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# Local boundedness for solutions of doubly nonlinear parabolic equations.

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**Abstract**

It is well known that the local and global boundedness require different conditions for solutions of singular parabolic equations. This phenomenon appeares for doubly nonlinear parabolic equation (1). In this paper local boundedness of solutions of (1) is proved by adding a suitable integrability condition on $u$.

## 1 Introduction

Let $G$ be a bounded domain in the $n(n \geq 2)$ dimensional Euclidean space $E^n, T > 0$ a real number. On $Q = G \times (0, T)$ consider the following doubly nonlinear parabolic equation:

$$\left(| u |^{\lambda-2}\right)_t - \operatorname{div} A(x,t,u,\nabla u) + B(x,t,u,\nabla u) = 0 \tag{1}$$

where $\lambda > 2$ is a constant. Suppose that $A(x,t,u,\xi)$ and $B(x,t,u,\xi)$ are defined on $Q \times E^1 \times E^n$, measurable in $(x,t)$, continuous in $u$ and $\xi$ satisfying the following conditions:

$$\xi \cdot A(x,t,u,\xi) \geq | \xi |^p - K | u |^l - f_0(x,t) \tag{2}$$
$$| A(x,t,u,\xi) | \leq K_1 | \xi |^{p-1} + K | u |^{l(l-1/p)} + f_1(x,t) \tag{3}$$
$$| B(x,t,u,\xi) | \leq b(x,t) | \xi |^{\gamma} + K | u |^{l-1} + f_2(x,t) \tag{4}$$

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where $p > 1, k_1 \geq 1, k \geq 0, l = p(1+\lambda/n)$ and

$$\gamma_0 = p - \frac{n+p}{n+\lambda} \leq \lambda < p$$

are all constants,

$$b(x,t) \in L^r(Q) \tag{5}$$

$$\begin{cases} \frac{1}{r} = 1 - \frac{\gamma}{p} - \frac{1}{l} & \text{as} \quad \gamma_0 \leq \gamma < \gamma_1 = p - \frac{n}{n+\lambda} \\ r = \infty & \text{as} \quad \gamma = \gamma_1 \\ \frac{1}{r} < 1 - \frac{\gamma}{p} - \frac{p-\gamma}{l}\frac{n+\lambda}{n+p} = \left(1 - \frac{\gamma}{p}\right)\frac{p}{n+p} & \text{as} \quad \gamma_1 < \gamma < p, \end{cases} \tag{6}$$

$$f_i(x,t) \in L^{s_i}(Q), \ s_0, s_2 > \frac{n+p}{p}, \ s_1 > \frac{n+p}{p-1}. \tag{7}$$

Denote

$$\begin{cases} t^* = l & \text{as} \quad \gamma_0 \leq \gamma \leq \gamma_1 \\ \frac{1}{t^*}\left[1 - (p-\gamma)\frac{n+\lambda}{n+p}\right] + \frac{p-\gamma}{l}\frac{n+\lambda}{n+p} + \frac{\gamma}{p} + \frac{1}{r} = 1 & \text{as} \quad \gamma_1 < \gamma < p. \end{cases} \tag{8}$$

Let

$$u \in C(0,T; L^\lambda(G)) \cap L^p(0,T; W^{1,p}(G)) \cap L^{t^*}(Q) \tag{9}$$

be a generalized solution of (1), this means that

$$\int_0^t \int_G \left[-v_\tau \mid u \mid^{\lambda-2} u + \nabla v \cdot A(x,\tau,u,\nabla u) + vB(x,\tau,u,\nabla u)\right] dx d\tau \tag{1'}$$

$$+ \int_G v(x,\tau) \mid u(x,\tau) \mid^{\lambda-2} u(x,\tau) \mid_{\tau=0}^{\tau=t} dx = 0$$

$$\forall t \in (0,T), v \in W^{1,\lambda}(0,T; L^\lambda(G)) \cap L^p(0,T; W_0^{1,p}(G)) \cap L^\infty(Q).$$

Under the restriction $2 < \lambda < \frac{np}{n-p}$ (as $1 < p < n$), i.e. $p > \frac{n\lambda}{n+\lambda}$, X. T. Liang & X. X. Liang in (1) proved that if there exists a constant $M$ such that

$$(u-M)^+ = \max(u-M, 0) \in L^p(0,T; W_0^{1,p}(G)) \text{ and } (u-M)^+ \mid_{t=0} = 0,$$

then $u$ globally bounded on $Q$. Without any boundedness restriction of $u$ on the parabolic boundary of $Q$ the local boundedness of $u$ in $Q$ is also proved in [1]. The reason of restricting $\lambda < \frac{np}{n-p}$ in [1] is to guarantee the following imbedding theorem holds:

$$\left(\int\int_Q \mid u \mid^l dxdt\right)^{q/l} \leq C\left(\operatorname*{Vrai\,max}_{t \in (0,T)} \int_G \mid u \mid^\lambda dx + \int\int_Q \mid \nabla u \mid^p dxdt\right) \tag{10}$$

$$\forall u \in L^\infty(0,T;L^\lambda(G)) \cap L^p(0,T;W_0^{1,p}(G))$$

where the constant $C > 0$ is independent of $u$, $G$ and $T$ and

$$q = p\frac{n+\lambda}{n+p} \tag{11}$$

However, in the case of $\lambda \geq \frac{np}{n-p}$ (that is $p \leq \frac{n\lambda}{n+\lambda}$) (10) is also true. This is an immediate consequence of Theorem 2.2 in Chapter II of Ladyzenskaja, O. A. Solonnikov, V. A. & Ural'ceva, N. N. [2]. thus the global boundedness of solution of (1) proving in [1] holds for $\lambda > 2, 1 < p \leq \frac{n\lambda}{n+\lambda}$ too. Because of $l = p\frac{n+\lambda}{n} \leq \lambda$ in the case $p \leq \frac{n\lambda}{n+\lambda}$, the proof of the local boundedness gives in [1] is no longer applicable. So, the local boundedness is unsolved for $\lambda > 2$ and $1 < p \leq \frac{n\lambda}{n+\lambda}$. Now we want filling the gap. For this we need the additional assumption:

$$u \in L^{\tilde{l}}_{loc}(Q)$$

$$\begin{cases} \tilde{l} > \lambda & \text{as} \quad l = p\frac{n-\lambda}{n} = \lambda \\ \tilde{l} > \frac{(\lambda-p)n}{p} & \text{as} \quad l = p\frac{n+\lambda}{n} \leq \lambda. \end{cases} \tag{12}$$

Our result is the following

**Theorem 1.** *Let $\lambda > 2, 1 < p \leq \frac{n\lambda}{n+\lambda}$. Let the conditions (2)-(7) be fulfilled. Let $u$ satisfying (9), (12) be a generalized solution of (1). Then $u$ is locally bounded in $Q$.*

In the case $\lambda = 2$ and $1 < p < 2$, E. Di Benedetto and M. A. Herrero [3], H. J. Choe [4,5] and M. M. Porzio [6] have discussed the local boundedness and the condition (12) is also needed (in the case $\lambda = 2$ and $1 < p < 2$, (12) coincides with that in [3]-[6] and is sharp in such a case as proved in [3], III.7). Our argument is different from that given in [3]-[6] and is an improvement of [1]. The result of boundedness is needed in regarding regularity of solutions.

By substituting $w = | u |^{\lambda-2} u$ the equation (1) changes into a class of doubly nonlinear parabolic equations. The prototype of this class of equations is

$$w_t - \operatorname{div}(| w |^m | \nabla w |^{p-2} \nabla w) = 0.$$

V. Vespri [7] and M. M. Porzio and V. Vespri [8] consider the Holder continuity of bounded solutions and some existence for the last class of equations.

**Remark 1.** In the definition of the generalized solution, the requirement of $u \in L^{t^*}(Q)$ in (9) is not a supplemental retriction on $u$ when $\gamma_0 \leq \gamma \leq \gamma_1$ (because every function in $C(0,T;L^\lambda(G)) \cap L^p(0,T;W^{1,p}(G))$ belongs to $L^l(Q)$).

**Remark 2.** When $\gamma_0 \leq \gamma \leq \gamma_1$ the conditions (6) and (8) is necessary even as in the time-independent case, see M. Giaquinta [9] pp. 148-149 and X. T. Liang [10].

## 2 Proof of Theorem 1

Under the assumptions of Theorem 1 the requirement on the test function, $v \in W^{1,\lambda}(0,T;L^\lambda(G)) \cap L^p(0,T;W_0^{1,p}(G)) \cap L^\infty(Q))$, can be replaced by $v \in W^{1,\lambda}(0,T;L^\lambda(G)) \cap L^p(0,T;W_0^{1,p}(G)) \cap L^{\tilde{l}}(G))$ where $\tilde{l}$ is as in (12). Using (10) and repeating the deducation of [1] formula (18) we arrive at that

$$\left(\int\int_{A(k,\tau_1,\rho_1)} (u-k)^l dxdt\right)^{q/l}$$
$$\leq C\left\{\int\int_{A(h,\tau_0,\rho_0)} \left[\frac{(u-k)^\lambda + k^\lambda}{\tau_1 - \tau_0} + \left|\frac{u-k}{\rho_0-\rho_1}\right|^p\right] dxdt\right.$$
$$+k^l \mid A(k,\tau_0,\rho_0) \mid + \mid A(k,\tau_0,\rho_0) \mid^{1-1/s_0}$$
$$+ \mid A(k,\tau_0,\rho_0) \mid^{\frac{q}{q-1}(1-1/s_2-1/l)}$$
$$\left. + (\rho_0-\rho_1)^{-\frac{q}{q-1}} \mid A(k,\tau_0,\rho_0) \mid^{\frac{q}{q-1}(1-1/s_1-1/l)}\right\} \tag{13}$$

$$\forall k \geq k_0,\ \frac{\rho}{2} \leq \rho_1 < \rho_0 \leq \rho,\ t_0 - \rho^p \leq \tau_0 < \tau_1 \leq t_0 - \left(\frac{\rho}{2}\right)^p$$

where $\mid A \mid$ denotes the $(n+1)-$dimensional Lebesque measure of $A$,

$$A(k,\tau_0,\rho_0) = \{B(x_0,\rho_0) \times (\tau_0,t_0)\} \cap \{u > k\}$$
$$B(x_0,\rho) = \{x \in E^n, \mid x - x_0 \mid < \rho\}$$
$$B(x_0,\rho) \times (t_0 - \rho^p, t_0) \subset Q, \quad 0 < \rho < 1,$$

$q$ is an in (11), $k_0$ is a constant large enough and constant $C > 0$ is independent of $k, \rho_0, \rho_1, \tau_0, \tau_1$ (for simplicity the norms of $f_i$ in $L^{s_i}(Q)$

are absorbed into the constant $C$). Because (10) holds for $1 < p \leq \frac{n\lambda}{n+\lambda}$, so does (13) for the solution $u$ appearing in Theorem 1.

We first consider the case $l < \lambda$. In this case we have

$$\begin{cases} \frac{1}{\lambda} = \frac{\alpha}{l} + \frac{1-\alpha}{\tilde{l}}, \\ (0,1) \ni \alpha = \left(\frac{1}{\lambda} - \frac{1}{\tilde{l}}\right) / \left(\frac{1}{l} - \frac{1}{\tilde{l}}\right) = \frac{l}{\lambda}\left(1 - \frac{\lambda - l}{\tilde{l} - l}\right) > \frac{q}{\lambda}, \end{cases} \tag{14}$$

$$\left(\int\int_{A(k,\tau_0,\rho_0)} (u-k)^\lambda dxdt\right)^{1/\lambda}$$
$$\leq \left(\int\int_{A(k,\tau_0,\rho_0)} (u-k)^l dxdt\right)^{\alpha/l} \left(\int\int_{A(k,\tau_0,\rho_0)} (u-k)^{\tilde{l}} dxdt\right)^{(1-\alpha)/\tilde{l}} \tag{15}$$

For any $k > h$, there holds

$$| A(k,\tau_0,\rho_0) | \leq \int\int_{A(h,\tau_0,\rho_0)} \left|\frac{u-h}{k-h}\right|^\lambda dxdt. \tag{16}$$

From (15), (16) it follows that

$$k^\lambda \mid A(k,\tau_0,\rho_0) \mid \leq \left|\frac{k}{k-h}\right|^\lambda \left(\int\int_{A(h,\tau_0,\rho_0)} (u-h)^l dxdt\right)^{\lambda\alpha/l}$$
$$\left(\int\int_{A(k,\tau_0,\rho_0)} | u |^{\tilde{l}} dxdt\right)^{\lambda(1-\alpha)/\tilde{l}} \tag{17}$$

Combining (15)-(17) with (13), we arrive at

$$\left(\int\int_{A(k,\tau_1,\rho_1)} (u-k)^l dxdt\right)^{q/l}$$
$$\leq C\Bigg\{(\tau_1-\tau_0)^{-1} \left(\int\int_{A(h,\tau_0,\rho_0)} (u-h)^l dxdt\right)^{\lambda\alpha/l}$$
$$\left(\int\int_{A(h,\tau_0,\rho_0)} | u |^{\tilde{l}} dxdt\right)^{\lambda(1-\alpha)/\tilde{l}}$$
$$+ (\tau_1-\tau_0)^{-1} \left(\frac{k}{k-h}\right)^\lambda \left(\int\int_{A(h,\tau_0,\rho_0)} (u-h)^l dxdt\right)^{\lambda\alpha/l}$$

$$
\begin{aligned}
& \left(\int\int_{A(h,\tau_0,\rho_0)} |u|^{\tilde{l}}dxdt\right)^{\lambda(1-\alpha)/\tilde{l}} \\
+ \quad & (\rho_0-\rho_1)^{-p}(k-h)^{p-l}\int\int_{A(h,\tau_0,\rho_0)}(u-h)^l dxdt \\
& + \left(\frac{k}{k-h}\right)^l \int\int_{A(h,\tau_0,\rho_0)}(u-h)^l dxdt \\
+ \quad & \left((k-h)^{-l}\int\int_{A(h,\tau_0,\rho_0)}(u-h)^l dxdt\right)^{1-1/s_0} \\
+ \quad & (\rho_0-\rho_1)^{-q/(q-1)} \\
& \left((k-h)^{-l}\int\int_{A(h,\tau_0,\rho_0)}(u-h)^l dxdt\right)^{(1-1/s_1-1/l)q/(q-1)} \qquad (18)
\end{aligned}
$$

$$
\forall k>h\geq k_0,\ \frac{\rho}{2}\leq\rho_1\leq\rho_0\leq\rho,\ t_0-\rho^p\leq\tau_0\leq\tau_1\leq t_0-\left(\frac{\rho}{2}\right)^p.
$$

Let $\epsilon>0$, a constant that will be determined later. From (16) and the absolute continuity of a Lebesque integre, we take $H>1$ so large that

$$
\int\int_{A(H,t_0-\rho^p,\rho)} |u|^{\tilde{l}}dxdt\leq\epsilon\rho^{n+p}. \qquad (19)
$$

For $m=0,1,2,\cdots$, take

$$
k_m=2H-\frac{H}{2^m},\ \rho_m=\frac{1}{2}\left(1+\frac{1}{2^m}\right)\rho,
$$

$$
\tau_m=t_0-\left(\frac{\rho}{2}\right)^p+\left(\left(\frac{\rho}{2}\right)^p-\rho^p\right)\Big/2^m,
$$

$$
J_m=\int\int_{A(k_m,\tau_m,\rho_m)}(u-k)^l dxdt.
$$

Because the constant $C$ appearing in (18) is independent of $k,h,\tau_0,\tau_1$ and $\rho_0,\rho_1$, replacing $k,h$, by $k_{m+1},k_m,\tau_0,\tau_1$ by $\tau_m,\tau_{m+1}$ and $\rho_0,\rho_1$ by $\rho_m,\rho_{m+1}$ respectively, we get from (18) that

$$
J_{m+1}^{q/l} \leq C\left\{\frac{2^{m+1}}{\left(1-\frac{1}{2^p}\right)\rho^p}J_m^{\lambda\alpha/l}\left(\epsilon\rho^{n+p}\right)^{\lambda(1-\alpha)/\tilde{l}}\right.
$$

$$
\begin{aligned}
&+ \frac{2^{m+1}}{\left(1-\frac{1}{2^p}\right)\rho^p}\left[\frac{2H}{\frac{H}{2^{m+1}}}\right]^{\lambda} J_m^{\lambda\alpha/l}\left(\epsilon\rho^{n+p}\right)^{\lambda(1-\alpha)/\tilde{l}} \\
&+ \left(\frac{2^{m+2}}{\rho^p}\right)^p\left(\frac{2^{m+1}}{H}\right)^{l-p} J_m + \left[\frac{2H}{\frac{H}{2^{m+1}}}\right]^{l} J_m \\
&+ \left[\left(\frac{2^{m+1}}{H}\right)^l J_m\right]^{1-1/s_0} + \left[\left(\frac{2^{m+1}}{H}\right)^l J_m\right]^{(1-1/s_2-1/l)q/(q-1)} \\
&+ \left(\frac{2^{m+1}}{\rho}\right)^{q/(q-1)}\left[\left(\frac{2^{m+1}}{H}\right)^l J_m\right]^{(1-1/s_1-1/l)q/(q-1)} \qquad (20)
\end{aligned}
$$

$$\forall m = 0,1,2,\cdots.$$

As $H > 1$ and

$$
\begin{aligned}
1-\frac{1}{s_0} = \frac{q}{l}+\frac{p}{n+p} &- \frac{1}{s_0}, \\
\frac{q}{q-1}\left(1-\frac{1}{s_2}-\frac{1}{l}\right) &= \frac{q}{q-1}\left(\frac{q}{l}+\frac{p}{n+p}-\frac{1}{s_2}-\frac{1}{l}\right) \\
&= \frac{q}{l}+\frac{q}{q-1}\left(\frac{p}{n+p}-\frac{1}{s_2}\right), \\
\frac{q}{q-1}\left(1-\frac{1}{s_1}-\frac{1}{l}\right) &= \frac{q}{q-1}\left(\frac{q}{l}+\frac{p}{n+p}-\frac{1}{s_1}-\frac{1}{l}\right) \\
&= \frac{q}{l}+\frac{q}{q-1}\frac{1}{n+p}+\frac{q}{q-1}\left(\frac{p-1}{n+p}-\frac{1}{s_1}\right),
\end{aligned}
$$

(20) can be wtitten as

$$
\begin{aligned}
J_{m+1}^{q/l} \leq\ & J_m^{q/l} C\left\{\frac{2^{m+1}}{\left(1-\frac{1}{2^p}\right)\rho^p} J_m^{(\lambda\alpha-q)/l}\left(\epsilon\rho^{n+p}\right)^{\lambda(1-\alpha)/\tilde{l}}\right. \\
&+ \frac{2^{m+1}}{\left(1-\frac{1}{2^p}\right)\rho^p} 2^{(m+2)\lambda} J_m^{(\lambda\alpha-q)/l} \\
&\left(\epsilon\rho^{n+p}\right)^{\lambda(1-\alpha)/\tilde{l}} + \frac{2^{(m+2)l}}{\rho^p} J_m^{p/(n+p)}
\end{aligned}
$$

$$
\begin{aligned}
&+\ 2^{(m+2)l} J_m^{p/(n+p)} + 2^{(m+1)l} J_m^{p/(n+p)-1/s_0} \\
&+\ 2^{(m+1)lq/(q-1)} J_m^{(p/(n+p)-1/s_2)q/(q-1)} \\
&+\ \frac{2^{(m+1)q/(q-1)}}{\rho^{q/(q-1)}} 2^{(m+1)lq/(q-1)} \\
&\quad \cdot J_m^{pq/[(q-1)(n+p)]+((p-1)/(n+p)-1/s_1)q/(q-1)} \Big\} \qquad (21)
\end{aligned}
$$

$$\forall m = 0, 1, 2, \cdots.$$

From (19) it follows that

$$
\begin{aligned}
J_0 &= \int\int_{A(H,t_0-\rho^p,\rho^p)} (u-H)^l dxdt \\
&\le \left(\int\int_{A(H,t_0-\rho^p,\rho^p)} (u-H)^{\widetilde{l}} dxdt\right)^{l/\widetilde{l}} \\
&\quad \left(\int\int_{B(x_0,\rho)\times(t_0-\rho^p,t_0)} dxdt\right)^{1-l/\widetilde{l}} \\
&\le \left(\epsilon\rho^{n+p}\right)^{l/\widetilde{l}} \left(\omega\rho^{n+p}\right)^{1-l/\widetilde{l}} = \epsilon^{l/\widetilde{l}} \omega^{1-l/\widetilde{l}} \rho^{n+p}
\end{aligned}
$$

where $\omega$ is the volume of the unit ball in $E^n$. Assume we have proved

$$J_m \le \delta^m \epsilon^{l/\widetilde{l}} \omega^{1-l/\widetilde{l}} \rho^{n+p} \qquad (22)$$

where $\delta$ is a small positive constant that will be especified, then combining (21) with (22) and using (14) we have

$$
\begin{aligned}
J_{m+1}^{q/l} &\le J_m^{q/l} C \left\{ \frac{2}{\left(1-\frac{1}{2^p}\right)\rho^p} \left(2\delta^{(\lambda\alpha-q)/l}\right)^m \right. \\
&\quad \epsilon^{[\lambda\alpha-q+\lambda(1-\alpha)]/\widetilde{l}} \widetilde{\omega}^{(1-l/\widetilde{l})(\lambda\alpha-q)/l} \left(\rho^{n+p}\right)^{1-q/l} \\
&+\ \frac{2^{1+2\lambda}}{\left(1-\frac{1}{2^p}\right)\rho^p} \left(2^{1+\lambda}\delta^{(\lambda\alpha-q)/l}\right)^m \\
&\quad \epsilon^{[\lambda\alpha-q+\lambda(1-\alpha)]/\widetilde{l}} \widetilde{\omega}^{(1-l/\widetilde{l})(\lambda\alpha-q)/l} \left(\rho^{n+p}\right)^{1-q/l} \\
&+\ \frac{2^{2l}}{\rho^p} \left(2^l \delta^{p/(n+p)}\right)^m \left(\epsilon^{l/\widetilde{l}} \omega^{1-l/\widetilde{l}} \rho^{n+p}\right)^{p/(n+p)} \\
&+\ 2^{2l} \left(2^l \delta^{p/(n+p)}\right)^m \left(\epsilon^{l/\widetilde{l}} \omega^{1-l/\widetilde{l}} \rho^{n+p}\right)^{p/(n+p)}
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad 2^l \left(2^l \delta^{p/(n+p)-1/s_0}\right)^m \left(\epsilon^{l/\widetilde{l}}\omega^{1-l/\widetilde{l}}\rho^{n+p}\right)^{p/(n+p)-1/s_0} \\
&+ \quad 2^{lq/(q-1)} \left(2^l \delta^{p/(n+p)-1/s_2}\right)^{mq/(q-1)} \\
&\qquad \left(\epsilon^{l/\widetilde{l}}\omega^{1-l/\widetilde{l}}\rho^{n+p}\right)^{(p/(n+p)-1/s_2)q/(q-1)} \\
&+ \quad \frac{2^{q(l+1)/(q-1)}}{\rho^{q/(q-1)}} \left[2^{l+1}\delta^{p/(n+p)-1/s_1}\right]^{mq/(q-1)} \\
&\bullet \left(\epsilon^{l/\widetilde{l}}\omega^{1-l/\widetilde{l}}\rho^{n+p}\right)^{q/[(q-1)(n+p)]+((p-1)/(n+p)-1/s_1)q/(q-1)} \Bigg\}. \qquad (23)
\end{aligned}
$$

In virtue of $\rho < 1$ and $1 - \frac{q}{l} = \frac{p}{n+p}$, if we take $\epsilon, \delta$ such that

$$
\begin{aligned}
&2^{1+\lambda}\delta^{(\lambda\alpha-q)/l} \le 1, \qquad 2^l \delta^{p/(n+p)-1/s_0} \le 1, \\
&2^l \delta^{p/(n+p)-1/s_2} \le 1, \qquad 2^{l+1}\delta^{p/(n+p)-1/s_1} \le 1, \\
&C \left\{ \frac{2^{2+2\lambda}}{\left(1-\frac{1}{2^p}\right)} \epsilon^{[\lambda\alpha-q+\lambda(1-\alpha)]/\widetilde{l}}\omega^{(1-l/\widetilde{l})(\lambda\alpha-q)/l} \right. \\
&+2^{1+2l}\left(\epsilon^{l/\widetilde{l}}\omega^{1-l/\widetilde{l}}\right)^{p/(n+p)} \\
&+2^l\left(\epsilon^{l/\widetilde{l}}\omega^{1-l/\widetilde{l}}\right)^{p/(n+p)-1/s_0} + 2^{lq/(q-1)}\left(\epsilon^{l/\widetilde{l}}\omega^{1-l/\widetilde{l}}\right)^{(p/(n+p)-1/s_2)q/(q-1)} \\
&\left. +2^{(l+1)q/(q-1)} \left(\epsilon^{l/\widetilde{l}}\omega^{1-l/\widetilde{l}}\right)^{q/[(q-1)(n+p)]+((p-1)/(n+p)-1/s_1)q/(q-1)} \right\} \\
&\le \delta^{q/l}.
\end{aligned}
$$

we deduce from (22), (23) that

$$
J_{m+1}^{q/l} \le J_m^{q/l}\delta^{q/l} = \left(\delta^{m+1}\epsilon^{l/\widetilde{l}}\omega^{1-l/\widetilde{l}}\rho^{m+p}\right)^{q/l}
$$

This means (22) holds for $m+1$. By induction it holds for any positive integer $m$. Then, we have

$$
0 = \lim_{m\to\infty} J_m = \int\int_{A(2H, t_0-(\frac{\rho}{2})^p, \frac{\rho}{2})} (u-2H)^l dx dt
$$

i.e.

$$
\operatorname*{ess\,sup}_{B(x_0,\frac{\rho}{2})\times(t_0-(\frac{\rho}{2})^p, t_0)} u < 2H < +\infty.
$$

The local boundedness of $u$ from above is thus proved for the case $\lambda > l$. In this case replacing $u$ by $(-u)$ we can prove the local boundedness of $u$ from below by same argument.

In this case $\lambda = l$ we have $\widetilde{l} > l$

$$\int\int_{A(k,\tau_0,\rho_0)} (u-k)^l dxdt \leq \left(\int\int_{A(k,\tau_0,\rho_0)} (u-k)^l dxdt\right)^{q/l}$$

$$\left(\int\int_{A(k,\tau_0,\rho_0)} | u |^l dxdt\right)^{p/(n+p)}$$

$$\leq \left(\int\int_{A(k,\tau_0,\rho_0)} (u-k)^l dxdt\right)^{q/l} \left(\int\int_{A(k,\tau_0,\rho_0)} | u |^{\widetilde{l}} dxdt\right)^{lp/[\widetilde{l}(n+p)]}$$

$$| A(k,\tau_0,\rho_0) |^{(1-l/\widetilde{l})p/(n+p)}$$

$$\leq \left(\int\int_{A(h,\tau_0,\rho_0)} (u-k)^l dxdt\right)^{q/l} \left(\int\int_{A(h,\tau_0,\rho_0)} | u |^{\widetilde{l}} dxdt\right)^{lp/[\widetilde{l}(n+p)]}$$

$$\bullet \left[\int\int_{A(h,\tau_0,\rho_0)} \left|\frac{u-h}{k-h}\right|^l dxdt\right]^{(1-l/\widetilde{l})p/(n+p)} \qquad \forall k > h \geq k_0 \qquad (15')$$

and

$$k^\lambda | A(k,\tau_0,\rho_0) | \leq (k^l | A(k,\tau_0,\rho_0) |)^{q/l} \left(\int\int_{A(k,\tau_0,\rho_0)} | u |^l dxdt\right)^{p/(n+p)}$$

$$\leq \left[\left(\frac{k}{k-h}\right)^l \int\int_{A(h,\tau_0,\rho_0)} (u-h)^l dxdt\right]^{q/l}$$

$$\left[\int\int_{A(h,\tau_0,\rho_0)} \left|\frac{u-h}{k-h}\right|^{\widetilde{l}} dxdt\right]^{(1-l/\widetilde{l})p/(n+p)}$$

$$\bullet \left[\int\int_{A(h,\tau_0,\rho_0)} | u |^{\widetilde{l}} dxdt\right]^{lp/[\widetilde{l}(n+p)]} \qquad \forall k > h \geq k_0. \qquad (17')$$

Replacing (15), (17) by (15'), (17'), respectively, we can repeat the demonstration above to get the local boundedness of $u$.

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