_{r,s}$ is any tangent common to q_r and q_s, we have that $d_{i,j} \cap d_{k,\ell} \cap q_m = \emptyset$ (in $\boldsymbol{P}^2(\boldsymbol{C})$).

$(\check{G}_3)$: $\forall$ distinct i, j, k, ℓ, m and $\forall x_{r,s} \in q_r \cap q_s$ the line through $x_{i,j}$ and $x_{k,\ell}$ is not tangent to q_m. (in $\boldsymbol{P}^2(\boldsymbol{C})$).

(G_4) : $\forall i \neq j$, q_i and q_j intersect transversally (in $\boldsymbol{P}^2(\boldsymbol{C})$) at points that are smooth both on q_i and q_j.

In other words, the configurations represented in figure 1 are excluded (as usual, we draw a real picture that represents objects in $\boldsymbol{P}^2(\boldsymbol{C})$).

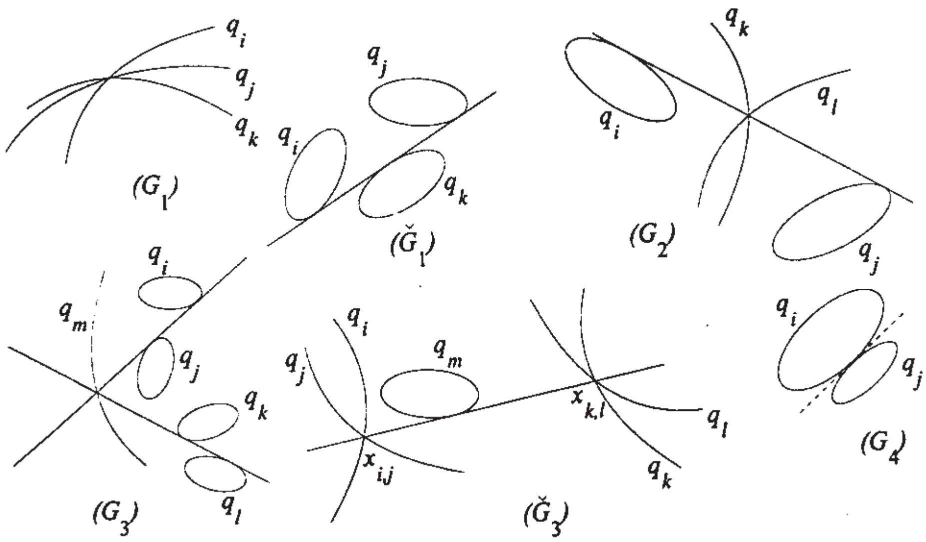

Figure 1. Configurations that we don't want in Section 1.

Let $\mathcal{U} \subset (\mathbb{P}Q)^5$ be the set of $([q_1], \dots, [q_5])$ satisfying the above genericity conditions. It is readily verified that $\mathcal{U}$ is a Zariski-open, nonempty subset of $(\mathbb{P}Q)^5$. Let

$$W(\mathcal{U}) = \{([q_1], \cdots, [q_5], [x_1], \cdots, [x_5], [q]) \in W \mid ([q_1], \cdots, [q_5]) \in \mathcal{U}\} \quad .$$

Proposition 2. *$F : W(\mathcal{U}) \to \mathcal{U}$ is proper*

Proof. We will use the space $\widehat{\mathbb{P}Q}$ of complete conics (see [4], example 22.27, page 297). Recall that $\widehat{\mathbb{P}Q}$ is the closure in $\mathbb{P}Q \times \mathbb{P}\check{Q}$ of the set

$$\{([q], [q']) \in \mathbb{P}Q_3 \times \mathbb{P}\check{Q} \mid q' \text{ is the dual of } q\} \quad ;$$

In fact, the natural projection $\widehat{\mathbb{P}Q} \to \mathbb{P}Q$ is the blowing up of $\mathbb{P}Q$ along $\mathbb{P}Q_1$. Set theoretically, $\widehat{\mathbb{P}Q}$ consists of pairs $[q], [q']$ where

- $[q'] = [\check{q}]$ if $[q]$ is of rank 2 or 3.
- in the case when $[q]$ is of rank 1, $[q']$ consists of a pair of lines (distinct or not) of $\check{\mathbb{P}}^2$ which are the lines of $\mathbb{P}^2$ going through one of two points (distinct or not) of $[q]$.

Denote by $\widehat{W}(\mathcal{U})$ the closed subvariety of $(\mathbb{P}Q)^5 \times \left(\check{\mathbb{P}}^2\right)^5 \times (\mathbb{P}Q)^5 \times \left(\mathbb{P}\check{Q}\right)^5 \times \widehat{\mathbb{P}Q}$ which consists of the $\hat{w} = (([q_i], [x_i], [\ell_i])_{i=1,\dots,5}, ([q], [q']))$ such that:

(1) $([q_1], \dots, [q_5]) \in \mathcal{U}$

(2) $x_i \in \ell_i$

(3) $q_i(x_i, y) = q(x_i, y) = 0, \ \forall y \in \ell_i$

(4) $q'(\ell_i, \ell_i) = 0$

for $i = 1, \dots, 5$. Consider the commutative diagram:

$$\begin{array}{ccc} \widehat{W}(\mathcal{U}) & \xrightarrow{\psi} & W(\mathcal{U}) \\ & \hat{F} \searrow \quad \swarrow F & \\ & \mathcal{U} & \end{array}$$

where $\widehat{F}$ and ψ are induced by the natural projections; now $\widehat{F}$ is proper. Therefore it suffices to show that the genericity conditions defining $\mathcal{U}$ imply that ψ is an isomorphism. But the lemma below implies that if $\hat{w} \in \widehat{W}(\mathcal{U})$, then $q \in \mathbb{P}\mathcal{Q}_3$, and so the map

$$([q_i], [x_i], [q]) \mapsto ([q_i], [x_i], [T_{[x_i]}q], ([q], [\check{q}])$$

is an inverse of ψ.

■

Lemma. *If* $\hat{w} = (([q_i], [x_i], [\ell_i])_{i=1,\dots,5}, ([q], [q'])) \in \widehat{W}(\mathcal{U})$, *then* $q \in \mathbb{P}\mathcal{Q}_3$.

Proof. Indeed, if $q \in \mathbb{P}\mathcal{Q}_2$, one of the conditions (G_1), $(\check{G}_1)$, (G_2) or G_3 is violated. If $q \in \mathbb{P}\mathcal{Q}_1$, the tangents ℓ_i to q_i at x_i belong to one of the sheaves of lines defined by q', but then one of the conditions (G_1), $(\check{G}_1)$, (G_2) or $\check{G}_3$ is violated .

■

Proposition 3. *The fibers of* $F = F(\mathcal{U}) : W(\mathcal{U}) \to \mathcal{U}$ *are finite.*

Proof. Consider the complexification $F_{\mathbb{C}}: W(\mathcal{U})_{\mathbb{C}} \to \mathcal{U}_{\mathbb{C}}$ of F and the projection $p : W(\mathcal{U})_{\mathbb{C}} \to (\mathbb{P}\mathcal{Q}_3)_{\mathbb{C}}$. Let $u \in \mathcal{U}_{\mathbb{C}}$; since $F_{\mathbb{C}}$ is proper and $(\mathbb{P}\mathcal{Q}_3)_{\mathbb{C}}$ is an affine variety, $p(F_{\mathbb{C}}^{-1}(u))$ consists of a finite number of points. Moreover, $p|F_{\mathbb{C}}^{-1}(u) \to (\mathbb{P}\mathcal{Q}_3)_{\mathbb{C}}$ has finite fibers because of (G_4).

■

Here comes a genericity condition that we will need later on. Let $k \in \{1, \dots, 5\}$ and denote by V_k the subvariety of $W(\mathcal{U})$ consisting of the $([q_i], [x_i], [q])$ such that the order of contact of q and q_k at x_k is at least 3. For example, let $[q_k] \in \mathbb{P}\mathcal{Q}_2$ and let x_k be the singular point of q_k; if q is tangent to one of the 2 lines through x_k that constitute q_k then the order of contact of q and q_k at x_k is 3 if q is smooth (see figure

2).

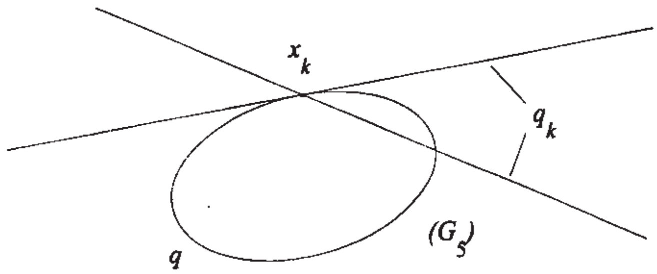

Figure 2. The order of contact is 3.

Since $W(\mathcal{U})$ and $\mathcal{U}$ have the same dimension, $F(V_k) \not\subseteq \mathcal{U}$ and $F(V_k)$ is closed, since F is proper. Our last genericity condition is the following:

(G_5) : $([q_1], \ldots, [q_5]) \notin \bigcup_{k=1,\ldots,5} F(V_k)$.

We shall denote again by $\mathcal{U}$ the set of $([q_1], \ldots, [q_5])$ that satisfy all the genericity conditions introduced so far.

Notice that $\mathcal{U}$ contains configurations of the form $([q_1], \cdots, [q_5])$ with $q_i \in \mathcal{Q}_2$, $i = 1, \ldots, 5$. Indeed, there is no problem in choosing $u^0 = (q_1^0, \ldots, q_5^0)$, $q_i^0 \in \boldsymbol{P}\mathcal{Q}_2$, $i = 1, \ldots, 5$, satisfying conditions (G_1) through (G_4) and $\check{G}_1$, $\check{G}_3$. For some $k \in \{1, \ldots, 5\}$, let y_k denote the singular point of q_k^0. Consider

$$F^{-1}(u^0)_k = \left\{(u^0, [x_i], [q]) \in W(\mathcal{U}) \mid x_k = y_k\right\} \quad :$$

this is a finite set which depends only on y_k, not on q_k. Therefore, we can deform u^0 into $u = ([q_1], \ldots, [q_5])$, where $q_i = q_i^0$ for $i \neq k$, and q_k is singular at y_k, but for all $(u, [x_i], [q]) \in F^{-1}(u)$ none of the two distinct lines composing q_k is tangent to $[q]$ at $x_k = y_k$, that is : $u \notin F(V_k)$.

2 On the singularities of the map F

Throughout this paragraph we shall assume that $u \in \mathcal{U}$.

Let $w = (u, [x_i], [q]) \in F^{-1}(u)$ and

$$s = s(w) = \left|\left\{x_i \mid x_i \in (q_i)_{\text{sing}}\right\}\right|$$

where $|X|$ denotes the cardinality of X. We shall see that the behaviour of F near w essentially depends only on $s(w)$.

Proposition 4. *Let $s \in \{0, \ldots, 5\}$ and assume that $x_i \in (q_i)_{sing}$ for $i \leq s$ and $x_i \in (q_i)_{reg}$ for $i > s$. Then the projection*

$$(\overline{x}_1, \ldots, \overline{x}_5, \overline{q}) \mapsto (\overline{x}_1, \ldots, \overline{x}_s)$$

induces an isomorphism

$$\phi : Ker(dF_w) \stackrel{\simeq}{\to} \{(\overline{x}_1, \ldots, \overline{x}_s) \mid q(x_i, \cdot) \wedge q_i(\overline{x}_i, \cdot) = 0 \,, \, i = 1, \ldots, s\} \quad .$$

If $\tau_i \in T_{[x_i]}q \setminus \{0\}$, then

$$Im\,\phi = \{(\overline{x}_1, \ldots, \overline{x}_s) \mid q_i(\tau_i, \overline{x}_i) = 0 \,, \, i = 1, \ldots, s\} \quad .$$

Corollary 5. $\dim \operatorname{Ker}(dF_w) = s(w)$

Proof. Indeed, since q_i, $i = 1, \ldots, s$ consists of 2 distinct lines, the linear map

$$T_{[x_i]}\mathbb{P}^2 \to \mathbb{R} \quad , \quad \overline{x}_i \mapsto q_i(\tau, \overline{x}_i)$$

has a kernel of dimension 1.

■

We give now a geometric description of $\operatorname{Im}\phi$. Let $\mathbb{P}\mathbb{R}^1_{[x_i]}$ denote the set of lines of $\mathbb{P}\mathbb{R}^2$ through $[x_i]$. Let us recall how two lines $\ell', \ell'' \in \mathbb{P}\mathbb{R}^1_{[x_i]}$ define a polarity among pairs of lines of $\mathbb{P}\mathbb{R}^1_{[x_i]}$. Let α be a homogeneous 2-form in 2 variables whose zeroes are ℓ' and ℓ''; if $v, w \in \mathbb{R}^2 \setminus \{0\}$ are such that $\alpha(v, w) = 0$, we say that the line through v is polar to the line through w with respect to the two lines ℓ', ℓ''. Choose $\tau_i \in T_{[x_i]}q \setminus \{0\}$; then $q_i(\overline{x}_i, \tau_i) = 0$ for $(\overline{x}_1, \ldots, \overline{x}_s) \in \operatorname{Im}\phi$. This means that $\overline{x}_i$ must lie on the polar line to $T_{[x_i]}(q)$ with respect to the 2 lines through $[x_i]$ defined by q_i (see figure 3).

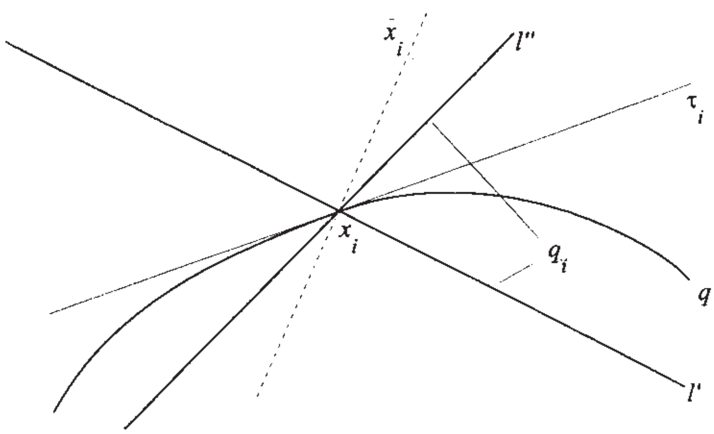

Figure 3. Geometric interpretation of the kernel of dF_w.

Proof of proposition 4. According to proposition 1, $\mathrm{Ker}\,(dF_w)$ is the subspace of $\left(\bigoplus_{i=1,\dots,5} T_{[x_i]}\mathbb{P}^2\right) \oplus T_{[q]}\mathbb{P}Q$ defined by the equations:

$$(A)\quad \begin{cases} \text{(I)} & q_i(x_i, \overline{x}_i) = 0 \\ \text{(II)} & 2q(x_i, \overline{x}_i) + \overline{q}(x_i, x_i) = 0 \qquad i = 1, \dots, 5 \\ \text{(III)} & q_i(\overline{x}_i, \cdot) \wedge q(x_i, \cdot) + q_i(x_i, \cdot) \wedge (\overline{q}(x_i, \cdot) + q(\overline{x}_i, \cdot)) = 0 \end{cases}$$

For $i \leq s$, since $q_i(x_i, \cdot) = 0$, this set of equations is equivalent to

$$(B)\quad \begin{cases} \text{(II)} & 2q(x_i, \overline{x}_i) + \overline{q}(x_i, x_i) = 0 \\ \text{(III)} & q_i(\overline{x}_i, \cdot) \wedge q(x_i, \cdot) = 0 \end{cases} \quad i \leq s$$

and for $i > s$ there exist scalars λ_i such that $q(x_i, \cdot) = \lambda_i q_i(x_i, \cdot)$. Therefore (A)(I) implies that $q(x_i, \overline{x}_i) = 0$ and the set of equations becomes:

$$(C) \quad \begin{cases} \text{(I)} & q_i(x_i, \overline{x}_i) = 0 \\ \text{(II)} & \overline{q}(x_i, x_i) = 0 \qquad i > s \quad . \\ \text{(III)} & q_i(x_i, \cdot) \wedge (q(\overline{x}_i, \cdot) + \overline{q}(x_i, \cdot) - \lambda_i q_i(\overline{x}_i, \cdot)) = 0 \end{cases}$$

Equation (B)(III) shows that ϕ is well defined.

ϕ is surjective. Let $\overline{x}_i \in T_{[x_i]}\mathbb{P}^2$ be such that $q(x_i, \cdot) \wedge q_i(\overline{x}_i, \cdot) = 0$ for $i \leq s$. Since q is non-singular, three of the x_i's are never aligned and so there exists $\overline{q} \in Q$ such that

$$\overline{q}(x_i, x_i) = \begin{cases} -2q(x_i, \overline{x}_i) & \text{if} \quad i \leq s \\ 0 & \text{if} \quad i > s \end{cases}$$

We choose $\overline{x}_i$, $i > s$, such that (C)(I) is satisfied. Then $\overline{x}_i = \xi_i \cdot \tau_i$, where τ_i is some fixed non zero element in $T_{[x_i]}q$ and ξ_i is a scalar.

We proceed now to choose ξ_i in order to satisfy (C)(III). Since the kernel of $q(x_i, \cdot)$, which equals the kernel of $q_i(x_i, \cdot)$, is generated by x_i and τ_i, we have to choose ξ_i in such a way that $q(\overline{x}_i, \cdot) + \overline{q}(x_i, \cdot) - \lambda_i q_i(\overline{x}_i, \cdot)$ also vanishes on x_i and on τ_i. It clearly vanishes on x_i; now $q_i(\tau_i, \tau_i) = 0$ and $q(\tau_i, \tau_i) \neq 0$. We may therefore take :

$$\xi_i = -\frac{\overline{q}(x_i, \tau_i)}{q(\tau_i, \tau_i)} \quad .$$

ϕ is injective. If $\overline{x}_i = 0$, $i \leq s$, then it follows from (B)(II) that $\overline{q}(x_i, x_i) = 0$ for $i \leq s$ and by (C)(II) $\overline{q}(x_i, x_i) = 0$ for $i > s$. Therefore, q and $\overline{q}$ have the 5 distinct points $[x_1], \ldots, [x_5]$ in common, and no three of these are aligned because q is non-singular, and so $\overline{q} = 0$ in $T_{[q]}\mathbb{P}Q$.

Now it follows from (C)(III) that for $i > s$

$$q_i(x_i, \cdot) \wedge (q(\overline{x}_i, \cdot) - \lambda_i q_i(\overline{x}_i, \cdot)) = 0$$

and therefore there are some scalars μ_i such that:

$$\heartsuit \qquad q(\overline{x}_i, \cdot) = \lambda_i q_i(\overline{x}_i, \cdot) + \mu_i q_i(x_i, \cdot) \quad .$$

Since $q_i(x_i, \overline{x}_i) = 0$ and $x_i \notin (q_i)_{\text{sing}}$ for $i > s$, $\overline{x}_i$ belongs to one of the 2 distinct lines that constitute q_i, say ℓ_i', and therefore $q_i(\overline{x}_i, \overline{x}_i) = 0$.

Replacing the dot by $\overline{x}_i$ in ♡ shows that $q(\overline{x}_i, \overline{x}_i) = 0$. But $q \cap \ell'_i = \{x_i\}$, therefore $\overline{x}_i = 0$ in $T_{[x_i]}\mathbb{P}^2$.

Since q is non-singular, $q(x_i, \cdot) \wedge q_i(\overline{x}_i, \cdot) = 0$ is equivalent to say that $q_i(\overline{x}_i, \cdot)$ vanishes on the kernel of $q(x_i, \cdot)$, which is generated by τ_i and x_i. Therefore :

$$q(x_i, \cdot) \wedge q_i(\overline{x}_i, \cdot) = 0 \Leftrightarrow q_i(x_i, \tau_i) = 0$$

■

We want now to study the second derivative of F. Recall that for a C^∞ map $G : X \to Y$ between C^∞ manifolds, the second intrinsic derivative, first introduced by Porteous [5], is the linear map

$$\spadesuit \qquad d^2\widetilde{G}_x : \mathrm{Ker}\,(dG_x) \otimes T_xX \to \mathrm{Coker}(dG_x)$$

which is obtained from the second derivative at x of G written in local coordinates. If $G : \mathbb{R}^n \to \mathbb{R}^p$ and $h : U \to \mathbb{R}^n$, $H^{-1} : V \to \mathbb{R}^p$ are local diffeomorphisms on $\mathbb{R}^n$ and $\mathbb{R}^p$ respectively, where $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^p$, $h(0) = x$, $H^{-1}(0) = G(x)$, then

$$\begin{aligned} d^2(HGh)_0 &= dH_{G(x)}(d^2G_x(dh_0, dh_0)) + dH_{G(x)}(dG_x(d^2h_0)) \\ &+ d^2H_{G(x)}(dG_x(dh_0), dG_x(dh_0)) \end{aligned}$$

from which it follows that the linear map $d^2\widetilde{G}_x : \mathrm{Ker}\,(dG_x) \otimes T_x\mathbb{R}^n \to \mathrm{Coker}(dG_x)$ is affected only by the linear part of the local diffeomorphisms h and H. This shows that the linear map of ♠ is well defined.

Let now L_1, L_2 and L_3 be open sets in $\mathbb{R}^{n_1}$, $\mathbb{R}^{n_2}$ and $\mathbb{R}^{n_3}$ respectively and let $\Phi : L_1 \times L_2 \to L_3$ be C^∞ and assume that $0 \in L_3$ is a regular value of Φ. Set $W = \Phi^{-1}(0)$ and let $F : W \to L_1$ be the map induced by the projection on the first factor. We want to express the second intrinsic derivative of F in terms of the derivatives of Φ. Denote by $\frac{\partial \Phi}{\partial w_1}(w)$ and $\frac{\partial \phi}{\partial w_2}(w)$ the derivatives of Φ in the direction L_1 and L_2 respectively at the point $w = (w_1, w_2)$.

Lemma 6. *The derivative $\frac{\partial \Phi}{\partial w_1}(w)$ induces an isomorphism:*

$$\theta : Coker(dF_w) \xrightarrow{\simeq} Coker\left(\frac{\partial \Phi}{\partial w_2}(w)\right) \quad .$$

We have a commutative diagram:

$$\begin{array}{ccc} Ker(dF_w)\otimes T_wW & \xrightarrow{d^2\Phi_w} & \boldsymbol{R}^{n_3} \\ \downarrow d^2\widetilde{F}_w & & \downarrow \\ Coker(dF_w) & \underset{\simeq}{\xrightarrow{(-1)\theta}} & Coker(\frac{\partial\Phi}{\partial w_2}(w)) \end{array}$$

from which $d^2\widetilde{F}_w$ can be expressed in terms of the derivatives of Φ.

Proof. The fact that θ is an isomorphism follows easily from the fact that Φ is a submersion and from the definition of F.

For the commutative diagram, let $h = (h^1, h^2) : U \to W \subset L_1 \times L_2$ be a local parametrisation of W, $h(0) = w$. Since $\Phi \cdot h = 0$, we have:

$$\begin{aligned} d^2\Phi_w(dh_0, dh_0) + d\Phi_w(d^2h_0) &= d^2\Phi_w(dh_0, dh_0) \\ &+ \frac{\partial\Phi}{\partial w_1}(w)(d^2h_0^1) + \frac{\partial\Phi}{\partial w_2}(w)(d^2h_0^2) = 0 \end{aligned}$$

and therefore, for $\overline{x}_i \in T_0U$, $i = 1, 2$, and $\overline{w}_i = dh_0(\overline{x}_i)$:

$$d^2\Phi_w(\overline{w}_1, \overline{w}_2) \equiv -\frac{\partial\Phi}{\partial w_1}(w)\left(d^2h_0^1(\overline{x}_1, \overline{x}_2)\right) \bmod \mathrm{Im}\frac{\partial\Phi}{\partial w_2}(w) \quad .$$

Since $h^1 = F \cdot h$, $d^2h_0^1(\overline{x}_1, \overline{x}_2) = d^2F_w(dh_0(\overline{x}_1), dh_0(\overline{x}_2)) + dF_w(d^2h_0(\overline{x}_1, \overline{x}_2))$ and so :

$$\begin{aligned} d^2\Phi_w(\overline{w}_1, \overline{w}_2) &\equiv -\frac{\partial\Phi}{\partial w_1}(w)\left(d^2F_w(dh_0(\overline{x}_1), dh_0(\overline{x}_2))\right) \\ &\quad -\frac{\partial\Phi}{\partial w_1}(w)\left(dF_w d^2h_0(\overline{x}_1, \overline{x}_2)\right) \\ &\equiv -\frac{\partial\Phi}{\partial w_1}(w)\left(d^2F_w(\overline{w}_1, \overline{w}_2)\right) \bmod \mathrm{Im}\ \frac{\partial\Phi}{\partial w_2}(w) \end{aligned}$$

from which our assertion follows.

■

We come back to our map $F : W(\mathcal{U}) \to \mathcal{U}$. Let L_1 be an open subset of $\mathcal{U} \subset (\boldsymbol{P}Q)^5$, L_2 an open subset of $\left(\boldsymbol{P}^2\right)^5 \times \boldsymbol{P}Q_3$ and $L_3 = \boldsymbol{R}^{15}$; we assume that L_1 and L_2 are contained in products of affine open sets, so that we have explicit representatives for $([q_i],[x_i],[q]) \in L_1 \times L_2$, and therefore it makes sense to write the map:

$$\Phi : L_1 \times L_2 \to L_3\,,\ ([q_i],[x_i],[q]) \mapsto$$
$$\Big((q_i(x_i,x_i))_{i=1,\dots,5},(q(x_i,x_i))_{i=1,\dots,5}(q_i(x_i,\cdot)\wedge q(x_i,\cdot))_{i=1,\dots,5}\Big)\,.$$

Note that because the projective spaces are replaced by affine spaces of the same dimension, we can also look at q_i and q as non-homogeneous polynomials of degree 2 on $\mathbb{R}^2$. Their derivatives at $x_i \in \mathbb{R}^2$ are linear maps: $\mathbb{R}^2 \to \mathbb{R}$, and if $q_i(x_i,x_i) = q(x_i,x_i) = 0$, the condition $d(q_i)_{x_i} \wedge dq_{x_i} = 0$ is equivalent to $q_i(x_i,\cdot)\wedge q(x_i,\cdot) = 0$. We know from proposition 1 that $0 \in \mathbb{R}^{15}$ is a regular value of Φ.

Recall that we assume that $[q_i] \in \mathbb{P}Q_2$, $i = 1,\dots,5$, $x_i \in (q_i)_{\text{sing}}$ for $i = 1,\dots,s$ and $x_i \in (q_i)_{\text{reg}}$ for $i = s+1.\dots,5$. For $w = ([q_i],[x_i],[q])$, we have that $\dim \operatorname{Ker}(dF_w) = s$, and so $\dim \operatorname{Coker}(dF_w) = \dim \operatorname{Coker}(\frac{\partial\Phi}{\partial w_2}(w)) = s$. Since

$$\frac{\partial\Phi}{\partial w_2}(w)(\overline{x}_1,\dots,\overline{x}_5,\overline{q}) = (q_1(x_1,\overline{x}_1),\dots,q_5(x_5,\overline{x}_5),\dots) = (\underbrace{0,\dots,0}_{s},*,\dots,*)$$

the first s coordinates of $\mathbb{R}^{15}$ represent $\operatorname{Coker}(\frac{\partial\Phi}{\partial w_2}(w))$ and so the restriction of the second intrinsic derivative of F to $\operatorname{Ker}(dF_w)\otimes\operatorname{Ker}(dF_w)$, that we still denote by $d^2\widetilde{F}_w$, can be identified using Lemma 6 to the bilinear map:

$$\begin{array}{rcl}\operatorname{Ker}(dF_w)\otimes\operatorname{Ker}(dF_w) & \to & \mathbb{R}^s \quad , \quad (\overline{x}_1,\dots,\overline{x}_5,\overline{q})\otimes(\overline{\overline{x}}_1,\dots,\overline{\overline{x}}_5,\overline{\overline{q}}) \\ & \mapsto & (-1)\cdot(q_1(\overline{x}_1,\overline{\overline{x}}_1),\dots,q_s(\overline{x}_s,\overline{\overline{x}}_s)) \quad .\end{array}$$

Recall from Proposition 4 that if $(\overline{x}_1,\dots,\overline{x}_5,\overline{q}) \in \operatorname{Ker}(dF_w)\setminus\{0\}$ then $q(x_i,\cdot)\wedge q_i(\overline{x}_i,\cdot) = 0$ for $i = 1,\dots,s$. If in addition $q_i(\overline{x}_i,\overline{x}_i) = 0$ for some $i = 1,\dots,s$, then $\overline{x}_i \in (q_i)_{\text{reg}}$ and so the tangent line to q at x_i is a component of q_i, which is excluded by the genericity condition (G_5).

In conclusion, we have proved the following result:

Theorem 7. *Let $u \in \mathcal{U}\cap(\mathbb{P}Q_2)^5$ and $w = (u,[x_1],\dots,[x_5],[q]) \in F^{-1}(u)$; assume that $x_i \in (q_i)_{sing}$ for $i \le s$ and $x_i \in (q_i)_{reg}$ for $i > s$. Then:*

- *dim* $Ker(dF_w) = s = \dim Coker(dF_w)$

- *Let* $(\overline{x}_1, \ldots, \overline{x}_5, \overline{q}) \in Ker dF_w$, *so that* $q_i(\overline{x}_i, \tau_i) = 0$, *for* $\tau_i \in T_{[x_i]}(q)$, $i = 1, \ldots, s$; *then*

  $$d^2\widetilde{F}_w(\overline{x}_1, \ldots, \overline{x}_5, \overline{q}; \overline{x}_1, \ldots, \overline{x}_5, \overline{q}) = (-1)\cdot(q_1(\overline{x}_1, \overline{x}_1), \ldots, q_s(\overline{x}_s, \overline{x}_s))$$

  and $q_i(\overline{x}_i, \overline{x}_i) \neq 0$ *for* $\overline{x}_i \neq 0$, $i = 1, \ldots, s$.

■

We will show in the next paragraph that the particular properties of the derivatives up to order 2 of F imply that there exists u' near u with 2^s non singular points in its fiber near the point w, where $s = \dim \operatorname{Ker} dF_w$.

3 A deformation theorem

We shall use the euclidean distance on $\mathbb{R}^n$; $B(0,r)$ will denote the open ball of radius r centered at 0.

Theorem 8. *Let* $f: \Omega \to \mathbb{R}^n$, $0 \in \Omega \subset \mathbb{R}^n$ *open,* $f(0) = 0$, *be a* C^∞ *map. Let* $s = \dim Ker(df_0)$ *and assume that*

$$d^2\widetilde{f}_0 : Ker(df_0) \otimes Ker(df_0) \to Coker(df_0)$$

is the product of s *quadratic forms of rank* 1 *with transversal kernels; that is, for a suitable choice of basis of* $Ker(df_0)$ *and* $Coker(df_0)$ *we can write:*

$$\text{for } (\alpha_1, \ldots, \alpha_s), (\beta_1, \ldots, \beta_s) \in Ker(df_0) \quad , \quad d^2\widetilde{f}_0((\alpha_1, \ldots, \alpha_s), (\beta_1, \ldots, \beta_s)) = (\alpha_1\beta_1, \ldots, \alpha_s\beta_s) \quad .$$

(1) *After a change of coordinates in the source and target,* f *can be written :*

$$f(x_1, \ldots, x_n) = (x_1^2, \ldots, x_s^2, x_{s+1}, \ldots, x_n) + g(x_1, \ldots, x_n)$$

for $\|x\| < 1$, *where* $g: B(0,1) \to \mathbb{R}^s$ *satisfies:*

$$g(0) = 0, \ \frac{\partial g}{\partial x_i}(0) = 0, i = 1, \ldots, n, \ \frac{\partial^2 g}{\partial x_i \partial x_j}(0) = 0, i, j = 1, \ldots, s.$$

(2) *In the coordinates of* (1), *let* $y_0 = (\underbrace{1, \ldots, 1}_{s}, 0, \ldots, 0)$. *There exists* $\delta > 0$ *such that for any* ε, $0 < \sqrt{\varepsilon} < \delta$, *the equation* $f(x) = \varepsilon y_0$ *has exactly* 2^s *solutions in the ball centered at* 0 *of radius* $\sqrt{2\varepsilon s}$, *at which the jacobian of* f *is non zero.*

Proof. (1) is a consequence of the hypothesis on $d^2\tilde{f}_0$.

Since $f(x) = \varepsilon y_0$ implies $x_{s+1} = \ldots = x_n = 0$, we might as well assume that $s = n$.

We have that for $t \in]-1,1[$, $g(tx_1, \ldots, tx_s) = t^3 g_1(x,t)$, where $g_1 \colon B(0,1) \times]-1,1[\to \mathbb{R}^s$ is C^∞. Let

$$\phi(x,t) = f(tx)/t^2 = (x_1^2, \ldots, x_s^2) + t g_1(x,t) \quad .$$

Set $\nu = \dfrac{1}{8s}$; the equation $\phi(x,0) = \nu \cdot y_0$ has 2^s solutions ξ_i^0, $i = 1, \ldots, 2^s$, of the form $(\pm\sqrt{\nu}, \ldots, \pm\sqrt{\nu})$, that lie in the ball $B(0, \frac{1}{2})$, and $\frac{\partial\phi}{\partial x}(\xi_i^0, 0)$ is invertible. It follows from the implicit function theorem that there exists $\delta' > 0$, $\eta > 0$ and 2^s functions $\xi_i(t) :]-\delta', \delta'[\to B(\xi_i^0, \eta) \subset B(0, \frac{1}{2})$, $i = 1, \ldots, 2^s$, $\xi_i(0) = \xi_i^0$, such that

$$\text{for } |t| < \delta',\ x \in \bigcup_{i=1,\ldots,2^s} B(\xi_i^0, \eta), \quad \phi(x,t) = \nu y_0 \iff \exists\, i \text{ such that } x = \xi_i(t)$$

and $\frac{\partial\phi}{\partial x}(\xi_i(t), t)$ is invertible. Since $\phi\left(B(0,1) \setminus \cup_{i=1,\ldots,2^s} B(\xi_i^0, \eta), 0\right)$ does not contain νy_0, there exists $\delta'' \leq \delta'$ such that for $|t| < \delta''$, $\nu y_0 \notin \phi\left(B(0, \frac{1}{2}) \setminus \cup_{i=1,\ldots,2^s} B(\xi_i^0, \eta), t\right)$, and therefore :

$$\text{for } |t| < \delta'',\ \|x\| < \frac{1}{2}, \qquad \phi(x,t) = \nu y_0 \iff \exists\, i \text{ such that } x = \xi_i(t) \quad .$$

Now

$$f(x) = \varepsilon y_0 \iff \phi\left(\frac{x}{\sqrt{\varepsilon/\nu}}, \sqrt{\varepsilon/\nu}\right) = \nu y_0 \quad .$$

If we set $\delta = \frac{\delta''}{2\sqrt{2s}}$, then

$$\sqrt{\varepsilon} < \delta \Leftrightarrow \sqrt{\varepsilon/\nu} < \delta'' \text{ and } \frac{\|x\|}{\sqrt{\varepsilon/\nu}} < \frac{1}{2} \Leftrightarrow \|x\| < \sqrt{2\varepsilon s} \quad .$$

Our assertion follows at once ∎

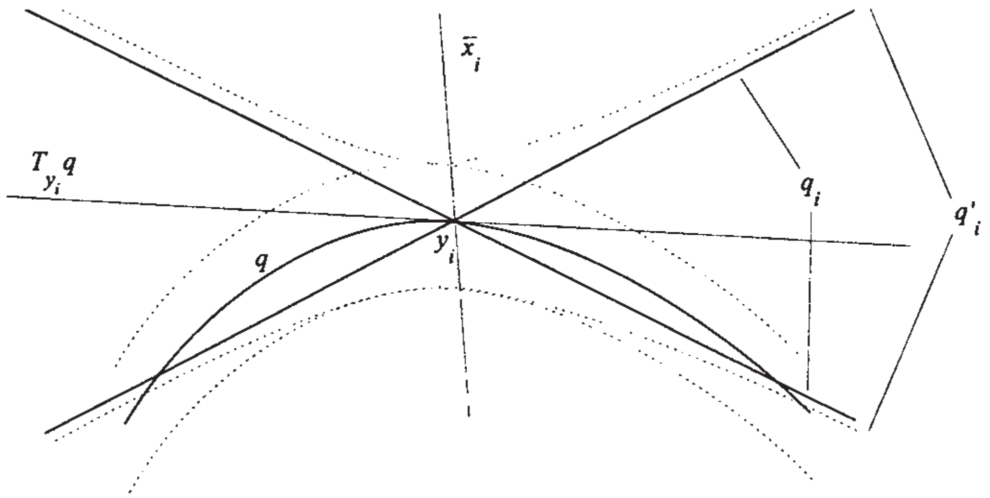

Figure 4. If we deform q_i to $q'_i = q_i + \varepsilon$ in such a way that q'_i appears in the sector not containing $T_{[y_i]}q$, we can guess that there are 2 conics near q tangent to q'_i.

Let us sketch how we will use this theorem to calculate the cardinality of a maximal generic fiber of the map F of paragraph 1. Let $u \in \mathcal{U} \cap (\mathbb{P}\mathcal{Q}_2)^5$, so that $u = ([q_1], \ldots, [q_5])$ where q_i is a degenerate conic that consists of 2 distinct lines meeting at a point $[y_i]$. For $s \in \{0, \ldots, 5\}$ we set

$$F^{-1}(u)_s = \left\{ w \in F^{-1}(u) \mid \dim \mathrm{Ker}\,(dF_w) = s \right\} \quad .$$

We restrict the equations q_i, $i = 1, \ldots, 5$ to some affine chart on $\mathbb{P}^2$ containing $[y_1], \ldots, [y_5]$, that we identify to $\mathbb{R}^2$. For $w \in F^{-1}(u)_s$, perhaps after renumeration, $w = ([y_1], \ldots, [y_s], [x_{s+1}], \ldots, [x_5], [q_1], \ldots, [q_5], [q])$. Recall that if $(\overline{x}_1, \ldots, \overline{x}_5, \overline{q}) \in \mathrm{Ker}\, dF_w$, then $\overline{x}_i$ is polar to $T_{[y_i]}q$ with respect to q_i, and so $\overline{x}_i$ and $T_{[y_i]}q$ lie on different components of the complement of q_i, for $i = 1, \ldots, s$; if we choose the equations q_i in such a way that $q_i(\overline{x}_i, \overline{x}_i) > 0$, or equivalently $q_i(\tau_i, \tau_i) < 0$ for $\tau_i \in T_{[y_i]}q$, then it follows from theorems 7 and 8 that if we replace q_i by $q'_i = q_i + \varepsilon$, $\varepsilon > 0$ small enough, then $F^{-1}(u')$, $u' = (q'_1, \ldots, q'_s, q_{s+1}, \ldots, q_5)$ will have 2^s points in a neighborhood of w. This can be confirmed intuitively, because then q'_i will have 2 sheets near $T_{[y_i]}q$ (see figure 4).

Remark. It can be shown in fact that F can be written locally, in the neighborhood of a w such that $\dim \mathrm{Ker}\,(dF_w) = s$, as $(t_1, \ldots, t_N) \mapsto (t_1^2, \ldots, t_s^2, t_{s+1}, \ldots, t_N)$. However, this cannot be detected from the properties of the derivatives of F at the point w, as shows the example $(t_1, t_2) \mapsto (t_1 + t_2^{2k+1}, t_2^2)$.

The next problem is that if $F^{-1}(u) = \{w_1, \ldots, w_t\}$, we will have to find a deformation u' as above, *valid for all* the $w_1, \ldots, w_t$. This means that whenever $([x_1], \ldots, [x_5], [q_1], \ldots, [q_5], [q]) \in F^{-1}(u)$ and $[x_i] = [y_i]$, then $q_i(\tau_i, \tau_i) < 0$ for $\tau_i \in T_{[y_i]}q$ (we will do this in Section 5). Then we will have :

$$\left|F^{-1}(u')\right| = \sum_{s=0}^{5} 2^s \left|F^{-1}(u)_s\right|$$

Finally, there are $\binom{5}{s}2^{5-s}$ ways of choosing a subset $I \subset \{1, \ldots, 5\}$ of cardinal s, and $5 - s$ lines, one among each pair of lines that constitute the q_i's. Therefore

$$\left|F^{-1}(u)_s\right| = \binom{5}{s} 2^{5-s} n_s$$

where n_s denotes the number of conics passing through s points and tangent to $5 - s$ lines. The number n_s depends on the mutual positions of the s points and the $5 - s$ lines and will be determined in the next paragraph.

4 Basic enumerations

Given a point $[x] \in \mathbb{P}^2$ and a line $\ell \subset \mathbb{P}^2$, we can define the 2 following divisors in $\mathbb{P}Q$:

$$D_x = \{[q] \in \mathbb{P}Q \mid x \in q\}$$

$$D_\ell = \{[q] \in \mathbb{P}Q \mid q \text{ is tangent to } \ell\} \quad .$$

The first divisor is a hyperplane, and some properties of the second are given in the following easy lemma, that we leave to the reader :

Lemma 9.

(1) *D_ℓ has degree 2*

(2) $(D_\ell)_{sing} = \{q \mid q \supset \ell\} \simeq \check{\mathbb{P}}^2$

(3) *if $q \in (D_\ell)_{reg}$ and $[x] = q \cap \ell$, we have:*

$$T_{[q]}D_\ell = \{[\overline{q}] \mid \overline{q}(x) = 0\}$$

■

We introduce now genericity conditions on the choice of s points and $5-s$ lines in $\mathbb{P}^2$: we define $\Omega_s \subset \left(\mathbb{P}^2\right)^s \times \left(\check{\mathbb{P}}^2\right)^{5-s}$ as the set of $([x_1], \ldots, [x_s], \ell_{s+1}, \ldots, \ell_5)$ that satisfy:

(1) 3 among the $[x_i]$'s are not aligned (in particular, $[x_i] \neq [x_j]$ for $i \neq j$).

(2) 3 among the ℓ_i's do not go through a same point (in particular, $\ell_i \neq \ell_j$ for $i \neq j$).

(3) $\forall i, j \; x_i \notin \ell_j$.

(4) $\forall i_1 \neq i_2$, $j_1 \neq j_2$ any line through x_{i_1} and x_{i_2} does not go through $\ell_{j_1} \cap \ell_{j_2}$.

(5) $\forall$ distinct i_1, i_2, i_3, i_4 and $\forall j$ the intersection of the line through $[x_{i_1}]$ and $[x_{i_2}]$ with the line through $[x_{i_3}]$ and $[x_{i_4}]$ does not belong to ℓ_j.

(6) $\forall$ distinct i_1, i_2, i_3, i_4 and $\forall j$, x_j does not belong to the line through $\ell_{i_1} \cap \ell_{i_2}$ and $\ell_{i_3} \cap \ell_{i_4}$.

In other words, the configurations shown in figure 5 are not allowed.

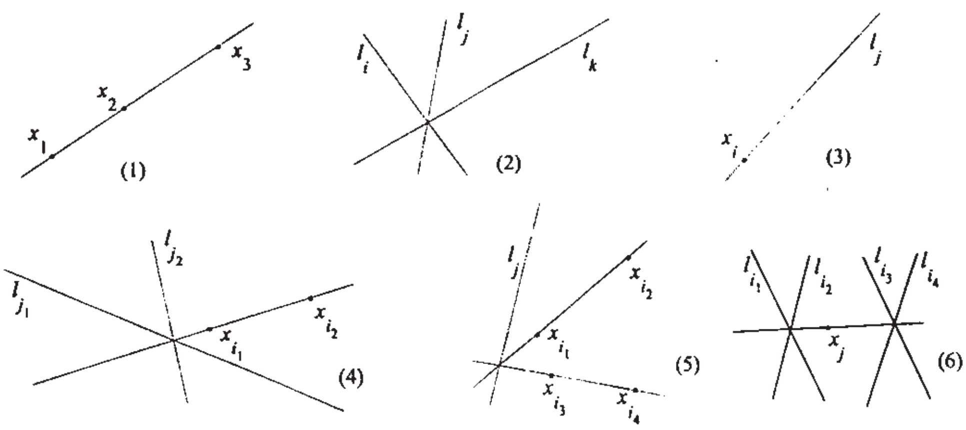

Figure 5. Configurations that we don't want in Section 3.

Lemma 10. *Let* $([x_1], \dots, [x_s], \ell_{s+1}, \dots \ell_5) \in \Omega_s$ *and* $[q] \in D_{x_1} \cap \dots \cap D_{x_s} \cap D_{\ell_{s+1}} \cap \dots \cap D_{\ell_5}$. *Then* $[q] \notin \mathbb{P}Q_2$ *and if* $[q] \in \mathbb{P}Q_3$, $D_{x_1}, \dots, D_{x_s}, D_{\ell_{s+1}}, \dots, D_{\ell_5}$ *intersect transversally at* $[q]$.

Proof. Assume that $[q] \in \mathbb{P}Q_2$ and let $[y]$ be its singular point. Then the genericity condition (1) implies that $s \leq 4$. Any tangent to q goes through y, and so condition (3) implies that $[x_i] \neq [y]$, $i = 1, \dots, s$, and condition (2) implies that $s \geq 3$.

If $s = 3$, condition (1) or (4) is contradicted, and if $s = 4$ condition (1) or (5) is contradicted.

Now let $[q] \in \mathbb{P}Q_3$; then by lemma 9 (2) $[q]$ is a smooth point of each divisor D_{x_i}, D_{ℓ_j} and the intersection of the tangent spaces of the divisors at $[q]$ is

$$\left\{\bar{q} \in T_{[q]}\mathbb{P}Q \mid \bar{q}(x_1) = \dots = \bar{q}(x_s) = \bar{q}(y_{s+1}) = \dots = \bar{q}(y_5) = 0\right\}$$

where $y_j = q \cap \ell_j$. Conditions (1), (2) and (3) imply that the points $[x_1] \dots, [x_s]$, $[y_{s+1}], \dots, [y_5]$ are 5 distinct points on q, and therefore 3 of them are never aligned. But there is exactly 1 conic going through 5 points, 3 of which are never aligned. ■

Let

$$V_s = \{(([x_1], \ldots, [x_s], \ell_{s+1}, \ldots, \ell_5), [q]) \in \Omega_s \times \boldsymbol{P}\mathcal{Q}_3 \mid q \in D_{x_1} \cap \ldots \cap D_{x_s} \cap D_{\ell_{s+1}} \cap \ldots \cap D_{\ell_5}\} \quad .$$

Proposition 11. *The variety V_s is smooth and the natural projection $\pi: V_s \to \Omega_s$ is a proper submersion with finite fibers.*

Proof. The facts that V_s is smooth and that π is a submersion follow from lemma 10.

If in the definition of V_s we allow $[q] \in \boldsymbol{P}\mathcal{Q}$, the corresponding projection π is obviously proper. Lemma 10 implies in this case that $q \notin \mathcal{Q}_2$, and if $s \geq 3$ the genericity condition (1) implies that $q \notin \mathcal{Q}_1$. Therefore π is proper for $s \geq 3$. The case $s \leq 2$ is obtained by observing that associating to a conic its dual induces an isomorphism $V_s \simeq V_{5-s}$.

■

Corollary 12. *The map*

$$\Omega_s \to \mathbb{N} \quad , \quad \omega \mapsto \left|\pi^{-1}(\omega)\right|$$

is locally constant.

■

We compute now $|\pi^{-1}(\omega)|$ for various connected components of Ω_s. By applying our results to the dual conics, the cases $s = 3, 4, 5$ will be deduced from the cases $s = 2, 1, 0$ respectively.

First of all, we compexify the situation. Then it follows from lemma 9 (1) that $|\pi^{-1}(\omega)| = 1, 2, 4, 4, 2, 1$ for all $\omega \in (\Omega_s)_{\mathbb{C}}$, $s = 0, 1, 2, 3, 4, 5$. We set $N_s = \left|\pi_{\mathbb{C}}^{-1}(\omega)\right|$. Back to the real case, we shall say that a component Ω_s^0 of Ω_s is *maximal* if $|\pi^{-1}(\omega^0)| = N_s$ for $\omega^0 \in \Omega_s^0$.

In what follows, we will make use of the action of the group $PGl(3, \mathbb{R})$ on Ω_s; since it is connected, it will preserve the connected components of Ω_s.

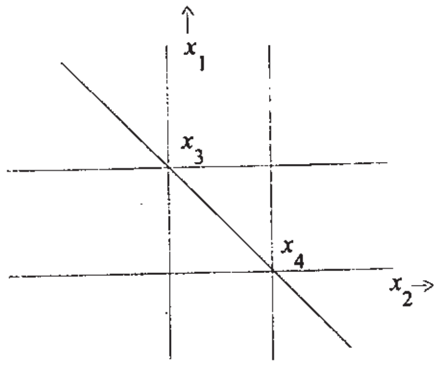

Figure 6. $s = 5$; the sixth forbidden line is at ∞.

$\boxed{s = 0 \text{ and } s = 5}$ There is exactly one (non-singular) conic through 5 points, 3 of which are never aligned, and so all the components of Ω_5 are maximal. Dually, it follows that all the components of Ω_0 are maximal.

In fact, the variety Ω_5 has 12 connected components: the set of 4-tuples of points of $\mathbb{P}^2$ 3 by 3 not aligned is connected because it is a homogeneous space for $PGl(3, \mathbb{R})$. Therefore we can fixe the first 4 points $[x_1], \ldots, [x_4]$; then for the fifth point there will be 6 lines forbidden by the genericity conditions, namely those through the pairs of the first 4 points. It is now easy to check on an explicit example that there are 12 connected components in the complement of such 6 lines (see figure 6, in which one of the forbidden lines is the line at ∞).

$\boxed{s = 1}$ The variety Ω_1 has 16 connected components. Indeed, using the action of $PGl(3, \mathbb{R})$ we can fix the four lines and $[x_1]$ must belong to the complement E of this 4 lines, but not to the lines joining pairwise intersections of the ℓ_i's. Among the components of E, there are 4 triangles T_i and 3 quadrangles Q_j. Clearly (see figure 7), the components of type Q are maximal, those of type T are not.

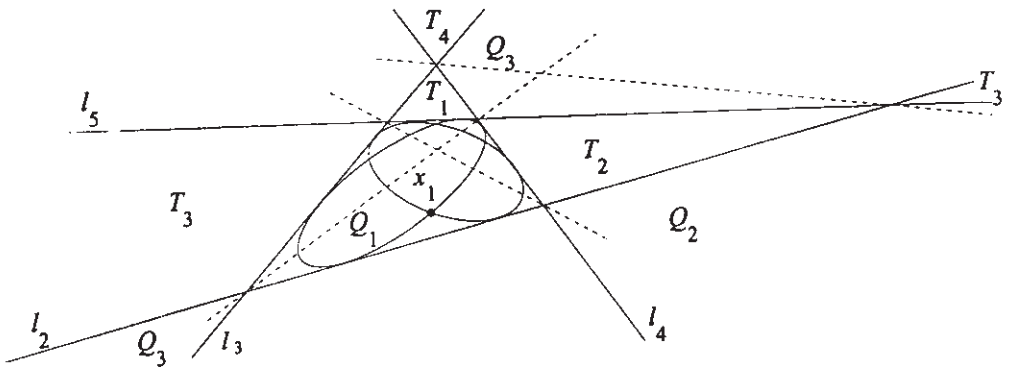

Figure 7. $s = 1$; choose x_1 in a quadrangle if you want to be in a maximal component.

$\boxed{s = 2}$ The variety Ω_2 has 12 connected components. Indeed, we can fix the 3 lines ℓ_1, ℓ_2, ℓ_3 and the point $[x_1]$; the point $[x_2]$ must be choosen in the complement of the 6 lines ℓ_1, ℓ_2, ℓ_3 and the three lines joining $[x_1]$ to the intersections $\ell_j \cap \ell_h$. The maximal components are those where $[x_1]$ and $[x_2]$ are in the same component of the complement of the 3 lines ℓ_1, ℓ_2, ℓ_3. Since the choice of ℓ_1, ℓ_2 and ℓ_3 is irrelevant, it suffices to check on a particular case. We take :

$$\begin{aligned} [x_1] &= [-1:0:1]\,,\ [x_2] = [1:0:1]\,, \ell_3 = \{y = -z\}\,,\ \ell_4 \\ &= \{x = 2z\}\,,\ \ell_5 = \{x = -2z\}\,. \end{aligned}$$

Let $q(x, y, z) = ax^2 + by^2 + cz^2 + dxy + exz + fyz = 0$ be a conic through $[x_1]$, $[x_2]$ and tangent to ℓ_1, ℓ_2 and ℓ_3. Then:

$$\left.\begin{aligned} q(x_1) = \; 0 \Longrightarrow \;\; a + c - e = 0 \\ q(x_2) = \; 0 \Longrightarrow \;\; a + c + e = 0 \end{aligned}\right\} \Rightarrow a = -c\,,\ e = 0$$

Then the conic $q = a(x^2 - z^2) + (by + dx + fz)y$ must be tangent to:

$$\begin{aligned} \ell_3 &\Longrightarrow \quad d^2 - 4a(-a + b - f) = 0 \\ \ell_4 &\Longrightarrow \quad (2d + f)^2 - 12ab = 0 \\ \ell_5 &\Longrightarrow \quad (-2d + f)^2 - 12ab = 0 \quad . \end{aligned}$$

It follows from the last 2 equations that $df = 0$.

If $d = 0$, we have

$$\begin{cases} (1) & a(-a + b - f) = 0 \\ (2) & f^2 - 12ab = 0 \quad ; \end{cases}$$

$a = 0$ gives the double line through $[x_1]$ and $[x_2]$, for which we don't care. Replacing $b = a + f$ in equation (2) above gives 2 distinct real solutions: $f = a(6 \pm 4\sqrt{3})$.

If $f = 0$, we have

$$\begin{cases} (1) & d^2 - 4a(-a + b) = 0 \\ (2) & 4d^2 - 12ab = 0 \end{cases}$$

which implies that $a(4a - b) = 0$, and replacing $b = 4a$ in equation (1) above gives 2 new real solutions: $d = \pm 2a\sqrt{3}$. If $a = 0$, we find again the double line through $[x_1]$ and $[x_2]$.

In conclusion, we have 4 good real solutions.

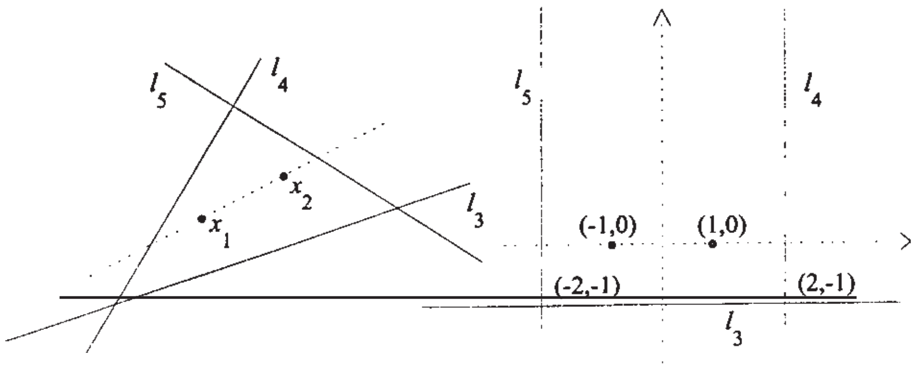

Figure 8. $s = 2$; the dashed line should not go through a vertex. At right, the particular case that we investigate.

$\boxed{s=3}$ This case is dual to $s = 2$. The maximal components of Ω_3 are those for which the 3 points $[x_1]$, $[x_3]$ and $[x_3]$ are in the same component of the complement of the 2 lines ℓ_1 and ℓ_2 (see figure 9).

$\boxed{s=4}$ This is dual to $s = 1$. If we let ℓ_5 be the line at ∞, its complement can be identified with $\mathbb{R}^2$, and it contains the 4 points $[x_1], \ldots, [x_4]$. The maximal components are those for which these 4 points are the vertices of a *convex* quadrangle in $\mathbb{R}^2$ (see figure 9).

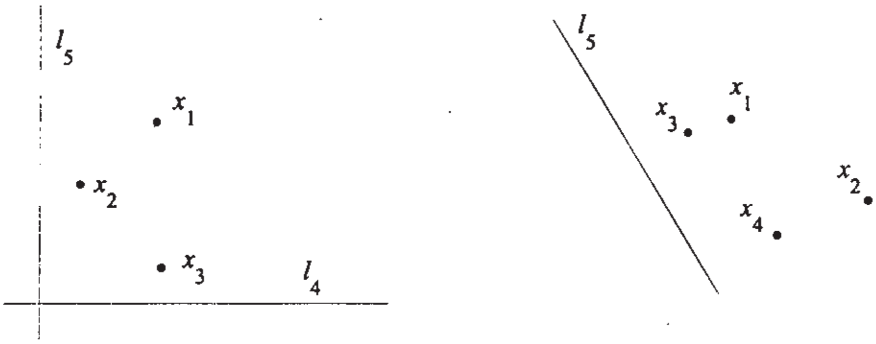

Figure 9. $s = 3$ and $s = 4$.

5 The final step

In this paragraph we shall work in some affine chart of $\mathbb{P}^2$ that we identify with $\mathbb{R}^2$. Let $y_1, \ldots, y_5 \in \mathbb{R}^2$ be the vertices of a regular pentagon and denote by Π the convex hull of $y_1, \ldots, y_5$ (i.e. the pentagon itself). Denote by $\check{\mathbb{P}}^2_{y_i}$ the space of lines through y_i and let $\ell_i^0 \in \check{\mathbb{P}}^2_{y_i}$, $i = 1, \ldots, 5$, be such that for all $I \subset \{1, \ldots, 5\}$ the configuration $((y_i)_{i \in I}, (\ell_j^0)_{j \in C(I)})$, where $C(I) = \{1, \ldots, 5\} \setminus I$, belongs to a maximal component of $\Omega_{|I|}$ (figure 10 shows such a configuration). Let L_i, $i = 1, \ldots, 5$ be open neighborhoods of the ℓ_i^0's such that for all $I \subset \{1, \ldots, 5\}$ the configurations $((y_i)_{i\in I}, (\ell_j)_{j\in C(I)})$ still belong to a maximal component of $\Omega_{|I|}$.

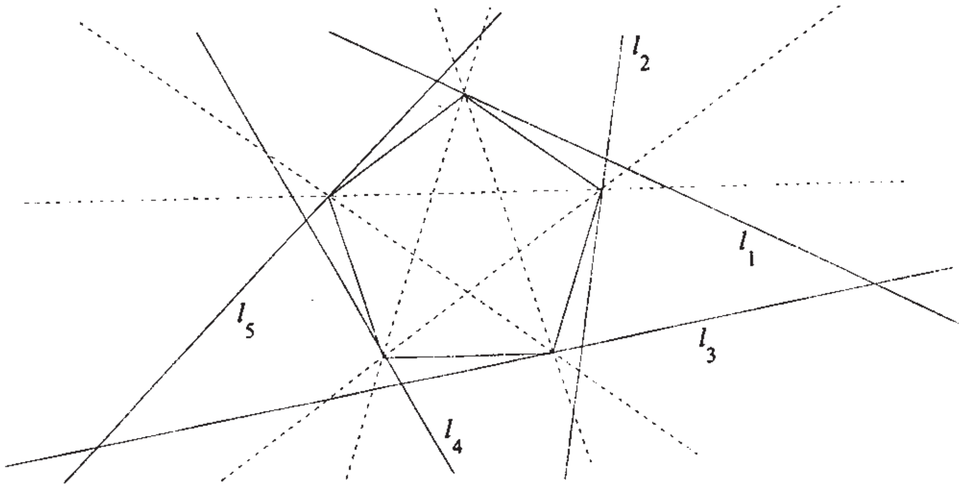

Figure 10. 5 generic lines that generate maximal configurations.

Set

$$V(I) = \Big\{\big((\ell_j)_{j\in C(I)}, q\big) \in \big(\textstyle\prod_{j\in C(I)} L_j\big) \times \mathbb{P}Q_3 \mid q(y_i) = 0\,,\ \forall i \in I \text{ and } \forall j \in C(I)\ q \text{ is tangent to } \ell_j\Big\}$$

The following lemma tells us that it is possible to make a good choice of lines and that this choice is stable, in some sense.

Lemma 13. *Let $U \subset L_1 \times \cdots \times L_5$ be defined as follows :*

$$U \ni (\ell_1, \ldots, \ell_5) \Leftrightarrow \begin{cases} \forall I \subset \{1\ldots, 5\}\,, \\ ((\ell_j)_{j\in C(I)}, q) \in V(I) \Longrightarrow \forall i \in I\,, T_{[y_i]}q \neq \ell_i \end{cases}.$$

Then :

(1) *U is open and dense in $L_1 \times \cdots \times L_5$.*

(2) *If $(\ell_1, \ldots, \ell_5) \in U$, there exist connected neighborhoods $U(\ell_h) = U_h$ of ℓ_h in L_h, $h = 1, \ldots, 5$ such that:*

$$\forall I \subset \{1, \ldots, 5\}, \forall \left(\ell'_j\right)_{j\in C(I)}, \ell'_j \in U_j$$
$$\text{we have : } \left(\left(\ell'_j\right)_{j\in C(I)}, q\right) \in V(I) \Longrightarrow T_{[y_i]}q \notin U_i\,, \forall i \in I \quad .$$

Proof.

(1) For $I \subset \{1, \ldots, 5\}$ and $i_0 \in I$ set

$$V'(I, i_0) = \{(\ell_1, \ldots, \ell_5), q \in \prod_{h=1}^{5} \check{\boldsymbol{P}}^2_{y_h} \times \boldsymbol{P}Q_3 \mid$$

$$\ell_j \in L_j \, \forall j \in C(I), q(y_i) = 0 \, \forall i \in I, q \text{ is tangent to } \ell_j \, \forall j \in C(I) \text{ and } T_{[y_{i_0}]}q = \ell_{i_0}\} \quad .$$

$V'(I, i_0)$ is a closed subset of codimension 1 of the set

$$V'(I) = \{((\ell_1, \ldots, \ell_5), q) \in \prod_{h=1}^{5} \check{\boldsymbol{P}}^2_{y_h} \times \boldsymbol{P}Q_3 \mid$$
$$\ell_j \in L_j \, \forall j \in C(I), q(y_i) = 0 \, \forall i \in I, q \text{ is tangent to } \ell_j \, \forall j \in C(I)\}$$

and it follows from proposition 11 that the natural projection

$$p_I : V'(I) \to \Big(\prod_{j \in C(I)} L_j\Big) \times \Big(\prod_{i \in I} \check{\boldsymbol{P}}^2_{y_i}\Big)$$

is proper, and therefore the set

$$X_{I,i_0} = p_I\left(V'(I, i_0)\right)$$

is closed, of codimension 1 in $\left(\prod_{j \in C(I)} L_j\right) \times \left(\prod_{i \in I} \check{\boldsymbol{P}}^2_{y_i}\right)$. Now :

$$U = L_1 \times \cdots \times L_5 \setminus \bigcup_{I \subset \{1, \ldots, 5\}, \, i_0 \in I} X_{I,i_0}$$

therefore U is open, dense in $L_1 \times \cdots \times L_5$.

(2) For $I \subset \{1, \ldots, 5\}$, consider the diagram :

$$\begin{array}{lcl} V(I) & \xrightarrow{\tau_I} & \prod_{i \in I} \check{\boldsymbol{P}}^2_{y_i} \\ p_{C(I)} \downarrow & & \\ \prod_{j \in C(I)} L_j & & \end{array}$$

where $\tau_I\left((\ell_j)_{j \in C(I)}, q\right) = \left(T_{[y_i]}q\right)_{i \in I}$. Let $u = (\ell_1, \ldots, \ell_5) \in U$ and set $z = (\ell_j)_{j \in C(I)}$, $w = (\ell_i)_{i \in I}$. Since $u \in U$, we have that $\tau_I^{-1}(w) \cap$

$p_{C(I)}^{-1}(z) = \emptyset$. It follows from the fact that $p_{C(I)}^{-1}(z)$ is finite and that $p_{C(I)}$ is a covering that there exist open sets :

$$U'_{C(I),j} \subset L_j \, , \, U'_{C(I),j} \ni \ell_j \, , \, \forall j \in C(I)$$

$$U''_{I,i} \subset L_i \, , \, U''_{I,i} \ni \ell_i \, , \, \forall i \in I$$

such that, setting $U'_{C(I)} = \prod_{j\in C(I)} U'_{C(I),j}$ and $U''_I = \prod_{i\in I} U''_{I,i}$:

$$p_{C(I)}^{-1}(U'_j) \cap \tau_I^{-1}(U''_I) = \emptyset \quad .$$

If we take U_h to be the connected component of :

$$\left(\bigcap_{C(I)\ni h} U'_{C(I),h}\right) \cap \left(\bigcap_{I\ni h} U''_{I,h}\right)$$

that contains ℓ_h then assertion (2) will be satisfied

■

If ℓ' and ℓ'' are lines through the point y in $\mathbb{R}^2$ that are not perpendicular then they determine two angles : one that is strictly smaller than $\pi/2$, another that is strictly larger than $\pi/2$. We shall call the *sector determined by ℓ' and ℓ''* the set of lines that go through y and lie in the smaller angle.

Choose $(\ell_1, \ldots, \ell_5) \in U$ and $\ell'_h \neq \ell''_h \in U_h(\ell_h)$, $h = 1, \ldots, 5$; then any pair (ℓ'_h, ℓ''_h) determines a sector as explained above, which is contained in U_h. We choose an equation q_h of the conic $\ell'_h \cup \ell''_h$, $h = 1, \ldots, 5$ in such a way that q_h takes negative values in the sector determined by (ℓ'_h, ℓ''_h). Set $u = ([q_1], \ldots, [q_5])$; we may assume also that $u \in \mathcal{U}$ (that is

: u satisfies conditions (G_1) through (G_5) of Section 1).

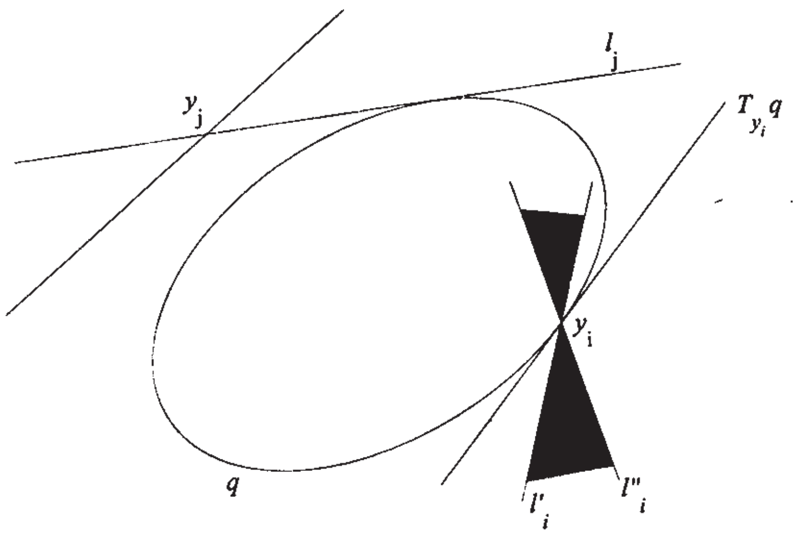

Figure 11. The sector defined by (ℓ'_i, ℓ''_i) does not contain the tangent to q at y_i.

It follows from the properties of the U_h's, $h = 1, \ldots, 5$, that if $w = ([q_1], \ldots, [q_5], [x_1], \ldots, [x_5], [q]) \in F^{-1}(u)_s$, then for all i such that $[x_i] = [y_i]$, $T_{[y_i]}q$ will lie outside the sector determined by ℓ'_i, ℓ''_i (see figure 11), and so its polar with respect to q_i will lie inside the sector. Therefore it follows from theorems 7 and 8 that if we replace q_i by $q'_i = q_i + \varepsilon$, where $\varepsilon > 0$ is small enough, then there are 2^s points of $F^{-1}([q'_1], \ldots, [q'_5])$ in a neighborhood of w. Note that the conics defined by the q'_i lie inside the sector defined by (ℓ'_i, ℓ''_i), which is what we expect intuitively.

Let $s \in \{0, \ldots, 5\}$ and

$$F^{-1}(u)_s = \left\{w \in F^{-1}(u) \mid \dim \operatorname{Ker}(dF_w) = s\right\}$$

as in Section 3. Then :

$$\left|F^{-1}(u)_s\right| = \binom{5}{s} 2^{5-s} n_s$$

where $n_s = 1, 2, 4, 4, 2, 1$ for $s = 0, 1, 2, 3, 4, 5$. Finally, we set $u' = (q'_1, \ldots, q'_5)$ and so :

$$\begin{aligned} |F^{-1}(u')| &= \sum_{i=0}^{5} 2^s 2^{5-s} \binom{5}{s} n_s \\ &= 2^5 \left(\tbinom{5}{0} 1 + \tbinom{5}{1} 2 + \tbinom{5}{2} 4 + \tbinom{5}{3} 4 + \tbinom{5}{4} 2 + \tbinom{5}{5} 1 \right) = 3264 . \end{aligned}$$

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