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Automorphisms of generic cyclic covers.

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Abstract

We generalize an argument of [Ma2] for proving a result about automorphisms of generic simple cyclic covers of smooth algebraic varieties. A finite map $\pi: S \to X$ is called a simple cyclic cover if there exists an invertible sheaf $\mathcal{L}$ on X such that $\pi_* \mathcal{O}_S = \oplus_{i=0}^{n-1} \mathcal{L}^{-i}$ (cf. [B-P-V] I.17). Here we prove under some "mild" assumptions on the triple $X, \mathcal{L}, n$ that for the generic cyclic cover S the group $Aut(S)$ of biregular automorphisms equals the group μ_n of automorphisms of the branched cover π.

0 Introduction

Recent examples ([F-P], [Ma2], [Ma3]) have shown that one of the simplest way to distinguish irreducible subvarieties $\{M_i\}$ of the moduli space of varieties with ample canonical bundle and fixed topological invariants is to look at the automorphism groups $\{G_i\}$ of their generic members. It is clear that this approach works only if at least one of the group G_i is different from 0, the typical situation where this condition is satisfied is the case of Galois covers of varieties. In this paper we are interested to the case of simple cyclic, or n-root, covers. More precisely we consider the following set-up:

Set-Up: Let X be a smooth connected compact complex variety of dimension $N \geq 2$, $L \xrightarrow{\pi} X$ a holomorphic line bundle on X, $n \geq 2$ an integer and $V \subset \mathbb{P}(H^0(X, L^{\otimes n}))$ a nonempty linear system.

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For every $f \in H^0(X, L^{\otimes n})$ we define the n-root covering of f as the branched covering $X(\sqrt[n]{f}, L) \longrightarrow X$ defined by the hypersurface in L of equation $z^n = f$, where $z \in H^0(L, \pi^* L)$ is the tautological section.

It is immediate to observe that $X(\sqrt[n]{f}, L)$ is smooth if and only if $D = div(f)$ is smooth, if n and L are fixed then the isomorphism class of the cover depends only on the divisor D, for notational convenience we shall denote from now on $X_D = X(\sqrt[n]{f}, L)$.

It is well known that if D is sufficiently ample and generic then $Aut(X_D) = \mu_n = \{\xi \in \mathbb{C}^* | \xi^n = 1\}$, i.e. every holomorphic automorphism of X_D is an automorphism of the cover.

In [F-P] Fantechi and Pardini give, in the more general situation of abelian covers, an effective condition on the ampleness of L in such a way the above fact is true. However their condition is quite restrictive and their results don't find application in many interesting cases as the ones, for example, described in [Cat], [Ma1], [Ma2], [Ma3].

We shall say that a linear system V on X is big and base point free if it is base point free and $D^N > 0$ for $D \in V$; this second condition is equivalent to requiring that $\dim \phi_V(X) = \dim X$, where $\phi_V : X \to V^\vee$ is the associated projective morphism.

We denote by $Aut(X, V)$ the subgroup of automorphisms g of X such that $gD = D$ for every $D \in V$, note that if V is big and base point free then the generically finite morphism ϕ_V is $Aut(X, V)$-invariant and then $Aut(X, V)$ is a finite group.

A linear system V is called *automorphisms generic* if there exists a nonempty Zariski open subset $A \subset V$ such that $Aut(X, D) = Aut(X, V)$ for every $D \in A$. Note that this is a significant condition only for "special polarized varieties X, nL", in fact if a polarized variety (X, H), $H \in \mathrm{Pic}(X)$, has finitely many automorphisms (according to [Mat] Corollary 1, this is true if H is ample and X is not a ruled variety) then every nonempty linear system $V \subset \mathbb{P}(H^0(X, H))$ is automorphisms generic. The cubics of $\mathbb{P}^2$ give an example of linear system which is not automorphisms generic, although for every smooth cubic $D \subset \mathbb{P}^2$, $Aut(\mathbb{P}^2, D)$ is a finite group.

If V_1, V_2, V are linear systems on X we shall write $V_1 + V_2 \subset V$ if $D_1 + D_2 \in V$ for every $D_1 \in V_1, D_2 \in V_2$.

This note is completely devoted to prove the following

Theorem A. *Let X, n, L, V be as in the Set-up and assume:*

1) $K_X \otimes L^{\otimes(n-1)}$ *is an ample line bundle.*

2) V *is automorphisms generic.*

3) Either

i) $n = 2$ *and there exists big base point free linear systems* V_1, V_2, V_3 *such that* $V_1 + V_2 + V_3 \subset V$.

ii) $n > 2$ *and there exists big base point free linear systems* V_1, V_2 *such that* $V_1 + V_2 \subset V$.

Then for the generic $D \in V$ *there exists an exact sequence of groups*

$$0 \longrightarrow \mu_n \longrightarrow Aut(X_D) \xrightarrow{q} Aut(X, D) = Aut(X, V).$$

Note that by A.3 V is also big and base point free.

Corollary B. *Assume the condition A.1, A.3 are satisfied and* $Aut(X, D) = 1$ *for generic* $D \in V$, *then* $Aut(X_D) = \mu_n$.

The main tools of the proof are the semicontinuity theorem of [F-P] which is used for "putting" some special singularities in X_D and the Cartan's lemma ([Car] pag. 97) which gives a very short and easy proof of the commutativity of certain automorphisms.

The statement of theorem A is particularly useful in the study of simple iterated cyclic covers (cf. [Ma2]), in fact if $S \xrightarrow{\pi} X$ is a smooth Galois cover with group G such that there exists an exact sequence

$$0 \longrightarrow G \longrightarrow Aut(S) \longrightarrow Aut(X)$$

and V is a linear system on X which satisfies conditions A.2 and A.3 then also $\pi^* V$ satisfies A.2 and A.3.

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1. The proof.

Let $f: (\mathbb{C}^{N+1}, 0) \to (\mathbb{C}, 0)$ be a germ of holomorphic function of multiplicity $m \geq 2$ and let $(X, 0) = f^{-1}(0)$. If f_m is the homogeneous part of degree m in the Taylor expansion of f, up to a linear change of coordinates in $\mathbb{C}^{N+1}$ we can write $f_m = f_m(x_0, ..., x_s)$, $s \leq N$, with the partial derivatives $\frac{\partial f_m}{\partial x_i}$, $i = 0, ..., s$, linearly independent over $\mathbb{C}$.

In this system of coordinates $C = \{f_m = 0\}$ is the tangent cone of the singularity $(X, 0)$ and $C^0 = \{x_0 = ... = x_s = 0\} \subset C$ is its vertex. For every analytic automorphism $\phi \in Aut(X, 0)$ we denote by $d\phi: \mathbb{C}^{N+1} \to \mathbb{C}^{N+1}$ its differential, it is clear that $d\phi(C) = C$ and then $d\phi(C^0) = C^0$.

Lemma 1.1. *In the notation above let $G \subset Aut(X, 0)$ be a finite group, $g \in G$ and let $\alpha, \beta \in \mathbb{C}^*$ be such that*

$$dgx_i = \begin{cases} \alpha x_i & \text{for } i = 0, ..., s \\ \beta x_i & \text{for } i = s+1, ..., N \end{cases}$$

then g is contained in the center of G.

Proof. According to Cartan's lemma [Car] the differential $d: G \to GL(\mathbb{C}^{N+1})$ is an injective homomorphism of groups and therefore, since G is finite and C^0 is G-stable, also the induced homomorphism $\hat{d}: G \to GL(C^0) \times GL(\mathbb{C}^{N+1}/C^0)$ is injective and it is clear that $\hat{d}g$ belongs to the center.

■

As an immediate consequence we get

Proposition 1.2. *Let $n \geq 2$ be an integer, $f: (\mathbb{C}^N, 0) \to (\mathbb{C}, 0)$ a germ of holomorphic map of multiplicity $m \geq 2$, $(X, 0) \subset (\mathbb{C}^{N+1}, 0)$ the hypersurface singularity of equation $x_0^n = f(x_1, ..., x_N)$ and $\tau \in Aut(X, 0)$ defined by $\tau x_0 = \xi x_0$, $\xi \in \mu_n$, $\tau x_i = x_i$ for every $i > 0$.*

Assume that either:

1. *i) $n = 2$ and $m \geq 3$, or*
2. *ii) $n > 2$, $m = 2$ and f is a Morse function.*

If $G \subset Aut(X, 0)$ is a finite group and $\tau \in G$ then τ is contained in the center of G.

We recall that a Morse function has a nondegenerate Hessian at every critical point.

Lemma 1.3. *Let X, n, V be as in the Set-up and assume the condition 3 of theorem A is satisfied. Then there exists a divisor $D_0 \in V$ with isolated singularities, such that $Aut(X,V)$ acts freely on $Sing(D_0)$ and:*

1. *i) if $n = 2$ there exists $p \in X$ with $mult_p(D_0) \geq 3$.*

2. *ii) if $n > 2$ there exists a point $p \in X$ where D_0 has a Morse singularity.*

Note that if $\phi_V(X) \subset V^\vee$ is a smooth variety with positive class then 1.3.ii) follows from the existence of Lefschetz pencils without any further assumption on V.

Proof. We prove here only the case $n > 2$, being the case $n = 2$ completely analogous. Let V_1, V_2 be linear systems as in A.3.ii) and let $U \subset X$ be a Zariski open subset where the finite group $Aut(X,V)$ acts freely.

V_1 and V_2 are big and base point free, it is therefore possible to find a point $p \in U$ such that, setting $B_i = \phi_{V_i}^{-1}\phi_{V_i}(p)$, $i = 1,2$, we have $B_1 \cup B_2 \subset U$ and the differential of ϕ_{V_i} has maximal rank at every point of B_i (in particular each B_i is finite).

For every such a point p we denote

$$V_i(-p) = \{D \in V_i | p \in D\}, \quad i = 1,2$$

$$V(-2p) = \{D \in V | mult_p(D) \geq 2\}$$

By Bertini theorem the generic $D_i \in V_i(-p)$ is smooth and therefore the singularities of a generic $D \in V(-2p)$ are contained in $B_1 \cap B_2$.

If $z_1, ..., z_N$ is a system of local coordinates at p, then for every pair $i, j \in 1, ..., N$ there exists a divisor $D_{i,j} \in V_1(-p) + V_2(-p)$ such that its local equation at p is $z_i z_j +$ *higher order terms* and therefore a generic element of the linear system generated by $D_{i,j}$ has a Morse singularity at p. The generic $D_0 \in V(-2p)$ clearly satisfies the lemma.

■

From now on let n, L, V and X be as in the statement of theorem A and let $D_0 \in V$, $p \in D_0$ as in lemma 1.3.

Let $\{D_t\} \subset V$ be a generic pencil containing D_0, since V is base point free and D_0 has isolated singularities, by Bertini's theorem for generic $t \in \mathbb{C}$ the divisor D_t is smooth and intersects transversally D_0; since V is automorphism generic we have moreover $Aut(X, D_t) = Aut(X, V)$. This implies in particular that the divisor $H = \cup_t D_t \times \{t\} \subset X \times \mathbb{C}$ is nonsingular and therefore the simple cyclic cover of degree n branched over H, $Z \xrightarrow{\pi} X \times \mathbb{C}$ is a nonsingular algebraic variety of dimension $N+1$ such that for every t, $Z_t = \pi^{-1}(X \times t) = X_{D_t}$.

Since $K_X \otimes L^{\otimes(n-1)}$ is an ample line bundle, by Hurwitz formula (cf. [B-P-V] Ch. I, [Re] p. 349) every normal Gorenstein variety $X_{D_t} = Z_t$ has ample canonical bundle and therefore we can apply the following semicontinuity result.

Theorem 1.4. *In the notation above there exists a disk $0 \in \Delta \subset \mathbb{C}$, a base change $\Delta \to \mathbb{C}$, $z \to z^m$, and a finite group G acting on the pull-back family $\hat{Z} = Z \times_{\mathbb{C}} \Delta \to \Delta$ preserving fibres, such that the natural restrictions $G \to Aut(\hat{Z}_t)$ are injective for every $t \in \Delta$ and bijective for every $t \neq 0$.*

Proof. Immediate consequence of Prop. 4.4 (with L_t the canonical bundle), Remark 7.1 and Prop. 7.3 (with $f = 1$) of [F-P].

■

The group μ_n of automorphisms of the covering $\hat{Z} \xrightarrow{\pi} X \times \Delta$ is, in the notation of 1.4, a subgroup of G, we claim that it is a *normal* subgroup. Let $\tau \in \mu_n$, $g \in G$ and let $q \in X_{D_0} = \hat{Z}_0$ be the point such that $\pi(q) = p$: q and $g(q)$ are singular points of X_{D_0} and then, since X is smooth, they belong to the ramification divisor of the covering and $\tau(q) = q, \tau g(q) = g(q), g^{-1}\tau g(q) = q$. By the proposition 1.2 the automorphism $h = g^{-1}\tau g$ commutes with τ and therefore for every t, $|t| << 1$, there exists an automorphism $h_t \in Aut(X, D_t)$ such that $h_t \circ \pi = \pi \circ h$.

For t generic $Aut(X, D_t) = Aut(X, V)$ which is finite and then closed, for continuity also $h_0 \in Aut(X, V)$, but $h_0(p) = p$, therefore h_0 must be the identity and h is an automorphism of the cover, i.e. $h \in \mu_n$. Since τ and g are arbitrary this proves the normality of μ_n.

It is now easy to prove theorem A, in fact for generic D, μ_n is a normal subgroup of $Aut(X_D)$, in particular for every automorphism $g: X_D \to X_D$ there exists a continuous map of pairs $q(g): (X, D) \to (X, D)$ commuting with π and g. By Riemann extension theorem $q(g)$

is analytic and therefore it is defined a group morphism $q: Aut(X_D) \to Aut(X, D)$ whose kernel is clearly μ_n.

Example 1.5. Let $S \to \mathbf{P}^1 \times \mathbf{P}^1$ be a $\mathbf{Z}_2 \times \mathbf{Z}_2$ covering defined by the equations $z^2 = f$, $w^2 = g$ where f, g are bihomogeneous polynomials of respective bidegrees $(2a, 2b), (2n, 2m)$. If $a, b, n, m \geq 3$ then for generic f, g as above $Aut(S) = \mathbf{Z}_2 \times \mathbf{Z}_2$.

In fact, considering the factorization $S \to X \to \mathbf{P}^1 \times \mathbf{P}^1$, where X is defined by the equation $z^2 = f$ then from theorem A and corollary B follows immediately that for generic f $Aut(X) = \mathbf{Z}_2$ and for generic g there exists an exact sequence

$$0 \longrightarrow \mathbf{Z}_2 \longrightarrow Aut(S) \longrightarrow Aut(X).$$

Some remarks on the hypotheses of theorem A. The conditions V base point free and A.1 are quite natural in the problem, while the conditions A.2 and A.3 are essentially technical, it is therefore natural to ask if we can improve A.2 and A.3. It is easy to see that we cannot avoid the bigness of V (cf. also the notion of flexible divisor in [Cat]). Consider for example the automorphisms of $\mathbf{P}^5$ defined by

$$\alpha(x_0, x_1, x_2, y_0, y_1, y_2) = (y_0, y_1, y_2, x_0, x_1, x_2)$$

$$\beta(x_0, x_1, x_2, y_0, y_1, y_2) = (x_1, x_2, x_0, y_1, y_2, y_0)$$

Note that α and β generate the dihedral group D_3.
Let $S \subset \mathbf{P}^5$ be the intersection of three generic D_3-invariant hypersurfaces of sufficiently high degree. Then S is a simply connected surface of general type with $Aut(S) = D_3$. The set of fixed points of α has codimension 3 in $\mathbf{P}^5$ and then $X = S/\alpha$ is a smooth surface of general type with universal cover S.

It is easy to see that $Aut(X) = 1$ and therefore the thesis of corollary B is not true in this case.

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