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A note on a theorem of Horikawa.

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Abstract

In this paper we classify the algebraic surfaces on $\mathbb{C}$ with $K_S^2 = 4$, $p_g = 3$ and canonical map of degree $d = 3$. By our result and the previous one of Horikawa [10] we obtain the complete determination of surfaces with $K^2 = 4$ and $p_g = 3$.

Introduction

The aim of this paper is to classify minimal surfaces S on $\mathbb{C}$ with $K^2 = 4$, $p_g = 3$ and canonical map of degree $d = 3$. The existence of such surfaces is claimed without proof in [10] [Section 2, p. 110]. In the same paper Horikawa showed that surfaces with $K_S^2 = 4$ and $p_g = 3$ have $d = 2, 3, 4$ and he classified the cases $d = 2$ and $d = 4$. We have already considered surfaces with $K_S^2 = 4$, $p_g = 3$ and $d = 3$ in [14], but in this article we adopt a different point of view. We will explicitely construct a birational model $X \subset \mathbb{P}^3$ of S where X is a quintic with only a singular point which is an elliptic Gorenstein singularity of type $\tilde{E}_8$ (cf. [12] and the first section below).

Main theorem

Let $A_s = \{(i_0, i_1, i_2) \in \mathbb{Z}_+^3 | \, i_0 + i_1 + i_2 = 5 - s, 3i_0 + 2i_1 + i_2 \geq 6\}$ and let $\mathcal{A}$ be the sublinear system of the quintics $X \subset \mathbb{P}^3$ with the following equation:

$$1) \; X = \{x \in \mathbb{P}^3 | \sum_{s=0}^{3} \sum_{I \in A_s} a_I x^I = 0\}$$

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where $x^I = x_0^{i_0}x_1^{i_1}x_2^{i_2}x_3^{s}$, $a_I \in \mathbb{C}$ and (x_0, x_1, x_2, x_3) is a projective system of coordinates on $\mathbb{P}^3$. Then there exists an open set $\mathcal{A}' \subset \mathcal{A}$ such that the minimal desingularization of a quintic $X \in \mathcal{A}'$ is a minimal surface S with $K_S^2 = 4$, $p_g = 3$ and canonical map of degree three. Reciprocally any minimal surface S with $K_S^2 = 4$, $p_g = 3$ and canonical map of degree three is obtained in this way. Moreover let $\phi : X - \to \mathbb{P}^2$ be the rational map induced by the projection from $(0,0,0,1) \in \mathbb{P}^3$, and let $\nu : S \to X$ be the desingularization map, then $\phi_{|K_S|} = \phi \circ \nu$.

Our theorem implies that the locus of surfaces with $p_g = 3$, $K_S^2 = 4$ and $deg_{|K_S|} = d = 3$ is irreducible, unirational and of dimension 29 (see 2.16). Moreover, together with [10] [Theorem 2.1 and 2.2], it gives a complete classification of surfaces with $p_g = 3$ and $K_S^2 = 4$. We also think that our point of view of considering the canonical map via a projection from elliptic points in some nice birational model X of S will shed some new light on this subject. In fact we hope to apply this technique to irregular surfaces with $d \geq 3$; a subject quite unknown: see section 2 of [3] for an interesting survey and [11] for some new results.

In section 1 we recall some results on elliptic singularities and we will prove that the minimal desingularization of a general quintic $X \in \mathcal{A}$ is a surface with $p_g = 3$, $K_S^2 = 4$ and $d = 3$. In section 2 we will study the canonical linear system of S and we will explicitely construct a birational morphism $S \to X$ where $X \in \mathcal{A}'$. I wish to thank the referee for helpufl comments, which led to an improvement in the arrangement of this paper.

1 Quintics with a singular point of type $\tilde{E}_8$

In this section we show that the minimal desingularization S of a general $X \in \mathcal{A}$ (see the statement of the main theorem in the introduction) has $p_g = 3$, $K_S^2 = 4$ and $d = 3$. We begin with a general result on elliptic singularities. It is well known (cf. [12], p. 288) that if X is a normal Gorenstein surface and $\nu : S \to X$ is the minimal resolution of an isolated singularity $x_0 \in X$ then there exists an effective divisor G_0 on S supported on $\nu^{-1}(x_0)$ such that $\omega_S = \nu^*(\omega_X) \otimes \mathcal{O}_S(G_0)$ and $K_S^2 = K_X^2 + G_0^2$. Moreover the spectral sequence $H^p(X, R^q\nu_*\mathcal{O}_S) \Rightarrow H^{p+q}(X, \mathcal{O}_X)$ implies $\chi(\mathcal{O}_X) - p_g(x_0) = \chi(\mathcal{O}_S)$ where $p_g(x_0) = h^0(X, R^1\nu_*\mathcal{O}_S)$. In the same paper we find that G_0 is an elliptic curve with $G_0^2 = -1$; then

$\nu_*(\omega_S) = \mathcal{M}_{x_0}\omega_X$ where $\mathcal{M}_{x_0}$ is the ideal of x_0 in X. This singularity is called simple elliptic singularity of type $\tilde{E}_8$.

Let $\mathcal{A}$ be the sublinear system given by the quintics with equation 1).
From now on X will be a general element of $\mathcal{A}$. Suppose that $(0,0,0,1) \in X$ is the unique singular point and also that it is an $\tilde{E}_8$- singular point. Then $\chi(\mathcal{O}_S) = 4$ and $K_S^2 = 4$. We will show that $deg(\phi_{|K_S|}) = d = 3$. In fact, since $\nu_*(\omega_S) = \mathcal{M}_{x_0}\omega_X$, we easily see that $H^0(S, \mathcal{O}_S(K_S)) \cong \nu^*\mathcal{V}$ where $\mathcal{V} \subset H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$ is the sublinear system of the hyperplanes containing x_0. In particular $\phi_{|K_S|}$ is induced by the projection from the point $(0,0,0,1)$; it is also easy to see that for the general $X \in \mathcal{A}$ the general straight line l containing x_0 intersects X in other three distinct points. We are led to the following result:

Lemma 1.1. *If $P_0 = (0,0,0,1) \in X$ is the only singular point of a general $X \in \mathcal{A}$ and if it is of type $\tilde{E}_8$ then the minimal desingularization $\nu : S \to X$ has the following invariants: $p_g = 3$ and $K_S^2 = 4$. Furthermore the canonical map $\phi_{|K_S|}$ has degree 3.*

Proof. By the previous analysis we know that $\chi(\mathcal{O}_S) = 4$, $K_S^2 = 4$ and $d = 3$. Thus we only need to show that $q(S) = 0$; but this is the content of [5][prop.5.1].

■

It remains to prove that the general $X \in \mathcal{A}$ satisfies the conditions of 1.1. The proof falls naturally in two parts which correspond to the two hypotheses of 1.1.

Lemma 1.2. *Let* Sing(X) *be the singular locus of X. If X is a general element of $\mathcal{A}$ then* Sing(X) $= \{(0,0,0,1)\}$.

Proof. It is rather obvious that $\{(0,0,0,1)\} \in$ Sing(X). Since other singularities impose closed conditions on $\mathcal{A}$ we need to show that there exists an element $X \in \mathcal{A}$ which satisfies the claim. Consider the quintic with the following equation:

$$F = \frac{2}{5}x_0^5 + \frac{2}{5}x_1^5 + x_1x_2^4 + \frac{1}{2}x_0^4x_3 - x_1^3x_3^2 + x_0^2x_3^3 = 0.$$

Obviously Sing(X) $= \{F = \frac{\partial F}{\partial x_i} = 0,\ i = 0,1,2,3\}$ and an easy computation shows that $(0,0,0,1)$ is the unique solution. ■

We recall now the following description of the points of type $\tilde{E}_8$:

Lemma 1.3. *The point $P_0 \in X$ is an $\tilde{E}_8$-point if and only if near to P_0 the normal Gorenstein surface X is biregular to the surface of $\mathbb{C}^3_{(x,y,z)}$ given by the following equation:*

$$2)\ x^2 + y^3 + g(y,z) = 0$$

where g is a nonzero linear combination of monomials yz^a with $a \geq 4$ and z^a with $a \geq 6$.

Proof. See [12] [prop. 2.9].

■

Remark 1.4. Let $(0,0,0) = P_0 \in \mathbb{C}^3_{(x,y,z)}$, $\mathcal{O}_{P_0,\mathbb{C}^3} =_{\text{def}} \mathcal{O}$, and $x^2+y^3+yz^4+z^6h(x,y,z) = f \in \mathcal{O}$ where $h \in \mathcal{O}$ and $h(0,0,0,) \neq 0$. It is easy to check that P is a point of type $\tilde{E}_8$ for the germ given by f.

We can now prove the final lemma of this section:

Lemma 1.5. *Let X be a general element of $\mathcal{A}$. Then the point $P_0 = (0,0,0,1) \in X$ is an $\tilde{E}_8$ singularity .*

The problem is local. We set $\mathcal{O}_{P_0,\mathbb{C}^3} = \mathcal{O}$. Let $X = \{F = 0\}$ as in the statement of the main theorem. We consider affine coordinates $x = \frac{x_0}{x_3}$, $y = \frac{x_1}{x_3}$, $z = \frac{x_2}{x_3}$, and we put $f_0(x,y,z) = F(x,y,z,1)$. The basic idea of the proof is to take successive "reduction" of $f_0 \in \mathcal{O}$ to obtain the $f \in \widehat{\mathcal{O}}$ of 1.4.

We wish to arrange the monomials of f_0 according to the occurrence of xy^2, x^2, y^3, yz^4 in it. First we group all monomials which are divisible by xy^2, then, among the remaining ones, those divisible by x^2, and so on by y^3, y^2z^2, xyz and finally by xz^3. In other words we can write:

$$f_0(x,y,z) = m_0xy^2 + p_0x^2 + q_0y^3 + r_0y^2z^2 + s_0xyz + t_0xz^3 + u_0yz^4$$

where $u_0, t_0, s_0, r_0, q_0, p_0, m_0 \in \mathbb{C}[x,y,z]$ and they do not vanish at $(0,0,0)$. We consider $\xi_1 : \widehat{\mathcal{O}} \to \widehat{\mathcal{O}}$ given by $x_1 = x + \frac{m_0}{2p_0}y^2 + \frac{s_0}{2p_0}yz + \frac{t_0}{2p_0}z^3$, $y_1 = y$ and $z_1 = z$. We denote $l_1(x_1, y_1, z_1) = \xi_1^{-1}l_0(x,y,z)$ for every $l_0 \in \mathbb{C}[[x,y,z]]$. If we put: $\widehat{r_1} = r_1 - \frac{s_1^2}{4p_1} - \frac{t_1m_1}{2p_1}z_1$, $\widehat{q_1} = q_1 - \frac{m_1^2}{4p_1}y_1 - \frac{s_1m_1}{2p_1}z_1$, $\widehat{u_1} = u_1 - \frac{t_1s_1}{2p_1}$, $\widehat{t_1} = -\frac{t_1^2}{4p_1}$, then we obtain:

$$\xi_1^{-1}f_0 = f_1 = p_1x_1^2 + \widehat{q_1}y_1^3 + \widehat{r_1}y_1^2z_1^2 + \widehat{u_1}y_1z_1^4 + \widehat{t_1}z_1^6.$$

We apply this argument again. We consider the automorphism: $\xi_2 : \widehat{\mathcal{O}} \to \widehat{\mathcal{O}}$ given by: $x_2 = x_1$, $y_2 = y_1 + \frac{\widehat{r_1}}{3\widehat{q_1}} z_1^2$, $z_2 = z_1$. We define $l_2(x_2, y_2, z_2) = \xi_2^{-1} l_1(x_1, y_1, z_1)$ for every $l_1 \in \mathbb{C}[[x_1, y_1, z_1]]$. We set $p = \xi_2^{-1} p_1$, $q = \xi_2^{-1} \widehat{q_1}$, $u = \widehat{u_2} - \frac{\widehat{r_2}^2}{3\widehat{q_2}}$, $t = \widehat{t_2} + 2\frac{\widehat{r_2}^3}{27\widehat{q_2}^2} - \frac{\widehat{r_2}\widehat{u_2}}{3\widehat{q_2}}$ then

$$\xi_2^{-1} f_1 = f_2 = p x_2^2 + q y_2^3 + u y_2 z_2^4 + t z_2^6.$$

We recall that we are working on an open set of $\mathcal{A}$ where we can take a fourth root of u; then through $\xi_3 : \widehat{\mathcal{O}} \to \widehat{\mathcal{O}}$, $x_3 = p^{\frac{1}{2}} x_2$, $y_3 = q^{\frac{1}{3}} y_2$, $z_3 = q^{\frac{1}{12}} u^{\frac{1}{4}} z_2$ we obtain:

$$\xi_3^{-1} f_2 = f = x_3^2 + y_3^3 + y_3 z_3^4 + a(x_3, y_3, z_3) z_3^6$$

where $a \in \mathcal{O}$ and $a(0,0,0) \neq 0$ which, by 1.4, is the desired conclusion.

■

2 Minimal surfaces with $K^2 = 4$, $p_g = 3$ and $d = 3$ as minimal models of quintics with a unique singular point

From now on S will be a minimal surface with $p_g = 3$, $K^2 = 4$ and $d = 3$. Moreover by [5][prop. 5.1] we have that $q(S) = 0$. In this section we will prove that there exists a birational morphism $\phi_{|K_S + G_0|} : S \to X \subset \mathbb{P}^3$ which contracts G_0 where G_0 is an elliptic curve with $G_0^2 = -1$ and X is a quintic in $\mathcal{A}$. We start with a lemma on the canonical map $\phi_{|K_S|}$ of S.

Lemma 2.1. *The canonical linear system $| K_S |$ is without fixed part and it has a unique base point P.*

Proof. Let $| K_S | = Z + | M |$ where Z and M are respectively the fixed part and the mobile part of $| K_S |$. Let $\sigma : \tilde{S} \to S$ be a minimal resolution of the base points of $\phi_{|K_S|} = \phi_{|M|}$ and let L be the mobile part of $| \sigma^* M |$. We first show that $Z = 0$. In fact since $d = 3$ and $p_g = 3$ then $M^2 \geq L^2 = 3$. Since S is of general type then $K_S Z \geq 0$ and since M is mobile then $MZ \geq 0$. By

$$4 = K_S^2 = M^2 + (M + K_S)Z \geq 3 + (M + K_S)Z \geq 3$$

we have $(M+K_S)Z \leq 1$. We need to consider: $(M+K_S)Z = 1$ or $(M+K_S)Z = 0$.

If $(M+K_S)Z = 1$ then $K_SZ = 1$ and $MZ = 0$ or $K_SZ = 0$ and $MZ = 1$. If $K_SZ = 1$ and $MZ = 0$ then $Z^2 = 1$ and it is impossible, since $Z^2K^2 \leq (ZK)^2$ by the Hodge index theorem. If $K_SZ = 0$ and $MZ = 1$ then K_S is not a 2-connected divisor contrary to [2] [lemma 1].

If $(M+K_S)Z = 0$ then $MZ = 0$ and $K_SZ = 0$ and by [2] [lemma 1] we have $Z = 0$. Since $Z = 0$ then $K_S = M$. In particular $M^2 = 4$ and the argument of [8][p. 45-46] yields the claim.

■

We have shown in 2.1 that $\mid K_S \mid$ has a simple base point P. Let $\sigma : \tilde{S} \to S$ be the blowing up of P, $E = \sigma^{-1}(P)$ and L the mobile part of $\mid \sigma^* K_S \mid$. In particular $\sigma^* K_S \equiv L + E$ and since $K_{\tilde{S}} =\mid \sigma^* K_S \mid +E$ we have $\mid K_{\tilde{S}} \mid \equiv \mid L \mid +2E$. Moreover $\mid L \mid$ defines a morphism $\phi_{|L|} : \tilde{S} \to \mathbb{P}^2$ such that $\phi_{|L|} = \phi_{|K_S|} \circ \sigma$. In the next lemma (see also [14]) we will find on $\tilde{S}$ a pencil $\mid \tilde{F} \mid$ of non-hyperelliptic curves of genus 3 and an effective divisor $\tilde{G}$ such that $K_S \equiv G + F$ where $G = \sigma_* \tilde{G}$ and $F = \sigma_* \tilde{F}$. The task will be to understand the structure of G (see 2.15).

Lemma 2.2. *There exists a point x of $\mathbb{P}^2$ such that the divisors L_x of the sublinear system $\tilde{\Lambda} \subset \mid L \mid$ induced by the lines containing x have the following form:*

$$L_x \equiv \tilde{G} + \tilde{F},$$

where $\tilde{G}$ is the fixed part of $\tilde{\Lambda}$, $\phi_{|L|}(\tilde{G}) = x$ and $\mid \tilde{F} \mid$ is a pencil of curves of genus 3 with a simple base point $\tilde{Q}$. Furthermore $\tilde{Q} \notin E$ and the following numerical identities hold:

(i) $L\tilde{G} = 0$, $L\tilde{F} = 3$, $\tilde{F}\tilde{G} = 2$, $\tilde{F}^2 = 1$ *and* $\tilde{G}^2 = -2$.

(ii) $\tilde{G}E = 1$, $\tilde{F}E = 0$.

Proof. We first show that:

Remark 2.3. $LE = 1$ and $\phi_{|L|}(E)$ is a line in $\mathbb{P}^2$.

Proof. Since $\sigma^*(K_S)E = 0$, $\sigma^* K_S = L + E$ and $E^2 = -1$ we obtain $0 = (L+E)E = LE + (-1)$, that is $LE = 1$. On the other hand since $\mid L \mid$ is base point free if $\phi_{|L|}(E)$ is a point then $LE = 0$. Moreover $\phi_{|L||_E} : E \to \phi_{|L|}(E)$ has degree 1. In particular $\phi_{|L|}(E)$ is a line.

■

We will apply the following claim to the special point $x \in \mathbb{P}^2$ whose existence is asserted in 2.2.

Claim. Let y be a point of $\mathbb{P}^2$. Let $\tilde{\Lambda}_y$ be the sublinear system induced on $\tilde{S}$ by the lines containing y and let $\tilde{F}$, $\tilde{G}$ be respectively the mobile part and the fixed part. If $\tilde{G} \neq 0$ then $L\tilde{G} = 0$ and $L\tilde{F} = 3$. In particular $E \not\prec \tilde{G}$.

Proof of the claim

We recall that on $\tilde{S}$ there is not an infinite family of rational curves since $\tilde{S}$ is of general type. In particular since $\phi_{|L|}(\tilde{F})$ is a line then $L\tilde{F} \geq 2$ and the general element of the pencil $|\tilde{F}|$ is irreducible. On the other hand $3 = L^2 = L\tilde{F} + L\tilde{G}$ thus $2 \leq L\tilde{F} \leq 3$, since $LD \geq 0$ for every effective divisor D on $\tilde{S}$. We can exclude the case $L\tilde{F} = 2$. In fact by the theorem of Bertini the general L is irreducible thus if $L\tilde{F} = 2$ then $L\tilde{G} = 1$. Let z be a general point of $\mathbb{P}^2$. Since $\deg(\phi_{|L|}) = 3$ then $\phi_{|L|}^{-1}(z) = \{z_1, z_2, z_3\}$ where $z_i \neq z_j$ if $i \neq j$. Let $l_{y,z}$ be the line containing y and z. Obviously there exists $\tilde{F}_z \in |\tilde{F}|$ such that $\phi_{|L|}^{*} l_{y,z} = \tilde{G} + \tilde{F}_z$ and $\phi_{|L|}$ induces a double cover $\tilde{F}_z \to l_{y,z}$. Since $\phi_{|L|}^{-1} l_{y,z} = \operatorname{supp}(\tilde{G} + \tilde{F}_z)$ there exist $i \in \{1,2,3\}$ such that $z_i \in \tilde{G}$: a contradiction since z is a general point of $\mathbb{P}^2$. Hence $L\tilde{F} = 3$ and $L\tilde{G} = 0$. Moreover since $LE = 1$ then $E \not\prec \tilde{G}$ and this proves our claim.

We turn to the proof of 2.2. Assume for a while that we can prove the existence of an irreducible reduced effective divisor C of $\tilde{S}$ such that $\phi_{|L|}(C) = x$ is a point and $CE > 0$. In this case the lemma is a consequence of Hodge index theorem and some easy numerical conditions. In fact if $\tilde{\Lambda}$ is the sublinear system of $|L|$ induced by the lines containing x then by our assumption the fixed part $\tilde{G}$ of $\tilde{\Lambda}$ is a non-zero effective divisor and $C \prec \tilde{G}$. Hence by the claim $L\tilde{G} = 0$ and $L\tilde{F} = 3$ where $\tilde{F}$ is the mobile part of $\tilde{\Lambda}$. Furthermore since $E \not\prec \tilde{G}$ and $EC > 0$ then $E\tilde{G} > 0$. On the other hand since the general element of $|\tilde{F}|$ is irreducible then $\tilde{F}E \geq 0$. By 2.3 we have $1 = LE = \tilde{G}E + \tilde{F}E > \tilde{F}E \geq 0$ then $E\tilde{G} = 1$ and $E\tilde{F} = 0$. We recall that $K_{\tilde{S}} \equiv L + 2E$. By the genus formula $2p_a(\tilde{F}) - 2 = \tilde{F}^2 + K_{\tilde{S}}\tilde{F} = \tilde{F}^2 + L\tilde{F} = \tilde{F}^2 + 3$, then $\tilde{F}^2$ is odd. We collect all these results in the following system:

$$\begin{cases} \tilde{G}^2 + \tilde{F}\tilde{G} = L\tilde{G} = 0 \\ \tilde{G}\tilde{F} + \tilde{F}^2 = L\tilde{F} = 3 \\ \tilde{F}\tilde{G} \geq 0,\ \tilde{F}^2 = 2k+1,\ E\tilde{G} = 1,\ E\tilde{F} = 0. \end{cases}$$

If $\tilde{F}\tilde{G} = 0$ then $\tilde{F}^2 = 3$, $\tilde{G}^2 = 0$ and by Hodge index theorem $\tilde{G}$ is numerically equivalent to 0: a contradiction since $E\tilde{G} = 1$. If $\tilde{F}\tilde{G} \geq 1$ thus $1 \leq \tilde{F}^2 \leq 2$. Hence $k = 0$ and it is easy to see that $\tilde{F}^2 = 1$, $\tilde{G}\tilde{F} = 2$ and $\tilde{G}^2 = -2$. In particular $\mid \tilde{F} \mid$ is a pencil of curves of genus 3 with a simple base point $\tilde{Q}$. Since the general element of $\mid \tilde{F} \mid$ is irreducible then it is also smooth and by the genus formula, $\tilde{F}$ is of genus 3. Finally $\tilde{F}$ is non-hyperelliptic. In fact $K_{\tilde{F}} = (K_{\tilde{S}} + \tilde{F})_{\tilde{F}}$ and $\mid K_{\tilde{S}} \mid$ cuts on $\tilde{F}$ a g^1_3 which is a sublinear system of the canonical system $\mid K_{\tilde{F}} \mid$.

To complete the proof of 2.2 it remains to show that there exists an irreducible reduced effective divisor C of $\tilde{S}$ such that $\phi_{|L|}(C) = x$ is a point and $CE > 0$. We will use some cohomological results that we collect in the following remark.

Remark 2.4. *i)* $h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(2L)) = 6$ and $h^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(2L)) = 1$.
ii) $h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(2L + E)) = 7$ and $h^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(2L + E)) = 0$.
iii) $h^0(S, \mathcal{O}_{\tilde{S}}(3L)) = 10$.

Proof of the remark

i). Notice that $h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(2L)) \geq h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)) = 6$.

We now prove that $h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(2L)) \leq 6$. By [[10] p.109] the general L is a non-hyperelliptic curve of genus 5. From 2.3 the general L intersects E in one point: $P_L = L \cap E$.

By adjunction $\omega_L = (2L + 2E)_{|L} = 2L_{|L} + 2P_L$ and therefore $h^0(L, \mathcal{O}_L(\omega_L - 2P_L) = 3$. Our assertion follows now by the 0-cohomology of the sequence $0 \to \mathcal{O}_{\tilde{S}}(L) \to \mathcal{O}_{\tilde{S}}(2L) \to \mathcal{O}_L(\omega_L - 2P_L) \to 0$. Finally by Serre duality $h^2(\tilde{S}, \mathcal{O}_{\tilde{S}}(2L)) = 0$ since $(K_{\tilde{S}} - 2L)L = -1$. Then by Riemann-Roch theorem it follows that $h^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(2L)) = 1$.

ii). By Ramanujam vanishing theorem $h^1(S, \mathcal{O}_S(2K_S)) = 0$ and by Riemann-Roch formula we have $h^0(S, \mathcal{O}_S(2K_S)) = 8$. Since $\sigma^* 2K_S \equiv 2L + 2E$ then $h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(2L + E)) = 7$ if and only if the bicanonical system $\mid 2K_S \mid$ is base point free and this is a known result [4][Theor. 4.1]. It is now easy to see that $h^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(2L + E)) = 0$.

iii) From 2.3 we know that $\phi_{|L|}(E)$ is a line, then there exists an effective divisor C_0 such that $(\phi_{|L|})^*(\phi_{|L|}(E)) = E + C_0 = L_0 \in \mid L \mid$. We need to study C_0. Since $1 = LE = (E + C_0)E = -1 + C_0 E$ then $C_0 E = 2$ and from $3 = L^2 = (E + C_0)L = 1 + C_0 L = 1 + C_0 E + C_0^2 = 3 + C_0^2$ we obtain $C_0^2 = 0$. Since $\mid K_S \mid$ is 2-connected then C_0 is a 1-connected effective divisor. Moreover by the genus formula $p_a(C_0) = 4$.

We notice now that $3L_{|C_0} = (L+2E+2C_0)_{|C_0} = \omega_{C_0} + C_{0_{|C_0}}$ where ω_{C_0} is the dualizing sheaf of C_0. Consider the two sequences:

$$0 \to \mathcal{O}_{\tilde{S}}(2L) \to \mathcal{O}_{\tilde{S}}(3L) \to \mathcal{O}_L(3L) \to 0$$

and

$$0 \to \mathcal{O}_{\tilde{S}}(2L+E) \to \mathcal{O}_{\tilde{S}}(3L) \to \omega_{C_0} + C_{0_{|C_0}} \to 0.$$

By the proof of *i)* we know that $g(L) = 5$ then $h^0(L, \mathcal{O}_L(3L)) = 5$. Hence by *i)* and the cohomology of the first sequence we obtain $10 \leq h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(3L)) \leq 11$. Now we argue by contradiction. Suppose that $h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(3L)) = 11$. Then by *ii)* and the second sequence we obtain $h^0(C_0, (\mathcal{O}_{C_0}(\omega_{C_0} + C_0)) = 4$. Then by Riemann-Roch theorem it follows $h^1(C_0, (\mathcal{O}_{C_0}(\omega_{C_0} + C_0)) = 1$. By Serre duality we obtain $h^1(C_0, (\mathcal{O}_{C_0}(\omega_{C_0} + C_0)) = h^0(C_0, (\mathcal{O}_{C_0}(-C_0)) = 1$. It is also easy to see that $deg(\mathcal{O}_{C_1}(-C_0)) = 0$ for each component $C_1 \prec C_0$. Hence by [1] 12.2 we have $C_{0_{|C_0}} = \mathcal{O}_{C_0}$. On the other hand $q(\tilde{S}) = 0$, $h^0(C_0, \mathcal{O}_{C_0}) = 1$ then by the 0-cohomology of the sequence $0 \to \mathcal{O}_{\tilde{S}} \to \mathcal{O}_{\tilde{S}}(C_0) \to \mathcal{O}_{C_0}(C_0) \to 0$ it would be $h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(C_0)) = 2$: a contradiction. This proves 2.4.

■

We continue discussing 2.2 and we mantain the notations introduced in the proof of 2.4 *iii)*. We argue by contradiction. Let us suppose that there exists no irreducible reduced effective divisor C of $\tilde{S}$ such that $\phi_{|L|}(C) = x$ is a point and $CE > 0$.

Since $EC_0 = 2$, we can take a point $z \in E \cap H$. Then we take an irreducible reduced component H of C_0 containing z, which, from our assumption, verifies $\phi_{|L|}(E) = \phi_{|L|}(H)$. We set $H' = C_0 \setminus H$. Let $\tilde{\Lambda}$ be the sublinear system induced on $\tilde{S}$ by the lines in $\mathbb{P}^2$ containing $\phi_{|L|}(z) = \tilde{x}$. We consider $\zeta \in H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(E))$, $h \in H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(H))$, $h' \in H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(H'))$; then $hh' = c_0$ where $div(c_0) = C_0$. Let $\langle x_0, x_1, x_2 \rangle$ be a basis of $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$, where $\tilde{x} = \{x_0 = x_1 = 0\}$ and $\phi_{|L|}(E) = \{x_0 = 0\}$. We set $X_i = \phi^\star_{|L|}(x_i) = x_i \circ \phi_{|L|}$, then $\langle X_0, X_1 \rangle$ is a basis of $\tilde{\Lambda}$, $X_0(z) = X_1(z) = 0$ and $X_0 = \zeta h h'$. By 2.4 *i)* we have $H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(2L)) = \phi^\star_{|L|} H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)))$. By 2.4 *ii)* and by the inclusion $H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(2L)) \overset{\otimes\zeta}{\to} H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(2L+E))$ there exists ψ such that $H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(2L+E)) = \zeta(H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(2L))) \oplus \psi\mathbb{C}$. We consider now the inclusion $H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(2L+E)) \overset{\otimes hh'}{\to} H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(3L))$. By 2.4 *iii)* it follows $H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(3L)) = \phi^*_{|L|}H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3)))$. In particular $\langle X_0^i X_1^j X_2^k \mid i+j+k=3\rangle$ is a basis of $H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(3L))$ and since $hh'\psi \in H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(3L))$ there is a linear combination with coefficients in $\mathbb{C}$

$$a)\ hh'\psi = \sum \alpha_{ijk} X_0^i X_1^j X_2^k$$

where $i+j+k=3$. We recall that $X_0 = \zeta hh'$. Obviously h does not divide neither X_1 nor X_2. Moreover by definition $X_1(z) = h(z) = 0$ and $X_2(z) \neq 0$. We evaluate *a)* in z and we see that $\alpha_{003}X_2^3(z) = 0$ that is $\alpha_{003} = 0$. Then $h \mid \sum \alpha_{0jk} X_1^{j-1} X_2^k$ where $j+k=3$. Evaluating again in z we obtain $\alpha_{012} = 0$ and $h \mid \alpha_{030}X_1 + \alpha_{021}X_2$. We repeat once more the argument and it yelds $\alpha_{021} = \alpha_{030} = 0$. This implies that $X_0 \mid hh'\psi$; but $X_0 = \zeta hh'$ then $\zeta \mid \psi$ or, in other words, ψ comes from $H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(2L))$ which is the desired contradiction. This proves 2.2.

■

We turn to our surface S and we recall that $G = \sigma_\star \tilde{G}$ and $F = \sigma_\star \tilde{F}$. We will see that $\mid F \mid$ is a pencil of curves of genus 3 with a simple base point $P' \neq P$, G is reducible and it has a component G_0 contained in a fibre $F_0 \in \mid F \mid$ such that P, P'supp(G_0) and $p_a(G_0) = 1$, these conditions will imply the theorem. We need some lemmas.

Lemma 2.5. *We use the notation of 2.2. We denote* $G = \sigma_\star(\tilde{G})$, $F = \sigma_\star(\tilde{F})$. *Then* $K_S \equiv G + F$ *and*
(j) $F^2 = 1$, $G^2 = -1$, $FG = 2$, *and* $P \in \mathrm{supp}(G)$*;*
(jj) $\mid F \mid$ *is a genus-3 pencil with a simple base point* $P' = \sigma(\tilde{Q})$. *Moreover* $P' \neq P$*;*
(jjj) G *is a 1-connected effective divisor with* $K_S G = 1$ *and* $p_a(G) = 1$.

Proof. *(j)*. From 2.2 *ii)* we see that $\sigma^\star(F) = \tilde{F}$ and $\sigma^\star(G) = \tilde{G} + E$. Then by 2.2 *i)* we have $F^2 = \tilde{F}^2 = 1$, $G^2 = (\sigma^\star(G))^2 = (\tilde{G}+E)^2 = \tilde{G}^2 + 2\tilde{G}E + E^2 = -2+2-1 = -1$, $FG = \tilde{F}(\tilde{G}+E) = \tilde{F}\tilde{G} + \tilde{F}E = 2+0 = 2$. By 2.2 *ii)* $\tilde{G}E = 1$ and since $E = \sigma^{-1}(P)$ we see that $P \in \mathrm{supp}(G)$.

(jj). From 2.2 we know that $\mid \tilde{F} \mid$ is a pencil of non-hyperelliptic curves of genus 3 with a simple base point $\tilde{Q} \notin E$. Then $P' \doteq \sigma(\tilde{Q}) \neq P = \sigma(E)$.

(jjj). Let $G = G_0 + G_1$ be a decomposition of G into two effective non-zero divisor such that $G_0 G_1 = 0$ and $FG_0 = 2 - i$, $FG_1 = i$ where

$0 \leq i \leq 1$. We set $A_0 = F + G_0$ and $A_1 = G_1$. Thus $K_S \equiv A_0 + A_1$ and $A_0A_1 = 1$; a contradiction since K_S is 2-connected. Moreover $K_SG = FG + G^2 = 2 - 1 = 1$ and by the genus formula we have $p_a(G) = 1$.

■

Corollary 2.6. *(a)* $h^0(S, \mathcal{O}_S(F)) = 2$, $h^1(S, \mathcal{O}_S(F)) = 0$ *and* $h^2(S, \mathcal{O}_S(F)) = 1$
(b) $h^0(S, \mathcal{O}_S(2F)) = 3$ *and* $h^1(S, \mathcal{O}_S(2F)) = h^2(S, \mathcal{O}_S(2F)) = 0$

Proof. *a)*. From 2.5 we have $K_S = F + G$ and by Serre duality $h^2(S, \mathcal{O}_S(F)) = h^0(S, \mathcal{O}_S(G)) = 1$. It is now easy to see that *a)* follows from 2.5 *jj)* and Riemann-Roch theorem.

b). We recall that the general element F of $|F|$ is a non-hyperelliptic curve of genus 3 and $F_{|F} = P'$. In particular $h^0(F, \mathcal{O}_F(2P')) = 1$ for the general F. It is easy to see that $h^0(S, \mathcal{O}_S(2F)) \geq 3$. On the other hand by *a)* and the cohomology of $0 \to \mathcal{O}_S(F) \to \mathcal{O}_S(2F) \to \mathcal{O}_F(2P') \to 0$ we obtain $h^0(S, \mathcal{O}_S(2F)) \leq 3$. Thus $h^0(S, \mathcal{O}_S(2F)) = 3$.

■

Corollary 2.7. *i)* $H^1(S, \mathcal{O}_S(G)) = 0$, *ii)* $H^0(S, \mathcal{O}_S(F - G)) = 0$, $H^1(S, \mathcal{O}_S(F - G)) = 0$ *and iii)* $H^1(S, \mathcal{O}_S(2G)) = 0$.

Proof. *i)*. By Serre duality and 2.6 *(a)* it follows *i)* while *iii)* is a consequence of Serre duality and the second equality of *ii)*.

ii). By contradiction. If $H^0(S, \mathcal{O}_S(F - G)) > 0$ then there exists $F_0 \in |F|$ such that $G \prec F_0$. We set $D = F_0 - G$. Then by 2.5 *j)* we have $1 = F^2 = F(G + D) \geq FG = 2$.

■

We will use the next lemma to show that $P' \in \text{supp}(G)$. This will play a central role to show that S is birational to a quintic $X \subset \mathbb{P}^3$.

Lemma 2.8. *With the notation of 2.5,* $h^0(S, \mathcal{O}_S(K_S + G)) = 4$ *and* $h^1(S, \mathcal{O}_S(K_S + G)) = 0$

Proof. We consider the cohomology of the adjunction sequence for G: $0 \to \mathcal{O}_S(K_S) \to \mathcal{O}_S(K_S + G) \to \omega_G \to 0$. By 2.5 *jjj)* $h^0(G, \omega_G) = 1$ and since $p_g(S) = 3$, $q(S) = 0$ then $h^0(S, \mathcal{O}_S(K_S + G)) = 3 + 1 = 4$ ■

A non intuitive fact is that $P' \in \operatorname{supp}(G)$. This result will give us the structure of the divisor G which will indicate how to prove the theorem. We need some preliminaries. Let $\tau : S' \to S$ be the blowing up in the point P'. Since P' is the simple base point of $\mid F \mid$ then $\tau^*(F) = F' + E'$ where $E' = \tau^{-1}(P')$ and $\mid F' \mid$ is a pencil without base point. In particular $\mid F' \mid$ induces a relatively minimal fibration $f : S' \to \mathbb{P}^1$ with fiber F'. We recall that the dualizing sheaf $\omega_{S'|\mathbb{P}^1} = \omega_{S'} \otimes (f^*\omega_{\mathbb{P}^1})^{-1}$ of f is the line bundle with associated divisor $K_{S'} + 2F'$.

Lemma 2.9. $h^0(S, \mathcal{O}_S(K_S + 2F)) = 10$.

Proof. Obviously $\mid K_S + 2F \mid$ does not have a fixed part. We need to show that P' is not a base point of $\mid K_S + 2F \mid$. It is sufficient to prove that P' is not a base point of $\mid K_S + F \mid$. If P' were a base point of $\mid K_S + F \mid$, since P' is a base point of $\mid F \mid$ and $q(S) = 0$, it would also be a base point of $\mid K_S + F \mid_{|F} = \mid K_F \mid$ for a general F, which is a contradiction. Then P' is not a base point of neither $\mid K_S + F \mid$ nor $\mid K_S + 2F \mid$. We notice now that $\tau^*(K_S + 2F) = \tau^*(K_S) + 2F' + 2E' \equiv K_{S'} + 2F' + E'$. Then by the cohomology of

$$0 \to \mathcal{O}_{S'}(K_{S'} + 2F') \to \mathcal{O}_{S'}(K_{S'} + 2F' + E') \to \mathcal{O}_{E'} \to 0$$

we see that $h^0(S', \mathcal{O}_{S'}(K_{S'} + 2F' + E')) = 1 + h^0(S', \mathcal{O}_{S'}(K_{S'} + 2F'))$. Thus we need to show only that $h^0(S', \omega_{S'|\mathbb{P}^1}) = 9$. The proof of this fact is a standard application of the relative duality and of the Leray spectral sequence for the morphism $f : S' \to \mathbb{P}^1$.

By [6][Prop.2.7] we know that $f_*\omega_{S'|\mathbb{P}^1}$ is a locally free sheaf of rank 3 and by [6][Prop.1.2] every invertible sheaf $\mathcal{L}$ which is a homomorphic image of $f_*\omega_{S'|\mathbb{P}^1}$ is of degree ≥ 0. It is well known that every locally free sheaf on $\mathbb{P}^1$ is decomposable. From these facts we see that $f_*\omega_{S'|\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) \oplus \mathcal{O}_{\mathbb{P}^1}(a_3)$ where $0 \leq a_1 \leq a_2 \leq a_3$. Then $h^1(\mathbb{P}^1, f_*\omega_{S'|\mathbb{P}^1}) = 0$ and by the Leray spectral sequence $h^1(S', \omega_{S'|\mathbb{P}^1}) = h^0(\mathbb{P}^1, R^1 f_*\omega_{S'|\mathbb{P}^1})$. From the relative duality it follows immediately $h^0(\mathbb{P}^1, R^1 f_*\omega_{S'|\mathbb{P}^1}) = h^0(\mathbb{P}^1, f_*\mathcal{O}_{S'}) = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 1$. Thus $h^1(S', \omega_{S'|\mathbb{P}^1}) = 1$. From Serre duality $h^2(S', \omega_{S'|\mathbb{P}^1}) = h^0(S', \mathcal{O}_{S'}(-2F')) = 0$ then by Riemann-Roch theorem we obtain $h^0(S', \omega_{S'|\mathbb{P}^1}) = 9$. ∎

Corollary 2.10. $h^0(S, \mathcal{O}_S(3F + G)) = 10$

Proof. Obvious since $K_S + 2F \equiv 3F + G$.

■

Now we will compare the canonical linear system K_F of a general F and the induced linear system $(K_S + G)_{|F}$. Then we will show that $P' \in \mathrm{supp}(G)$. We prove first that $| 3F | \neq 3 | F |$.

Remark 2.11. $h^0(S, \mathcal{O}_S(3F)) = 5$.

Proof. We will use the following *Claim:* $h^0(G, \mathcal{O}_G(3F + G)) = 5$. In fact by 2.5 *jjj)* we see that G is 1-connected and $\deg\omega_G = 0$. Moreover by 2.5 *j)* $\deg\mathcal{O}_G(3F + G) = 5$ and $\deg\mathcal{O}_G(-2F + G) < 0$. By Serre duality we have $h^1(G, \mathcal{O}_G(3F+G)) = h^0(G, \mathcal{O}_G(K_S+G-(3F+G)) = h^0(G, \mathcal{O}_G(-2F + G)) = 0$. Hence by Riemann-Roch theorem for curves $h^0(G, \mathcal{O}_G(3F + G)) = 5$ and this proves the Claim.

We show now that $h^0(S, \mathcal{O}_S(3F)) = 5$ Since F has genus 3 then $1 \leq h^0(F, \mathcal{O}_F(3P')) \leq 2$. Now the cohomology of $0 \to \mathcal{O}_S(2F) \to \mathcal{O}_S(3F) \to \mathcal{O}_F(3P') \to 0$ and 2.6 *b)* show that $h^0(S, \mathcal{O}_S(3F)) \leq 5$. On the other hand the cohomology of $0 \to \mathcal{O}_S(3F) \to \mathcal{O}_S(3F + G) \to \mathcal{O}_G(3F + G) \to 0$, 2.10 and the claim imply $h^0(S, \mathcal{O}_S(3F)) \geq 5$.

■

It is useful to remark this easy consequence of 2.11.

Corollary 2.12. *Let F be a general element of $| F |$ then there exists a point $P_F \in F$ such that $P_F + 3P' \equiv K_F$.*

Proof. By 2.11 and the first exact sequence in the proof of 2.11 we see that $h^0(F, \mathcal{O}_F(3P')) = 2$ then we have $h^1(F, \mathcal{O}_F(3P')) = h^0(F, \mathcal{O}_F(K_F - 3P')) = 1$. In other words there exists $P_F \in F$ which satisfies the statement.

■

Now we can prove

Lemma 2.13. $P' \in \mathrm{supp}(G)$

Proof. Let F be a general element of $| F |$ and we set $(K_S+G)_{|F} = D_F$. Since $K_S + G \equiv F + 2G$ then $D_F \equiv P' + 2G_{|F}$. On the other hand

$(K_S+F)_F = G_{|F} + 2P' \equiv K_F$. We now consider F as a smooth quartic in $\mathbb{P}^2$. Then $| K_F |$ is the linear system induced on F by the lines of $\mathbb{P}^2$. By 2.12 we have $3P' \prec K_F$, then the tangent l at P' of F has contact order 3 in P' that is $l_{|F} = 3P' + P_F$. Since $K_F \equiv (2F+G)_F = 2P' + G_{|F} = l_{|F} = 3P' + P_F$ then $P' \prec G_{|F}$; in particular $P' \in \mathrm{supp}(G)$.

■

We remark the following easy consequence of 2.12 and 2.13:

Remark 2.14. Let F be a general element of $| F |$ and P_F as in 2.12. Then $P_F + P' = G_{|F}$.

We now make clear the structure of the elliptic cycle G. For the reader's benefit we include in the next proposition some results which we have just proved. In this way we can collect all the results we will use to show our theorem.

Proposition 2.15. (The structure of G). *Let S be a minimal surface over $\mathbb{C}$ with $K_S^2 = 4$, $p_g = 3$ and $d = 3$. Then the canonical system is without fixed part, it has only a simple base point P and there exists a 2-dimensional sublinear system Λ such that the divisors K of Λ have the following form:*

$$K \equiv G + F$$

where G and F are respectively the fixed part and the mobile part of Λ. The linear system $| F |$ is a pencil of non-hyperelliptic curves of genus 3 with a simple base point $P' \neq P$. The divisor G is a 1-connected reducible divisor and

$$G = G_0 + G_1$$

where $p_a(G_0) = 1$, $P, P' \in \mathrm{supp}(G_0)$, there exists an $F_0 \in | F |$ such that $G_0 \prec F_0$ and G_1 is a chain of -2-rational curves. The following numerical identities hold: $G_0^2 = -1, G_1^2 = -2$, $G_0G_1 = 1$, $FG_0 = FG_1 = 1$. Moreover the map $\phi_{|K_S+G_0|} : S \to \mathbb{P}^3$ induces a birational morphism on the image $\phi_{|K_S+G_0|}(S) = X$ and X is a quintic.

Proof. We have shown the first part of the proposition in 2.1 and in 2.5. It remains to study G. In our discussion we will distinguish two cases: (i) G is irreducible and (ii) G is reducible. We want to exclude the case (i). If G is irreducible then, by 2.5 $jjj)$ it is also reduced. From 2.7 $ii)$ we see that G is not contained in any element $F \in | F |$ and by 2.5

j) we know that $P \in G$. We consider the sequence: $0 \to \mathcal{O}_S(F-G) \to \mathcal{O}_S(F) \to \mathcal{O}_G(F) \to 0$. By 2.7 ii we obtain that the restriction map $H^0(S,\mathcal{O}_S(F)) \to H^0(G,\mathcal{O}_G(F))$ is an isomorphism. Then by 2.6 (a) and by 2.13 the pencil $\mid F \mid$ cuts on G a complete linear system of degree two with one base point P'. We show now that P' is a smooth point of G. Otherwise let $\pi : X \to S$ be the blowing up of P', $D = \pi^*(F) - E$ and $C = \pi^*(G) - 2E$ where $E = \pi^{-1}(P')$. We remark that since $FG = 2$ then P' has multiplicity 2 on G. Since $DC = 0$ then there exists $D_0 \in \mid D \mid$ such that $C \prec D$. In particular $\pi_* D_0 = F_0$ is an element of $\mid F \mid$ such that $G \prec F_0$. But $1 = F^2 = FF_0 = F(G + (F-G)) \geq FG = 2$: a contradiction. The same argument shows that $F_{|G} = P_F + P'$ where $P_F \neq P'$ for the generic F. By $\mid P_F \mid$ we can construct a birational morphism $G \to \mathbb{P}^1$. Then 2.5 jjj) shows that G is a rational curve with a singular point Q of multiplicity 2. In particular $Q \neq P'$ and then $Q \neq P_F$ for every $F \in \mid F \mid$; but this is impossible.

ii). *G is reducible*

We first show the following

Claim. $\mid K_S + G \mid$ has a fixed part.

By contradiction we suppose that $\mid K_S + G \mid$ is without a fixed component. From this assumption it follows that it is also without base points. In fact by 2.5 j), jjj) we see that $(K_S + G)G = 0$. On the other hand by 2.8 $h^0(S, \mathcal{O}_S(K_S + G)) = 4$ and since $p_g(S) = 3$ then there exists $H \in \mid K_S + G \mid$ such that $H \cap supp(G) = \emptyset$. Then $\mid K_S + G \mid$ is without base points.

Consider now $0 \to \mathcal{O}_S(2G) \to \mathcal{O}_S(K_S + G) \to \mathcal{O}_F(K_S + G) \to 0$. By 2.7 iii) we see that $H^0(S, \mathcal{O}_S(K_S + G)) \to H^0(F, \mathcal{O}_F(K_S + G))$ is surjective. On the other hand by 2.13 and 2.14 we have $K_F \prec (K_S + G)_F$. We recall that $\mid K_F \mid$ is a g^2_4, while $\mid K_S + G)_F \mid$ is a g^2_5. Then $\mathcal{O}_S(K_S + G)$ induces a complete linear system on F with a base point. This is true for the general F, hence $\mid K_S + G \mid$ has a fixed part. This proves the claim.

Let G_1 be the fixed component. Since $\mid K_S \mid$ has only a base point then $G_1 \prec G$. We can split $G = G_1 + (G - G_1)$ and we denote $G_0 = G - G_1$. We show now that $\mid K_S + G_0 \mid$ is without base points. It is obvious that $FG_0 \geq 0$ and $0 \leq K_S G_0 \leq 1$. Moreover since G is 1-connected then $G_1 G_0 \geq 1$. Thus by $0 \leq K_S G_0 = FG_0 + G_0^2 + G_1 G_0 \leq 1$

we obtain $G_0^2 < 0$. In fact if $G_0^2 = 0$ then $G_1G_0 = 1$ and $K_SG_0 = 1$. In particular G_0 remains 1-connected but by adjunction we obtain $G_0^2 \neq 0$.

By definition $| K_S + G_0 |$ does not have any fixed component then $K_SG_0 \geq -G_0^2 > 0$. It is now easy to prove that $K_SG_0 = 1$, $G_0^2 = -1$, $K_SG_1 = 0$ and $(K_S + G_0)G_0 = 0$. In particular by 2.5 j) it follows that $P \in supp(G_0)$ since $K_SG_1 = 0$. Moreover $K_SG_1 = 0$ implies that $supp(G_1)$ is an union of -2-rational curves.

The same argument used in the proof of the last claim shows that there exists $H \in | K_S + G_0 |$ such that $H \cap supp(G_0) = \emptyset$. Then $| K_S + G_0 |$ does not have any base point. In fact if P_1 is a base point then $P_1 \in H$ and $P_1 \notin supp(G_0)$. Then $P_1 \in supp(K_S)$ for each $K_S \in | K_S |$; that is $P_1 = P$, thus $P_1 \in supp(G_0)$; a contradiction. We have shown that $\phi_{|K_S+G_0|}$ is a morphism. We set $n = \deg \phi_{|K_S+G_0|}$ and $X = \phi_{|K_S+G_0|}(S)$. Notice that since $(K_S + G_0)^2 = 5$ we have $5 = n\deg(X)$ then $n = 5$ and $\deg(X) = 1$ or $n = 1$ and $\deg(X) = 5$. The first case is clearly impossible since $\phi_{|K_S+G_0|}$ is induced by a complete linear system of dimension 4. Thus X is a quintic in $\mathbb{P}^3$ birational to S. We give now the desired decomposition of G. Since $G_0^2 = -1$ and $K_S \equiv F + G_0 + G_1$, by $K_SG_0 = 1$ and $K_SG_1 = 0$ we obtain:

$$\begin{cases} FG_0 + G_1G_0 = 2 \\ FG_1 + G_0G_1 + G_1^2 = 0. \end{cases}$$

By the first equation we obtain $0 \leq FG_0 \leq 1$. We exclude the case $FG_0 = 0$. In fact if $FG_0 = 0$ then $(K_S + G_0)_{|F} = K_{S|F}$. In particular $\phi_{|K_S+G_0|}(F)$ is a line and the image $X = \phi_{|K_S+G_0|}(S)$ has a one parameter family of rational curves. A contradiction since S is of general type and X is birational to S.

If $FG_0 = 1$ then $FG_1 = 1$ since $FG = 2$ and $G = G_1 + G_0$. Moreover by the first equation we have $G_1G_0 = 1$ and by the second equation $G_1^2 = -2$. We have shown above that $P \in G_0$. We prove now that $P' \in G_0$. Let F be a general element of $| F |$. Since $FG_0 = 1$ it is sufficient to show that $(G_0)_F = P'$. By contradiction we suppose that $(G_0)_F = Q_F \neq P'$. Thus $(K_S + G_0)_F = (K_S)_F + Q_F$ and since $(K_S + F)_F = (K_S)_F + P'$ then $h^0(F, \mathcal{O}_F(K_S + G_0)) = 2$ otherwise $1 = h^1(F, \mathcal{O}_F(K_S + G_0)) = h^0(F, \mathcal{O}_F(P' - Q_F))$ that is $Q_F = P'$. On the other hand by 2.8 and by the 0-cohomology of

$$0 \to \mathcal{O}_S(G+G_0) \to \mathcal{O}_S(K_S+G_0) \to \mathcal{O}_F(K_S+G_0) \to 0$$

it follows that $h^0(S,\mathcal{O}_S(G+G_0)) \geq 2$: a contradiction. Incidentally we have proved that $(G_0)_F = P'$ and this implies $(K_S+G_0)_F = K_F$. To finish the proof of the proposition we will show that $h^0(S,\mathcal{O}_S(F-G_0)) = 1$. By Serre duality this is equivalent to prove that $h^2(S,\mathcal{O}_S(G+G_0)) = 1$. This will follow by the cohomology of the above exact sequence. In fact by 2.8 $h^1(S,\mathcal{O}_S(K_S+G_0)) = 0$ and $h^2(S,\mathcal{O}_S(K_S+G_0)) = 0$. Then $H^1(F,\mathcal{O}_F(K_S+G_0)) \to H^2(S,\mathcal{O}_S(G+G_0))$ is surjective. On the other hand $(K_S+G_0)_F = K_F$. In particular $h^1(F,\mathcal{O}_F(K_S+G_0)) = 1$

■

Proof of the main theorem

We need to show only that the quintic X obtained in the proof of 2.15 belongs to $\mathcal{A}$. Let $\mathcal{B} \subset H^0(S,\mathcal{O}_S(5(K_S+G_0)))$ be the sublinear system given by the sections which vanish on G_0 with order six, that is $\mathcal{B} \simeq H^0(S,\mathcal{O}_S(5K_S-G_0))$. Since $K_S \equiv G_0+G_1+F$ then $5K_S-G_0 \equiv 4K_S+F+G_1$. We want to compute the dimension of $\mathcal{B}$. The cohomology of $0 \to \mathcal{O}_S(4K_S+F) \to \mathcal{O}_S(4K_S+F+G_1) \to \mathcal{O}_{G_1}(-1) \to 0$ yields $h^0(S,\mathcal{O}_S(5K_S-G_0)) = h^0(S,\mathcal{O}_S(4K_S+F))$. The Ramanujam vanishing theorem gives $h^1(S,\mathcal{O}_S(4K_S)) = 0$. Thus the cohomology of $0 \to \mathcal{O}_S(4K_S) \to \mathcal{O}_S(4K_S+F) \to \mathcal{O}_F(4K_S+F) \to 0$ and the theorem of Riemann-Roch imply that $h^0(S,\mathcal{O}_S(4K_S+F)) = 39$; that is $\dim_{\mathbb{C}}\mathcal{B} = 39$. We study now $\phi_{|K_S+G_0|}$. By the proof of 2.15 we see that there exist $g_i \in H^0(S,\mathcal{O}_S(G_i))$ with $i = 0,1$ such that $g_0g_1 = g \in H^0(S,\mathcal{O}_S(G))$ and a basis $\langle t_0, t_1\rangle$ of $H^0(S,\mathcal{O}_S(F))$ such that $g_0 \mid t_0$. Thus by 2.5 we have the following basis of $H^0(S,\mathcal{O}_S(K_S))$: $\langle t_0g, t_1g, z_2\rangle$ where $\mathrm{div}(z_2)$ is irreducible reduced and $P' \notin \mathrm{div}(z_2)$. In the proof of 2.15 we have constructed an effective divisor $H \equiv K_S+G_0$ such that $\mathrm{supp}(H) \cap \mathrm{supp}(G_0) = \emptyset$. Thus by the inclusion $\otimes g_0 : H^0(S,\mathcal{O}_S(K_S)) \to H^0(S,\mathcal{O}_S(K_S+G_0))$ we see that there exists $v \in H^0(S,\mathcal{O}_S(K_S+G_0))$ such that g_0 does not divide v and $\langle t_0gg_0, t_1gg_0, z_2g_0, v\rangle$ is a basis. We set $\phi_{|K_S+G_0|} = \psi$. We can chose a system (x_0,x_1,x_2,x_3) of coordinates on $\mathbb{P}^3$ such that $\psi^*x_0 = t_0g_0g$, $\psi^*x_1 = t_1g_0g$, $\psi^*x_2 = g_0z_2$ and $\psi^*x_3 = v$.

Let α_s the cardinality of A_s. It is obvious that $\alpha_3 + \alpha_2 + \alpha_1 + \alpha_0 = 1 + 6 + 13 + 20 = 40$. On the other hand if $x^I \in H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(5))$ as in the statement of the theorem then $\mathrm{ord}_{G_0}(\mathrm{div}(\psi^* x^I)) \geq 6$ since $\mathrm{ord}_{G_0}(\mathrm{div}(\psi^* x_i)) = 3 - i$, $i = 0, 1, 2, 3$. The forty sections $\psi^* x^I$ are in $\mathcal{B}$ for $I \in A_s$, $s = 0, 1, 2, 3$. Then there exists a non trivial relation on S: $\sum_{s=0}^{3} \sum_{I \in A_s} a_I \psi^* x^I = 0$. It is now obvious that this relation gives the equation of X and then $X \in \mathcal{A}$. This proves the main theorem.

We conclude our paper with the following easy consequence of the theorem:

Corollary 2.16. *Let $\mathcal{X}^3_{3,4}$ be the locus of surfaces with $K^2_S = \underline{4}$, $p_g = 3$ and canonical map of degree three. Then $\mathcal{X}^3_{3,4}$ is irreducible, unirational and it has dimension 29.*

Proof. By the main theorem there exists an open set $\mathcal{A}'$ in the linear system $\mathcal{A}$ and a rational dominant map $\pi : \mathcal{A}' \to \mathcal{X}^3_{3,4}$. If $S \in \mathcal{X}^3_{3,4}$ then it is easy to see that $\pi^{-1}(S)$ is the orbit by the action on $\mathcal{A}'$ of the subgroup G of $\mathbb{P}GL(5, \mathbb{C})$ given by the transformations of the following form: $x_0 \mapsto a_0 x_0$, $x_1 \mapsto a_1 x_0 + b_1 x_1$, $x_2 \mapsto a_2 x_0 + b_2 x_1 + c_2 x_2$ and $x_3 = a_3 x_0 + b_3 x_1 + c_3 x_2 + d_3 x_3$, where $a_i, b_j, c_k, d_3 \in \mathbb{C}$, $i = 0, 1, 2, 3$, $j = 1, 2, 3$, $k = 2, 3$. Since $\dim(G) = 9$ and the projective dimension of $\mathcal{A}$ is 38 then $\dim \mathcal{X}^3_{3,4} = 29$.

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