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# On minimal non $CC$–groups.

A. OSMAN ASAR and A. ARIKAN

**Abstract**

In this work it is shown that a locally graded minimal non $CC$–group $G$ has an epimorphic image which is a minimal non $FC$–group and there is no element in $G$ whose centralizer is nilpotent-by-Chernikov. Furthermore Theorem 3 shows that in a locally nilpotent $p$–group which is a minimal non $FC$–group, the hypercentral and hypocentral lengths of proper subgroups are bounded.

## 1 Introduction

Let $G$ be a group. As is well-known, $G$ is called an $FC$–group ($CC$–group) if for all $x \in G$,

$$[G : C_G(x)] < \infty \; (G/C_G(x^G) \text{ is Chernikov }).$$

$G$ is called a **minimal non $FC$–group** if every proper subgroup of $G$ is an $FC$–group but $G$ itself does not have this property. A minimal non $CC$–group is defined similarly.

Belyaev in [2] showed that if $G$ is a perfect locally finite minimal non $FC$–group then either $G/Z(G)$ is simple or $G$ is a $p$–group ($p$ is always a prime number). Recently Kuzucuoğlu and Phillips in [6] have shown that, in fact, $G$ must be a $p$–group. More recently F. Leinen and O. Puglisi in [7] have shown that a perfect locally nilpotent

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$p-$group which is a minimal non $FC-$group can be embedded in the McLain group $M(\mathbb{Q}, GF(p))$. However it is still an open question whether or not a perfect locally nilpotent minimal non $FC-$group can exist.

Otal and Peña in [9] extended some of the properties of a minimal non $FC-$group to a locally graded minimal non $CC-$group. Later the same authors and B. Hartley in [5] have shown that such a group is locally nilpotent $p-$group for some prime $p$. (A group is called **locally graded** if every nontrivial finitely generated subgroup has a proper subgroup of finite index).

In this work it is shown that a locally nilpotent $p-$group which is a minimal non $CC-$group contains a proper epimorphic image which is a minimal non $FC-$group in which every proper normal subgroup is nilpotent of finite exponent and there is no element in such a group whose centralizer is nilpotent-by-Chernikov. Also Theorem 3 shows that in a perfect locally nilpotent $p-$group which is a minimal non $FC-$group, the hypercentral length and hypocentral length of proper subgroups are bounded.

The main results of this work are stated below.

**Theorem 1.** *Let $G$ be a locally nilpotent $p-$group in which every proper subgroup is a $CC-$group. Then every proper subgroup of $G/G^0$ is an $FC-$group. Furthermore if $G$ is perfect then it contains a proper normal subgroup $K$ such that $Z(G)G^0 \leq K$ and every proper subgroup of $G/K$ is nilpotent of finite exponent.*

*(For a group $X$, $X^0$ denotes the unique maximal radicable abelian subgroup of $X$ whenever it exists).*

**Corollary 1.** *Let $G$ be a locally nilpotent $p-$group which is a minimal non $CC-$group. Then $G$ contains a proper normal subgroup $K$ such that $G/K$ is a minimal non $FC-$group and every proper normal subgroup of $G/K$ is nilpotent of finite exponent.*

**Proof**. By the Corollary on p. 1232 of [9], $G$ is perfect. Therefore the assertion follows from Theorem 1.

**Theorem 2.** *Let $G$ be a locally nilpotent $p-$group which is a minimal non $CC-$group. Then $C_G(x)$ is not nilpotent-by-Chernikov (an $NC-$group) for any $x \in G$.*

**Theorem 3.** *Let $G$ be a perfect locally nilpotent $p-$group which is a* *minimal non $FC-$group. Then for every proper subgroup $X$ of $G$ the following holds.*

(i) $Z_\omega(X) = X$.

(ii) $K_\omega(X) \leq Z(G)$.

(As usual for each ordinal $\alpha$, $K_\alpha(X)$ and $Z_\alpha(X)$ denote respectively the $\alpha$ th term of the lower and upper central series of $X$).

## 2 Proof of the Theorems

**Lemma 2.1.** *Let $A$ be a periodic $CC-$group and $B$ be a normal abelian subgroup of $A$ such that $A/B$ is radicable abelian. Then $A$ is abelian. If in addition $B$ is radicable abelian then so is $A$.*

**Proof**. The first part of the lemma follows from Lemma 2 of [3] and the second part is trivial.

Note that if in the following Lemma $H$ is a $CC-$group, then the conclusion follows from Lemma 1 of [3], but this result is not needed in the proof.

**Lemma 2.2.** *Let $H$ be a locally nilpotent $p-$group in which every proper subgroup is a $CC-$group. Then $H^0$ exists and $(H/H^0)^0 = 1$.*

**Proof**. By (1.1)(3) of [9] every proper subgroup of $H$ is hypercentral, which implies that every subgroup of $H$ is ascendant in $H$. By Zorn's Lemma, $H$ contains maximal radicable abelian subgroups. Let $P$ and $Q$ be two maximal radicable abelian subgroups. Then $[P,Q] = 1$ by Lemmas 3.2 and 3.4 of [8], since $P$ and $Q$ are ascendant in $G$. Hence $PQ$ is abelian and so $P = Q$ by the choice of $P$ and $Q$, which implies that $H$ has a unique maximal radicable abelian subgroup; that is, $H^0$ exists.

Next let $T/H^0 = (H/H^0)^0$. Then $T$ is radicable abelian by Lemma 2.1 and so $T = H^0$ by the maximality of $H^0$, which was to be shown.

**Lemma 2.3.** *Let $H$ be a locally nilpotent $p-$group in which every proper subgroup is a $CC-$group. Then every proper subgroup of $H/H^0$ is an $FC-$group.*

**Proof**. Since $(H/H^0)^0 = 1$ by Lemma 2.2, we may suppose without loss of generality that $H^0 = 1$. Let $K$ be any proper subgroup of $H$. Then $K/K^0$ is an $FC-$group by Lemma 1 of [3]. Moreover since $H$ contains a unique maximal radicable abelian subgroup by Lemma 2.2, it follows that $K^0 = 1$ and so $K$ is an $FC-$group which completes the proof of the Lemma.

**Lemma 2.4.** *Let $H$ be a perfect locally nilpotent $p-$group which is a minimal non $FC-$group. Then $H$ contains a proper normal subgroup $K$ such that $Z(H) \leq K$, and every proper normal subgroup of $H/K$ is nilpotent of finite exponent.*

**Proof**. By hypothesis $H = H'$, so without loss of generality we may syppose that $Z(H) = 1$. Let $1 \neq a \in H$ and put $C = C_H(a)$. Then $C \neq H$. Let $N$ be a proper normal subgroup of $H$ and put $D = C \cap N$. Then $[N : D]$ is finite, since $N < a >$ is an $FC-$group. Hence if

$$L = \bigcap_{x \in N} D^x$$

then $L$ is normal in $N$ and $N/L$ is finite. Next let

$$Y = \bigcap_{h \in H} L^h.$$

Then $Y$ is normal in $H$ and $N/Y$ is nilpotent of finite exponent, since it is embedded into the unrestricted direct product

$$\prod_{h \in H} (N/L^h)$$

where $N/L^h \cong N/L$ for all $h \in H$.

Finally let

$$K = \bigcap_{h \in H} C^h.$$

Then $K \neq H$ since $C \neq H$. Also $Y \leq K$ since $Y \leq L \leq D \leq C$. Therefore $NK/K$ is nilpotent since $N/Y$ is nilpotent. Since $N$ is any proper normal subgroup of $H, K$ is a desired subgroup of $H$.

**Proof of Theorem 1.** By Lemma 2.3 every proper subgroup of $G/G^0$ is an $FC-$group. Now suppose also that $G$ is perfect. Then $G/G^0$ is also perfect and so it is a minimal non $FC$–group. Therefore by Lemma 2.4 it contains a proper normal subgroup $K/G^0$ such that $Z(G/G^0) \leq K/G^0$ and every proper normal subgroup of $G/K$ is nilpotent of finite exponent. Obviously $Z(G)G^0 \leq K$. This complete the proof of the theorem.

**Proof of Theorem 2.** By the Corollary and by (1.1)(3) of [9] $G$ is perfect and every proper subgroup of $G$ is hypercentral. Assume that $C = C_G(a)$ is nilpotent-by-Chernikov ($NC$–group for short) for some $a \in G$. Then $C \neq G$ since $G$ is perfect by hypothesis. First we show that every proper subgroup of $G$ is an $NC$–group. So let $X$ be a proper subgroup of $G$. Clearly $G$ can be expressed as a union of an ascending chain of proper normal subgroups since it is perfect and locally nilpotent. Hence it follows that $a^G \neq G$. Then also $a^G X \neq G$ since $G$ is perfect but $a^G$ and $X$ both are hypercentral. Put $L = a^G X$. Since $L$ is a $CC$–group, $L/C_L(a^L)$ is Chernikov. Let $R = C_L(a^L)$. Since $R \leq C$, $R$ has a normal nilpotent subgroup $K$ such that $R/K$ is Chernikov. By Lemma 4.7 (i) of [4] we may suppose that $K$ is normal in $L$ since $R$ is normal in $L$. Also $L/K$ is Chernikov since $L/R$ and $R/K$ are Chernikov. Hence it follows that $L$ is an $NC$–group and then also $X$ has the same property. Consequently it follows that every proper subgroup of $G$ is both a $CC$–group and $NC$–group. But then $G$ is an $NC$–group by the Corollary to Theorem $B$ of [1], which is a contradiction since $G = G'$. This completes the proof of the theorem.

**Proof of Theorem 3.** (i) Let $X$ be a proper subgroup of $G$. By hypothesis $X$ is an $FC$–group. Therefore applying Theorem 4.38 of [10] yields that

$$Z_n(X) \leq X \leq Z_\omega(X)$$

For all $n \geq 1$, since $X$ is an $FC$–group. Hence it follows that $X = Z_\omega(X)$.

To show the second assertion first suppose that $Z(G) = 1$. By Lemma 2.21 of [10]

$$[K_m(X),\, Z_m(X)] = 1$$

for all $m \geq 1$. Hence since $K_\omega(X) \leq K_m(X)$ it follows that

$$[K_\omega(X),\, Z_m(X)] = 1$$

for all $m \geq 1$. But since

$$X = \bigcup_{m=1}^{\infty} Z_m(X)$$

by the first part of the proof it follows that $K_\omega(X) \leq Z(X)$.

Next applying Lemma 6 of [2] repeatedly we can write $G$ as

$$G = \bigcup_{i=1}^{\infty} X_i$$

where $X < X_i < X_{i+1}$ for all $i \geq 1$ since $G$ is countably infinite by the Corollary on p.1232 of [9]. Also it follows from the first part of the proof that

$$K_\omega(X_i) \leq Z(X_i)$$

for all $i \geq 1$. But since $K_\omega(X) \leq K_\omega(X_i)$ it follows that

$$K_\omega(X) \leq Z(X_i)$$

for all $i \geq 1$ which yields that

$$K_\omega(X) \leq Z(G) = 1.$$

Now in the general case put $\bar{G} = G/Z(G)$. Then $Z(\bar{G}) = 1$ by hypothesis. Therefore $\overline{K_\omega(X)} = K_\omega(\bar{X}) = 1$ and hence $K_\omega(X) \leq Z(G)$ by the preceding paragraph. This completes the proof of the theorem.

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Gazi Egitim Fakültesi
Teknikokullar
ANKARA-TÜRKİYE

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