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The full automorphism group of the Kulkarni surface.

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Dedicated to the memory of Sheela Phansalkar (1966-1990)

Abstract

The full automorphsim group of the Kulkarni surface is explicitly determined. It is employed to give three defining equations of the Kulkarni surface; each equation exhibits a symmetry of the surface as complex conjugation.

In [1] and [6], Accola and Maclachlan determined that for each genus $g \geq 2$, there exists a Riemann surface which admits $8g+8$ automorphisms. They also proved that an automorphism group of larger order cannot be uniformly constructed for every genus. In [5], Kulkarni analyzed whether the family constructed by Accola and Maclachlan is the only family of Riemann surfaces whose members posses $8g+8$ automorphisms. He proved that if $g \equiv 0, 1, 2 \bmod 4$ and sufficiently large, then the family constructed by Accola is unique, however if $g \equiv 3 \bmod 4$ there exists an additional family. Members of this family have subsequently been named Kulkarni surfaces. Kulkarni showed that these Riemann surfaces have an automorphism group isomorphic to the group with the following presentation:

$$< A, B \mid A^{2g+2} = B^4 = (AB)^2 = 1, \ \ B^2AB^2 = A^{g+2} > . \qquad (1)$$

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In [4], a defining equation for the Kulkarni surface is computed. After a birational change of coordinates, the defining equation exhibited in [4] becomes:

$$z^{2g+2} - (x-1)x^{g-1}(x+1)^{g+2} = 0. \tag{2}$$

Recall that a symmetry of a Riemann surface is a bijective, antiholomorphic involution. In [2], it was determined that, in addition to the automorphism group of order $8g+8$, each Kulkarni surface possesses three conjugacy classes of symmetries with fixed points. As is well known, to each class of symmetry a defining equation with real coefficients can be given which exhibits the symmetry as complex conjugation. In addition, since the three symmetries have fixed points, the defining equations for the surface admit real solutions.

In this paper we explicitly determine the full automorphism group of (2). We then employ our results to yield three defining equations of the Kulkarni surface in which the symmetries are exhibited by conjugation.

Throughout the paper, let g be a fixed integer congruent to 3 mod 4. Let G be the group defined in (1) and let X denote the Kulkarni surface of genus g. Let K be the subgroup of G generated by A and let H be the subgroup of G generated by A and B^2. Thus $[G:H] = [H:K] = 2$. Note that $K \triangleleft H$, however K is not a normal subgroup of G. Let U denote the upper half plane. To each of the subgroups defined above, we can associate a Fuchsian group. From (1), $G \cong \Lambda/\Gamma$ where Λ is the triangle group $(2,4,2g+2)$, and Γ is the normal subgroup of Λ generated by $B^2AB^2A^g$. There are three points of $X/G = U/\Lambda$ ramified in $X = U/\Gamma$. To conform with notation later in the paper, let R, R_∞, and R_0 be the points of ramification 2,4 and $2g+2$ respectively. It is easy to determine the following information:

(1) In the covering of X/G by X/H, both R and R_∞ are ramified, however R_0 is unramified. Let Q_0, and Q_1 denote the points lying above R_0 and let Q_∞ denote the point lying above R_∞.

(2) In the covering of X/H by X/K, Q_∞ is ramified. In addition, exactly one of Q_0 and Q_1 is ramified; we choose our notation so that Q_0 is ramified, and Q_1 is unramified. Let P_∞ denote the point lying over Q_∞, let P_{-1} and P_1 denote the points lying over Q_1, and let P_0 denote the point lying over Q_0.

(3) The only ramification that occurs in the covering of X/K by X is at P_{-1}, P_1, and at P_0. The ramification indices at these points are $2g+2, 2g+2$ and $g+1$ respectively.

The determination of the automorphism group of X is equivalent to finding the group of automorphisms of $C(x,z)$, the function field of X. We will freely work with both the function fields and the actual surfaces and use the same name for an automorphism of a surface and its induced automorphism on the function field of the surface. From equation (2) we see that X (and consequently $C(x,z)$) has the automorphism $A(x)=x$, $A(z)=\epsilon z$, where ϵ is a $2g+2$th root of unity. Note that the Riemann sphere, with the associated function field $C(x)$, is the orbit space X/K. Clearly the points $x=0, x=-1$, and $x=1$ of the orbit space are ramified in X with ramification indices $g+1, 2g+2$ and $2g+2$ respectively. From 3) these points correspond to P_0, P_{-1}, and P_1.

We now determine the automorphism B^2; this will allow us to determine the automorphism B. Recall that $< A, B^2 >= H$, and H/K induces automorphisms of X/K and of the function field $C(x)$. But from 2) above, we know in the cover of X/H by X/K, that P_{-1} and P_1 both lie over Q_1, and Q_0 is ramified. Thus the automorphism B^2K switches $x=1$ and $x=-1$ and fixes $x=0$. Recall that the automorphism group of $C(x)$ is the group of linear fractional transformations, thus $B^2(x)=-x$. The Riemann surface with function field $C(x^2)$ is the orbit space X/H. Define $y=x^2$. Then Q_0 and Q_1 are the points $y=0$, and $y=1$. Note from 2) that the point P_∞ is fixed by B^2. Thus Q_∞ corresponds to $y=\infty$.

Since $[G:H]=2$, the coset BH induces an automorphism of X/H and of the function field $C(y)$. From 1) we see that $y=0$ and $y=1$ are interchanged by BH and y_∞ is fixed by BH. Thus BH corresponds to the linear fractional transformation $y\mapsto 1-y$. Recalling that $y=x^2$ this yields that $BH(x^2)=1-x^2$. But examining (2) we see that if we define q and v by

$$q=\frac{z^2}{x(x+1)},\qquad v=q^{(g+1)/4}=\frac{z^{(g+1)/2}}{[x(x+1)]^{(g+1)/4}},\tag{3}$$

then from (2) we obtain

$$v^4=\Big(\frac{z^{g+1}}{[x(x+1)]^{(g+1)/2}}\Big)^2=\frac{x^2-1}{x^2}.\tag{4}$$

Thus $B(x) = \pm ixv^2$, where $i = \sqrt{-1}$. We choose $B(x) = -ixv^2$; we will see later that $B^3(x) = ixv^2$.

Note that the Galois group of $C(x,z)$ over $C(x)$ is the cyclic group generated by A. Thus for each divisor d of $2g+2$, $C(x, z^{(2g+2)/d})$ is the unique subfield of $C(x,z)$ which contains $C(x)$ as a subfield of index d. Note, in particular, that $C(x,v)$ is the unique field which contains $C(x)$ as a subfield of index 4.

It is left to determine $B(z)$. Note that

$$B\left(\frac{x^2-1}{x^2}\right) = \frac{x^2}{x^2-1}. \tag{5}$$

Applying B to (4) and using (5) easily yields that

$$(zB(z))^2 = \epsilon^r x(x+1)(-ixv^2)(-ixv^2+1), \tag{6}$$

for some integer r. Thus to determine $B(z)$ we must find a square root of $x(x+1)(-ixv^2)(-ixv^2+1)$. Let u denote such a square root. Then $[C(x,v^2,u) : C(x,v^2)] = 2$, and the previous paragraph yields that $C(x,v^2,u) = C(x,v)$. Thus $u \in C(x,v)$. Using this fact, a direct calculation yields that if

$$u = cx^2v^3 + icxv(x+1) \tag{7}$$

where $c = (i+1)/2$, then

$$u^2 = x(x+1)(-ixv^2)(-ixv^2+1). \tag{8}$$

We can now prove the following theorem.

Theorem. *Let (2) define the Kulkarni surface of genus g. Let the functions q, v and u be defined as in (3) and (7) and let ϵ be a $2g+2$th root of unity. Then the automorphism group of the Kulkarni surface is generated by A and B where*

$$A(x) = x, \quad A(z) = \epsilon z, \quad B(x) = -ixv^2, \quad B(z) = u/z.$$

In addition, these automorphisms satisfy the relations (1).

Proof. A is obviously an automorphism of (2). To prove B is an automorphism it is sufficient to show that

$$B\left(z^{2g+2} - (x-1)x^{g-1}(x+1)^{g+2}\right) = 0. \tag{9}$$

But the left hand side of (9) is

$$\frac{u^{2g+2}}{z^{2g+2}} - (-ixv^2 - 1)(-ixv^2)^{g-1}(-ixv^2+1)^{g+2}. \tag{10}$$

Using (2) and (8), (10) becomes

$$\frac{[x(x+1)(-ixv^2)(-ixv^2+1)]^{g+1}}{(x-1)x^{g-1}(x+1)^{g+2}} - (-ixv^2-1)(-ixv^2)^{g-1}(-ixv^2+1)^{g+2}$$

$$= (-ixv^2)^{g-1}(-ixv^2+1)^{g+1}\Big(\frac{x^2(-ixv^2)^2}{x^2-1} - (-ixv^2-1)(-ixv^2+1)\Big). \tag{11}$$

However, (4) yields that the third factor of line (11) equals 0. Thus B is an automorphism of (2).

Note that

$$B(q) = B\Big(\frac{z^2}{x(x+1)}\Big) = \frac{u^2}{z^2(-ixv^2)(-ixv^2+1)}. \tag{12}$$

Thus (8) implies

$$B(q) = \frac{x(x+1)}{z^2} = 1/q. \tag{13}$$

This immediately implies that $B(v) = 1/v$ and, since $B(x) = -ixv^2$, that $B^2(x) = -x$. Note also from (3) that $A(v) = iv$.

We now show that A and B satisfy the relations (1). Clearly $B^4(x) = x$, so we show that $B^4(z) = z$. But

$$B^4(z) = \frac{B^3(u)B(u)z}{B^2(u)u}. \tag{14}$$

But if we define $\alpha = cx^2v^3$ and $\beta = cxv$, then

$$u = \alpha - iB(\alpha) + i\beta. \tag{15}$$

Observe that $B^2(\alpha) = \alpha$ and $B(\beta) = -i\beta$ and, since $A(v) = iv$, that $A(\alpha) = -i\alpha$, $AB(\alpha) = iB(\alpha)$ and $A(\beta) = i\beta$. A short calculation yields

$$B^3(u)B(u) - uB^2(u) = -2\alpha^2 + 2B(\alpha)^2 - 2\beta^2 = 0.$$

The last equality occurs because of (4). This combined with (14) yields that $B^4(z) = z$.

We now show that BA has order 2. Clearly $BA(z) = \epsilon u/z$, and from (15) it is easy to deduce that $BA(u) = u$. This clearly yields $BABA(z) = z$. It is straightforward to see that $BABA(x) = x$, thus BA has order 2.

Finally we verify the last relation. Clearly $A(x) = x$ and $B^2(x) = -x$ so $B^2AB^2(x) = A^{g+2}(x) = x$. To complete the verification note that

$$B^2A(z) = \epsilon B(u)z/u, \quad A^{g+2}B^2(z) = A^{g+2}(B(u)z/u) \tag{17}$$

Since $g + 1 \cong 0 \bmod 4$, A^{g+1} fixes both x and v, thus A^{g+1} fixes both u and $B(u)$. Thus $A^{g+2}B^2(z) = -\epsilon zAB(u)/A(u)$. A short calculation shows

$$uAB(u) + A(u)B(u) = 2(-\alpha^2 + B(\alpha)^2 - \beta^2) = 0.$$

Thus the two quantities in (17) are equal and the last relation is verified. This proves the theorem.

In [2] it is shown that the Kulkarni surface admits three nonconjugate symmetries with fixed points. To each of these symmetries, a defining equation for the Kulkarni surface can be found for which the symmetry is given by conjugation. In addition, since each symmetry has fixed points, the defining equations admit real solutions. We now determine the symmetries and their associated defining equations.

For the remainder of the paper, we work exclusively with the function field $C(x, z)$ of the Kulkarni surface. Recall that a symmetry of a Riemann surface induces a symmetry of the function field of the surface, in other words, a field automorphism of order 2 which fixes the real, but not the complex, numbers. The symmetry given by conjugation of (2) determines the following automorphism of $C(x, z)$ over R :

$$x \mapsto x, \quad z \mapsto z, \quad i \mapsto -i. \tag{18}$$

We denote this automorphism of $C(x, z)$ by σ. Note that v and q are real rational functions of x and z, thus $\sigma(v) = v$ and $\sigma(q) = q$. It can easily be determined that $A\sigma$ and σB are also symmetries of $C(x, z)$. We show now that they each lie in distinct conjugacy classes.

(1) To show σ is not conjugate to σB; assume that $B^rA^s\sigma A^{-s}B^{-r} = \sigma B$. Since A and σ both fix x, we obtain $x = A^s\sigma A^{-s}(x) = B^{-r}\sigma B^{r+1}(x)$. By substituting $r = 0, 1, 2, 3$ we obtain a contradiction.

(2) To show σ and $A\sigma$ are not conjugate, assume that $B^rA^s\sigma A^{-s}B^{-r} = A\sigma$. Then $i^sB^r(v) = B^rA^s\sigma(v) = A\sigma B^rA^s(v) = i^sA\sigma B^r(v)$. Recall that $B^r(v) = v$ if r is even, and $B^r(v) = 1/v$ if r is odd. In either case we obtain a contradiction.

(3) To show that $A\sigma$ and σB are not conjugate, assume $B^rA^s\sigma^kA\sigma\sigma^kA^{-s}B^{-r} = \sigma B$. Then $A^s\sigma^kA\sigma\sigma^kA^{-s}(x) = B^{-r}\sigma B^{r+1}(x)$, and a contradiction is obtained as in (1) above.

We now determine a defining equation which yields $A\sigma$ as conjugation. Note that $A\sigma(x) = x$ and that $A\sigma((1+\epsilon)z) = (1+\epsilon)z$. Define $\kappa = (1+\epsilon)^{2g+2}$ and $\zeta = (1+\epsilon)z$. Multiplying (2) by the real, negative number κ yields

$$((1+\epsilon)z)^{2g+2} - (1+\epsilon)^{2g+2}(x-1)x^{g-1}(x+1)^{g+2} = 0. \tag{19}$$

Thus

$$\zeta^{2g+2} - \kappa(x-1)x^{g-1}(x+1)^{g+2} = 0. \tag{20}$$

Thus (20) is an equation for the Kulkarni surface with real coefficients, and for which complex conjugation is the map:

$$\zeta = (1+\epsilon)z \mapsto \zeta, \quad x \mapsto x \quad i \mapsto -i. \tag{21}$$

This is precisely the same automorphism as $A\sigma$. In addition, since κ is negative, whenever $0 < x < 1$, (20) admits a real solution for ζ. Thus (20) is a defining equation for a symmetry with fixed points.

We now determine an equation which exhibits the symmetry σB as conjugation. Recall that $g \cong 3 \bmod 4$. For this section, we assume that $g \geq 7$. Observe that A^{g+1} and B commute. This follows from (1), since

$$\begin{aligned} A^{g+1}BA^{g+1} &= A^{-1}A^{g+2}BA^{g+2}A^{-1} = A^{-1}(B^2AB^2)B(B^2AB^2)A^{-1} \\ &= A^{-1}B^2ABAB^2A^{-1} = A^{-1}BB^2A^{-1} = B. \end{aligned} \tag{22}$$

Thus A^{g+1} and B generate an abelian subgroup $M \cong Z_4 \times Z_2$ of order 8. Let E denote the subfield of $C(x,z)$ which is fixed by M. The only elements of order 2 in M are A^{g+1}, B^2, and B^2A^{g+1}. Thus there are exactly three maximal subfields of $C(x,z)$ which contain E.

We make the following definitions:

$$t = q + 1/q, \quad r = v + 1/v, \quad s = \frac{v}{q} + \frac{q}{v}. \tag{23}$$

Note that if $g = 3$, then $t = r$ and $s = 2$. For this reason, we assume that $g \geq 7$. Since v is a power of q, (13) yields that each of these elements is fixed by M. In addition, it is clear that each of r and s can be expressed as a polynomial in t. Note that $[C(q) : C(t)] = 2$, and from (4) and (3), we obtain that $[C(x, q) : C(q)] = 2$ and $[C(z, x) : C(x, q)] = 2$. Thus $[C(x, z) : C(t)] = 8$, and thus $C(t) = E$.

We note that $\{1, x, z, xz\}$ is a basis for $C(x, z)$ over $C(q)$. From $z^2 = qx(x+1)$, and $x^2 = 1/(1 - v^4)$ we deduce that

$$\frac{1}{x(x+1)} = \frac{(1 - v^4)(x(v^4 - 1) + 1)}{v^4},$$

$$1/z = \frac{z}{qx(x+1)} = \frac{(1 - v^4)(x(v^4 - 1) + 1)z}{qv^4},$$

and

$$B(z) = \frac{u}{z} = \frac{zx(v^4 - 1)(1 + i) + z(i(v^2 + 1) - v^2 + 1)}{2qv}.$$

We note that $C(x, q)$ consists of the elements of $C(x, z)$ which are fixed by A^{g+1}. If $h_0 \in C(x, z)$, then $B^4(h_0) = h_0$. This leads us to consider whether there exists $h_0 \in C(x, q)$ such that $B(h_0) = ih_0$. There are many such elements. Define

$$h = \frac{\sqrt{2}(i - 1)(v^4 - 1)x}{2v}. \tag{24}$$

We note that $B(h) = ih$ and $\sigma B(h) = h$. In addition, since h is fixed by A^{g+1}, but not by B^2, we obtain that $[C(h, t) : C(t)] = 4$ and

$$h^4 = \frac{-(v^4 - 1)^2}{v^4} = -r^2(r^2 - 4). \tag{25}$$

We now determine an element which is fixed by B but not by A^{g+1}. A natural candidate is

$$j_0 := z + B(z) + B^2(z) + B^3(z) = z + \frac{zx(v^4 - 1)(1 + i) + z(i(v^2 + 1) - v^2 + 1)}{2qv}$$

$$-\frac{z(x(v^4-1)+1)}{v^2}+\frac{z(x(v^4-1)-(v^2-1))-iz(x(v^4-1)+v^2+1)}{2qv}$$

$$=\frac{z(v^2-1)(v^2x+x-1)(v-q)}{qv^2}$$

However
$$(j_0)^2=\frac{2(v-q)^2}{q(v^2+1)}=\frac{2(s-2)}{r}. \tag{26}$$

In order to avoid fractions, we instead consider

$$j:=\frac{\sqrt{2}rj_0}{2}=\frac{\sqrt{2}z(v^2+1)(v-q)(x(v^4-1)-v^2+1)}{2qv^3}. \tag{27}$$

Thus $B(j)=j$, and $\sigma B(j)=j$. Since $A^{g+1}(z)=-z$, we have $A^{g+1}(j)=-j$. In addition,

$$j^2=\frac{(v^2+1)(q-v)^2}{qv^2}=r(s-2). \tag{28}$$

Note that $j\notin C(h,t)$ since j is not fixed by A^{g+1}. Thus $C(j,h,t)=C(x,z)$. A basis for $C(h,j,t)$ over $C(t)$ is given by

$$\{1,h,h^2,h^3,j,jh,jh^2,jh^3\}.$$

Define $\zeta=h+j$. Recall that the maximal subfields of $C(x,z)$ which contain $C(t)$ are fixed by an element of G of order 2. However $B^2(\zeta)=B^2(h+j)=-h+j$, $A^{g+1}(\zeta)=A^{g+1}(h+j)=h-j$ and $B^2A^{g+1}(\zeta)=-h-j$. Thus $C(\zeta,t)=C(z,x)$. The minimal polynomial of ζ over $C(t)$ will yield a defining equation for the Kulkarni surface. It is easily computed as follows. We define

$$F_2=(T-\zeta)(T-A^{g+1}(\zeta))$$

$$=T^2-2hT+h^2-j^2=T^2-2hT+h^2-r(s-2).$$

We define

$$F_4=(F_2)B^2(F_2)=(T^2-2hT+h^2-r(s-2))(T^2+2hT+h^2-r(s-2))$$

$$=T^4-2(h^2+r(s-2))T^2-2h^2r(s-2)-r^2(r^2-s^2+4(s-2)).$$

The minimum polynomial is $F = (F_4)B(F_4)$. Thus

$$F(\zeta) = \zeta^8 - 4r(s-2)\zeta^6 + 2r^2(r^2+3s^2-12s+8)\zeta^4 \tag{29}$$
$$+4r^3(s-2)(3r^2-s^2+4s-16)\zeta^2 + r^4(r^4+2r^2s(s-4)+s^4-8s^3+16s^2).$$

This polynomial can be written more compactly as:

$$((\zeta^2 - r(s-2))^2 + r^2(r^2-4))^2 + 16r^3(s-2)(r^2-4)\zeta^2 = 0, \tag{30}$$

or recalling that $j^2 = r(s-2) \in C(t)$ and $h^4 = -r^2(r^2-4) \in C(t)$, as

$$((\zeta^2 - j^2)^2 - h^4)^2 - 16h^4j^2\zeta^2 = 0. \tag{31}$$

It remains to express r and s in terms of t. Define the polynomials $f_n(t)$ by

$$f_n(t) = q^n + \frac{1}{q^n}.$$

Note that $f_n(t)$ satisfies the recurrence relation

$$f_{n+1}(t) = tf_n(t) - f_{n-1}(t), \qquad f_0 = 2, \quad f_1 = t. \tag{32}$$

Using the elementary theory of recurrence relations (32) has the closed form:

$$f_n(t) = \frac{\left(t+\sqrt{t^2-4}\right)^n + \left(t-\sqrt{t^2-4}\right)^n}{2^n}. \tag{33}$$

But (33) is easily seen to yield

$$f_n(t) = \frac{1}{2^{n-1}} \sum_{k=0}^{[n/2]} \binom{n}{2k} t^{n-2k}(t^2-4)^k \tag{34}$$

where $[n/2]$ is the greatest integer less than or equal to $n/2$. With this notation, $r = f_{(g+1)/4}$ and $s = f_{(g-3)/4}$, and thus (29) can be expressed entirely in terms of ζ and t.

Thus (29) is a defining equation for the Kulkarni surface. Note that complex conjugation is the map:

$$\zeta \mapsto \zeta, \quad q + 1/q = t \mapsto t, \quad i \mapsto -i.$$

This is precisely the automorphism σB. Thus (29) is an equation for the Kulkarni surface which exhibits σB as conjugation.

We verify that (30) admits real solutions, and thus that complex conjugation of (30) has fixed points. Let θ be any real number such that $\pi/2 < \theta < \pi$. In (23) define

$$q = e^{\frac{4\theta i}{g+1}}.$$

Thus

$$r = e^{\theta i} + e^{-\theta i}, \qquad s = e^{\frac{(g-3)\theta i}{g+1}} + e^{\frac{-(g-3)\theta i}{g+1}}.$$

Thus $-2 < r < 0$ and, since $g \geq 7$, $-2 < s < \sqrt{2}$. In (30), let r and s be defined as above, and let $\zeta^2 = r(s-2)$. Since $r(s-2)$ is positive, ζ is real. With these substitutions, the left hand side of (30) equals

$$r^4(r^2-4)(r^2-4+16(s-2)^2) < 0.$$

With the same definitions for r and s, the left hand side of (30) is positive for large, real values of ζ. Thus (30) admits real solutions.

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