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A short intervals result for linear equations in two prime variables.

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Abstract

Given A and B integers relatively prime, we prove that almost all integers n in an interval of the form $[N, N+H]$, where $N^{1/3+\varepsilon} \leq H \leq N$, can be written as a sum $Ap_1 + Bp_2 = n$, with p_1 and p_2 primes and ε an arbitrary positive constant. This generalizes the results of $[PP]$ established in the classical case $A = B = 1$ (Goldbach's problem).

1 Introduction

Let us consider the linear equation $Ap_1 + Bp_2 = n$, where p_1 and p_2 are prime numbers and A, B, n are positive integers satisfying the following conditions:

1) A, B are relatively prime, $(A, B) = 1$,

2) $n \in \mathcal{A} = \{n : (AB, n) = 1, \ ABn \equiv 0 \pmod 2\}$.

In the classical case $A = B = 1$ (Goldbach's problem) Perelli and Pintz [PP] proved that the above equation has solutions for all even integers $2n$ in an interval of the form $[N, N+H]$ with $O(HL^{-E})$ exceptions

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and

$$R(2n) = 2n\mathfrak{S}(2n) + O(NL^{-C}),$$

where

$$R(2n) = \sum_{h+k=2n} \Lambda(h)\Lambda(k), \quad \mathfrak{S}(2n) = 2\prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right)\prod_{\substack{p|n \\ p>2}}\left(\frac{p-1}{p-2}\right),$$

Λ is von Mangoldt's function, $L = \log N$, $N^{1/3+\varepsilon} \le H \le N$ and E, $C > 0$, $0 < \varepsilon < 2/3$ are arbitrary constants.

In this paper we want to establish the same results in the general case for A and B. Let us denote

$$R(n) = R(n, A, B) = \sum_{Ah+Bk=n} \Lambda(h)\Lambda(k),$$

$$M(n) = M(n, A, B) = \sum_{Ah+Bk=n} 1,$$

$$\mathfrak{S}(n) = \mathfrak{S}_{A,B}(n) = \prod_{p|ABn}\left(1 + \frac{1}{p-1}\right)\prod_{p\nmid ABn}\left(1 - \frac{1}{(p-1)^2}\right).$$

Theorem 1. *Let $0 < \varepsilon < 2/3$, $C > 0$ be arbitrary constants and $N^{1/3+\varepsilon} \le H \le N$. Then*

$$\sum_{\substack{N<n\le N+H \\ n\in\mathcal{A}}} |\, R(n) - M(n)\mathfrak{S}(n)\,|^2 \ll_{\varepsilon,A,B,C} HN^2L^{-C}.$$

Clearly Theorem 1 implies

Corollary. *Let E, $C > 0$ be arbitrary constants and $N^{1/3+\varepsilon} \le H \le N$. Then, for all $n \in [N, N+H] \cap \mathcal{A}$ with at most $O(HL^{-E})$ exceptions, the equation $Ap_1 + Bp_2 = n$ is solvable and we have*

$$R(n) = M(n)\mathfrak{S}(n) + O_{\varepsilon,A,B,C}(NL^{-C}).$$

Theorem 2. *Let $0 < \varepsilon < 5/6$, $E > 0$ be arbitrary constants. Then, for all $n \in [N, N + H] \cap \mathcal{A}$ with at most $O_{\varepsilon,A,B,C}(HL^{-E})$ exceptions, the equation $Ap_1 + Bp_2 = n$ is solvable, provided $N^{7/36+\varepsilon} \le H \le N$.*

Remarks.

I. The condition $(A, B) = 1$ is natural. In fact, if $(A, B) = d > 1$, then the equation $Ap_1 + Bp_2 = n$ has no solutions when $d \nmid n$. If $d|n$, then the equation is equivalent to $A'p_1 + B'p_2 = n'$, where $A' = A/d$, $B' = B/d$, $n' = n/d$ with $(A', B') = 1$. If $n \notin \mathcal{A}$, then our equation has at most one solution, and our method is not able to detect it.

II. It is easy to see [A; ch.2, ex.9] that for fixed A and B, $(A, B) = 1$, we have

$$|\,[N, N+H] \cap \mathcal{A}\,| \sim \frac{\varphi(AB)}{AB} H,$$

where φ is Euler's function. Hence the exceptional set in the above results is of order smaller than $|\,[N, N+H] \cap \mathcal{A}\,|$.

III. The number of non-negative integer solutions of the linear equation $Ax + By = n$, with $(A, B) = 1$, is given by

$$M(n) = [\frac{n}{AB}] \quad \text{or} \quad [\frac{n}{AB}] + 1.$$

This result can be proved by appealing to the following theorem (see [NZM; Theorem 5.1]) and recalling that $[\alpha] - [\beta] = [\alpha - \beta]$ or $[\alpha - \beta] + 1$ (where $[\alpha]$ denotes the integer part of α):

Theorem 3. *Let (x_o, y_o) be an integer solution of the linear equation $Ax + By = n$, with $(A, B) = 1$. Then the solutions of $Ax + By = n$ are given by:*

$$x = x_o + Bt, \quad y = y_o - At,$$

where t is an integer.

IV. Following the proofs of Theorem 1 and 2 below and computing the implicit constant in $\ll_{\varepsilon, A, B, C}$, it is easy to remark that the forementioned results holds uniformly for $A, B \ll L^{C_1}$, for a suitable constant $C_1 = C_1(C) > 0$.

2 Notation

C - an arbitrary positive constant,
ε - an arbitrarily small positive constant,
$\delta = 1/3 + \varepsilon$ or $7/36 + \varepsilon$,
N, H - positive integers such that $N^\delta \le H \le N$, $N > N_0 = N_0(A, B, C, \varepsilon)$,
$D = \frac{3}{\varepsilon}(2C + 25)$, $Q = \frac{\sqrt{H}}{2}$.

$$I_{q,a} = \left\{\frac{a}{q} + \eta, \eta \in \xi_{q,a}\right\}, \quad \text{where} \quad \xi_{q,a} \subset \left(-\frac{1}{qQ}, \frac{1}{qQ}\right),$$

$$I'_{q,a} = \left\{\frac{a}{q} + \eta, \eta \in \xi'_q\right\}, \quad \text{where} \quad \xi'_q = \left(-\frac{L^{4D}}{qY}, \frac{L^{4D}}{qY}\right),$$

$$\mathfrak{M} = \bigcup_{q \le L^D} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} I'_{q,a} \quad \text{major arcs,} \quad \mathfrak{m} = \left[\frac{1}{Q}, 1 + \frac{1}{Q}\right] \setminus \mathfrak{M} \quad \text{minor arcs,}$$

$$I'_{B,q,a} = \{\alpha \in [0,1] \mid \exists \beta \in I'_{q,a} : B\alpha \equiv \beta \pmod 1\} = B^{-1} \bigcup_{r=0}^{B-1} (r + I'_{q,a}),$$

$$\mathfrak{M}_B = \bigcup_{q \le L^D} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} I'_{B,q,a}, \qquad \mathfrak{m}_B = [0,1] \setminus \mathfrak{M}_B.$$

$$R^*(n) = R^*(N, Y, n, A, B) = \sum_{\substack{Ah+Bk=n \\ N-Y<Ah\le N \\ Bk \le Y}} \Lambda(h)\Lambda(k),$$

$$M^*(n) = M^*(N, Y, n, A, B) = \sum_{\substack{Ah+Bk=n \\ N-Y<Ah\le N \\ Bk \le Y}} 1.$$

$$e(\alpha) = e^{2\pi i \alpha}, \quad e_q(a) = e(a/q),$$

$$S_A(\alpha) = \sum_{N-Y<Ah\le N} \Lambda(h) e(Ah\alpha), \quad S_B(\alpha) = \sum_{Bk \le Y} \Lambda(k) e(Bk\alpha),$$

$$S(\beta) = \sum_{Bk \le Y} \Lambda(k) e(k\beta), \quad T(\eta) = \sum_{Bk \le Y} e(k\eta),$$

$$R(\eta, q, a) = S\left(\frac{a}{q} + \eta\right) - \frac{\mu(q)}{\varphi(q)} T(\eta),$$

$$W(\chi,\eta) = \sum_{Bk\le Y} \Lambda(k)\chi(k)e(k\eta) - \delta_\chi T(\eta),$$

with $\delta_\chi = \begin{cases} 1 & \text{if } \chi = \chi_o, \\ 0 & \text{if } \chi \neq \chi_o. \end{cases}$

$\sum_{a=1}^{q}{}^{*} = \sum_{\substack{a=1\\(a,q)=1}}^{q}$, $\|\beta\|$ = distance of β from the nearest integer, $d(q) = \sum_{d|q} 1$.

$$c_q(m) = \sum_{a=1}^{q}{}^{*} e\left(\frac{ma}{q}\right), \text{ Ramanujan's sum}$$

$$\tau(\chi) = \sum_{a=1}^{q}{}^{*} \chi(a)e\left(\frac{a}{q}\right), \text{ Gauss' sum}$$

$$\psi(x,\chi) = \sum_{n\le x} \Lambda(n)\chi(n).$$

The constants in the $\ll$ and O-symbols might depend on ε, A, B, C, even in an ineffective way.

3 Lemmas

Lemma 1. *Let $0 < \varepsilon < 5/12$, D, $G > 0$ be arbitrary constants, and N be a sufficiently large integer. Suppose that $N^{7/12+\varepsilon} \le Y \le N$ and $q \le L^D$. We have*

$$\sum_{\substack{N-Y<h\le N\\ h\equiv a \ (\mathrm{mod}\ q)}} \Lambda(h) = \frac{Y}{\varphi(q)} + O_{\varepsilon,D,G}(YL^{-G}),$$

for $(a,q) = 1$.

Proof. We have (see [PPS])

$$\psi(N,\chi) - \psi(N-Y,\chi) = \delta_\chi Y + O_{\varepsilon,D,G}(YL^{-G}), \quad \text{for every } \chi \ (\mathrm{mod}\ q),$$

with $G > 0$, $N^{7/12+\varepsilon} \le Y \le N$, $q \le L^D$.

Then the Lemma follows from the property

$$\sum_{\substack{N-Y<h\leq N\\ h\equiv a\,(\mathrm{mod}\,q)}} \Lambda(h) = \frac{1}{\varphi(q)} \sum_{\chi} \overline{\chi}(a)(\psi(N,\chi) - \psi(N-Y,\chi)).$$

Lemma 2. *If $(A,B)=1$ and $n \in \mathcal{A}$, we have*

$$\prod_{p|An}\left(1+\frac{1}{p-1}\right)\prod_{p\nmid ABn}\left(1-\frac{1}{(p-1)^2}\right) = \sum_{q=1}^{+\infty} \frac{\mu(q)}{\varphi(q)^2} c_q(n) f_{A,B}(q),$$

where $f_{A,B}(q) = \mu(q(q,B))\mu(q,A)\varphi(q,A)$.

Proof. As in [V; p.34], using the well-known property $c_q(n) = \frac{\mu(q/(q,n))\varphi(q)}{\varphi(q/(q,n))}$, we obtain

$$\sum_{X<q\leq Y} \frac{\mu(q)}{\varphi(q)^2} c_q(n) f_{A,B}(q) = \sum_{\substack{t|n\\ u|A}} \frac{\mu(tu)^2}{\varphi(tu)} \sum_{\substack{\frac{X}{tu}<q\leq \frac{Y}{tu}\\ (q,ABn)=1}} \frac{\mu(q)}{\varphi(q)^2} \ll$$

$$\ll \sum_{d|An} \frac{\mu(d)^2}{\varphi(d)} \min\left(\frac{d}{X},1\right).$$

Hence $\sum_{q=1}^{+\infty} \frac{\mu(q)}{\varphi(q)^2} c_q(n) f_{A,B}(q)$ converges. Furthermore, it is easy to see that

$$\frac{\mu(p^s)}{\varphi(p^s)^2} c_{p^s}(n) f_{A,B}(p^s) = \begin{cases} 1, & \text{if } s=0,\\ \frac{1}{\varphi(p)}, & \text{if } s=1 \text{ and } p|An,\\ -\frac{1}{\varphi(p)^2}, & \text{if } s=1 \text{ and } p\nmid ABn,\\ 0, & \text{if } s=1 \text{ and } p|B \text{ or } s>1\,. \end{cases}$$

Then, the Lemma is proved.

Lemma 3. *Let X and Y be natural numbers such that $X<Y$. If $(A,B)=1$ then for every real $\xi \neq 0$ we have that*

$$\sum_{\substack{X<t\leq Y\\ t\in\mathcal{A}}} e(t\xi) \ll_{A,B} \min\left(\frac{Y-X}{\hat{\delta}AB}, \frac{1}{\|\hat{\delta}AB\xi\|}\right),$$

where $\hat{\delta} = \frac{3-(-1)^{AB}}{2}$.

Proof. This is well-known for $A = B = 1$ (see [Vi]). Let $h_1, h_2, \dots, h_{\varphi(\hat{\delta}AB)}$ be a reduced residue system mod $\hat{\delta}AB$. Define $h'_i = \min\{X < t \le Y : t \equiv h_i \pmod{\hat{\delta}AB}\}$, for every $i = 1, 2, \dots, \varphi(\hat{\delta}AB)$. Then we write

$$\sum_{\substack{X<t\le Y \\ t\in\mathcal{A}}} e(t\xi) = \sum_{i=1}^{\varphi(\hat{\delta}AB)} e(h'_i\xi) \sum_{0\le s\le \frac{Y-X}{\hat{\delta}AB}} e(s\hat{\delta}AB\xi) \ll$$

$$\ll \varphi(\hat{\delta}AB) \min\left(\frac{Y-X}{\hat{\delta}AB}, \frac{1}{\| \hat{\delta}AB\xi \|}\right).$$

Thus, the Lemma is proved.

Lemma 4. *Let us denote*

$$I''_{q,a} = \begin{cases} I_{q,a} \setminus I'_{q,a} & \text{if } q \le L^D, \\ I_{q,a} & \text{if } q > L^D. \end{cases}$$

We have that

$$\max_{\substack{q\le Q \\ (a,q)=1}} \int_{I''_{q,a}} | S(\beta) |^2 \, d\beta \ll Y L^{-2C-3}.$$

Proof. See [PP; §5].

4 Proof of Theorems 1 and 2

We note that $M^*(n) = \frac{N+Y-2n+1}{AB} + O(1)$. Moreover, in the case $Y = N$, for $N < n \le N + H$ we have that $R^*(n) = R(n) + O_{A,B}(HL)$, $M^*(n) = M(n) + O_{A,B}(HL)$. Hence, from $\mathfrak{S}(n) \ll L$ we get that

$$\sum_{\substack{N<n\le N+H \\ n\in\mathcal{A}}} | R(n) - M(n)\mathfrak{S}(n) |^2 \ll$$

$$\sum_{\substack{N<n\le N+H \\ n\in\mathcal{A}}} | R^*(n) - M^*(n)\mathfrak{S}(n) |^2 + H^3L^2 =$$

$$= \sum_{\substack{N<n\le N+H \\ n\in\mathcal{A}}} \left| R^*(n) - \frac{Y}{AB}\mathfrak{S}(n) \right|^2 + O(H^3L^2).$$

Moreover, writing

$$\sum_{\mathfrak{m}_B} = \sum_{\substack{N<n\le N+H \\ n\in\mathcal{A}}} \left| \int_{\mathfrak{m}_B} S_A(\alpha)S_B(\alpha)e(-n\alpha)d\alpha \right|^2,$$

$$\sum_{\mathfrak{M}_B} = \sum_{\substack{N<n\le N+H \\ n\in\mathcal{A}}} \left| \int_{\mathfrak{M}_B} S_A(\alpha)S_B(\alpha)e(-n\alpha)d\alpha - \frac{Y}{AB}\mathfrak{S}(n) \right|^2,$$

we have

$$\sum_{\substack{N<n\le N+H \\ n\in\mathcal{A}}} \left| R^*(n) - \frac{Y}{AB}\mathfrak{S}(n) \right|^2 \ll \sum_{\mathfrak{m}_B} + \sum_{\mathfrak{M}_B}.$$

In order to prove Theorems 1 and 2 it sufficies to choose $H = Y^{1/3+\varepsilon}$, where $Y = N$ in Theorem 1 and $Y = N^{7/12+\varepsilon}$ in Theorem 2 and show that

$$\sum_{\mathfrak{M}_B} \ll HY^2L^{-C}, \tag{1}$$

$$\sum_{\mathfrak{m}_B} \ll HY^2L^{-C}. \tag{2}$$

Proof of (1). In this section we show that (1) holds for $N^{7/12+\varepsilon} \le Y \le N$. For any $\alpha \in I'_{B,q,a}$, we have that

$$S_B(\alpha) = S(\beta) = \sum_{Bk\le Y} \Lambda(n)e(k\beta),$$

where $\beta \in I'_{q,a}$ with $B\alpha \equiv \beta \pmod 1$. Then, for $\beta = \frac{a}{q} + \eta$, $\eta \in \xi'_q$, we write

$$S_B(\alpha) = S\left(\frac{a}{q} + \eta\right) = \frac{\mu(q)}{\varphi(q)}T(\eta) + R(\eta, q, a).$$

Now, we have that

$$R(\eta, q, a) = \sum_{Bk\le Y} \Lambda(k)e\left(k\left(\frac{a}{q} + \eta\right)\right) - \frac{\mu(q)}{\varphi(q)}T(\eta) =$$

$$\sum_{b=1}^{q}{}^{*} e\left(\frac{ab}{q}\right) \sum_{\substack{Bk\le Y\\ k\equiv b\,(\mathrm{mod}\,q)}} \Lambda(k)e(k\eta) - \frac{\mu(q)}{\varphi(q)}T(\eta) + O(\log^2 Y) =$$

$$\sum_{b=1}^{q}{}^{*} e\left(\frac{ab}{q}\right) \frac{1}{\varphi(q)} \sum_{\chi} \overline{\chi}(b) \sum_{Bk\le Y} \Lambda(k)\chi(k)e(k\eta) - \frac{\mu(q)}{\varphi(q)}T(\eta) + O(\log^2 Y) =$$

$$\frac{1}{\varphi(q)} \sum_{\chi} \chi(a)\tau(\overline{\chi})W(\chi,\eta) + O(\log^2 Y).$$

By partial summation, using the Siegel-Walfisz theorem, we have that

$$W(\chi,\eta) \ll YL^{-6D},$$

uniformly for $q \le L^D$, $\frac{a}{q}+\eta \in I'_{q,a}$. Thus, from the well-known estimate $\tau(\chi) \ll q^{1/2}$, we obtain

$$R(\eta,q,a) \ll YL^{-\frac{11D}{2}}, \tag{3}$$

uniformly for $\eta \in \xi'_{q,a}$, $q \le L^D$, $(a,q)=1$. Hence, we write

$$\int_{\mathfrak{M}_B} S_A(\alpha)S_B(\alpha)e(-n\alpha)d\alpha = \frac{1}{B}\sum_{q\le L^D}\sum_{a=1}^{q}{}^{*}\sum_{r=0}^{B-1} e\Big(-\frac{n}{B}(\frac{a}{q}+r)\Big)\times$$

$$\int_{\xi'_q} S_A\Big(\frac{1}{B}(\frac{a}{q}+r+\eta)\Big)\Big(\frac{\mu(q)}{\varphi(q)}T(\eta)+R(\eta,q,a)\Big)e(-\frac{n}{B}\eta)d\eta \ll$$

$$\sum_1 + \sum_2 + \sum_3,$$

where

$$\sum_1 = \frac{1}{B}\sum_{q\le L^D}\frac{\mu(q)}{\varphi(q)}\sum_{a=1}^{q}{}^{*}\sum_{r=0}^{B-1} e\Big(-\frac{n}{B}(\frac{a}{q}+r)\Big)\times$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} S_A\Big(\frac{1}{B}(\frac{a}{q}+r+\eta)\Big)T(\eta)e(-\frac{n}{B}\eta)d\eta,$$

$$\sum_2 = \frac{1}{B}\sum_{q\le L^D}\frac{\mu(q)}{\varphi(q)}\sum_{a=1}^{q}{}^{*}\sum_{r=0}^{B-1} e\Big(-\frac{n}{B}(\frac{a}{q}+r)\Big)\times$$

$$\int_{\frac{L^{4D}}{qY}}^{1/2} S_A\Big(\frac{1}{B}(\frac{a}{q}+r+\eta)\Big)T(\eta)e(-\frac{n}{B}\eta)d\eta,$$

$$\sum_3 = \frac{1}{B}\sum_{q\le L^D}\sum_{a=1}^{q}{}^{*}\sum_{r=0}^{B-1} e\Big(-\frac{n}{B}(\frac{a}{q}+r)\Big)\times$$

$$\int_{\xi'_q} S_A\Big(\frac{1}{B}(\frac{a}{q}+r+\eta)\Big)R(\eta,q,a)e(-\frac{n}{B}\eta)d\eta.$$

Recalling that

$$\frac{1}{B}\sum_{r=0}^{B-1} e_B(r(Ah-n)) = \begin{cases} 1 & \text{if } B|Ah-n,\\ 0 & \text{if } B\nmid Ah-n,\end{cases}$$

we have

$$\sum_1 = \sum_{q\le L^D}\frac{\mu(q)}{\varphi(q)}\sum_{N-Y<Ah\le N} c_{qB}(Ah-n)\Lambda(h)\times$$

$$\frac{1}{B}\sum_{r=0}^{B-1} e_B(r(Ah-n))\sum_{Bk\le N}\int_{-\frac{1}{2}}^{\frac{1}{2}} e_B(\eta(Ah+Bk-n))d\eta$$

$$= \sum_{q\le L^D}\frac{\mu(q)}{\varphi(q)}\sum_{\substack{N-Y<Ah\le N\\ Ah\equiv n\,(\mathrm{mod}\,B)}} c_q(B^{-1}(Ah-n))\Lambda(h). \qquad (4)$$

Using the property $c_q(m) = \sum_{d|(q,m)} d\mu\left(\frac{q}{d}\right)$, we write

$$\sum_{\substack{N-Y<Ah\le N\\ Ah\equiv n(\mathrm{mod}\,B)}} c_q(B^{-1}(Ah-n))\Lambda(h) = \sum_{d|q} d\mu\left(\frac{q}{d}\right)\sum_{\substack{N-Y<Ah\le N\\ Ah\equiv n\,(\mathrm{mod}\,Bd)}}\Lambda(h).$$

We observe that, since $(A,B)=1$ and $(AB,n)=1$, the linear congruence $Ah\equiv n\,(\mathrm{mod}\,Bd)$ has a solution if and only if $(A,d)=1$ and, if it exists, this solution is unique. Then, from Lemma 1 we obtain

$$\sum_{\substack{N-Y<Ah\le N\\ Ah\equiv n\,(\mathrm{mod}\,B)}} c_q(B^{-1}(Ah-n))\Lambda(h) =$$

$$\frac{Y}{A}\sum_{\substack{d|q\\ (d,An)=1}}\frac{d}{\varphi(dB)}\mu\left(\frac{q}{d}\right) + O_{\varepsilon,D,G}(qd(q)YA^{-1}L^{-G}) =$$

$$\frac{Y}{A\varphi(B)} \sum_{\substack{d|q \\ (d,An)=1}} \frac{\varphi(d,B)}{(d,B)} \frac{d}{\varphi(d)} \mu\left(\frac{q}{d}\right) + O_{\varepsilon,D,G}(qd(q)YA^{-1}L^{-G}). \tag{5}$$

For every square-free q, we have

$$\sum_{\substack{d|q \\ (d,An)=1}} \frac{\varphi(d,B)}{(d,B)} \frac{d}{\varphi(d)} \mu\left(\frac{q}{d}\right) = \frac{c_q(n)}{\varphi(q)} f_{A,B}(q), \tag{6}$$

where $f_{A,B}(q) = \mu(q(q,B))\mu(q,A)\varphi(q,A)$ (see Lemma 2).

Since $T(\eta) \ll \min(YB^{-1}, \| \eta \|^{-1})$, using the Cauchy-Schwarz inequality, we have that

$$\left| \int_{\frac{L^{4D}}{qY}}^{\frac{1}{2}} S_A(B^{-1}(a/q + r + \eta))T(\eta)e(-nB^{-1}\eta)d\eta \right| \le$$

$$\left| \int_{\frac{L^{4D}}{qY}}^{\frac{1}{2}} \left| S_A(B^{-1}(a/q + r + \eta)) \right|^2 d\eta \right|^{1/2} \times$$

$$\left| \int_{\frac{L^{4D}}{qY}}^{\frac{1}{2}} \| \eta \|^{-2} d\eta \right|^{1/2} \ll q^{1/2} Y L^{1-2D}.$$

Analogously, from (3) we have that

$$\int_{\xi'_q} S_A(B^{-1}(a/q + r + \eta))R(\eta, q, a)e(-nB^{-1}\eta)d\eta \ll q^{-1/2} Y L^{1-\frac{7D}{2}}$$

Therefore, the contribution of $\sum_2$ and $\sum_3$ to $\sum_{\mathfrak{M}_B}$ is

$$\ll_{A,B} HY^2 \max(L^{2-D}, L^{2-4D}) \ll HY^2 L^{-C}. \tag{7}$$

Recalling that $\frac{B}{\varphi(B)} = \prod_{p|B} \left(1 + \frac{1}{p-1}\right)$, from (4)-(7) it follows that

$$\sum_{\mathfrak{M}_B} \ll \sum_{\substack{N<n\le N+H \\ n\in A}} \left| \sum_1 - \frac{Y}{AB}\mathfrak{S}(n) \right|^2 + O(HY^2L^{-C}) \ll$$

$$\sum_{\substack{N<n\le N+H \\ n\in A}} \left| \frac{Y}{AB} \Big(\frac{B}{\varphi(B)} \sum_{q\le L^D} \frac{\mu(q)}{\varphi(q)^2} f_{A,B}(q) c_q(n) - \mathfrak{S}(n) \Big) \right|^2 +$$

$$O\Big(\sum_{\substack{N<n\le N+H\\ n\in\mathcal{A}}}\Big|YA^{-1}L^{-G}\sum_{q\le L^D}\overline{\frac{qd(q)}{\varphi(q)}}\Big|^2\Big)+O(HY^2L^{-C}).$$

The contribution of the first O-term is

$$\ll HY^2L^{3D-2G}\ll HY^2L^{-C},$$

provided $2G>3D+C$. Then, from the proof of Lemma 2, it is easy to see that

$$\sum_{\substack{N<n\le N+H\\ n\in\mathcal{A}}}\left|\frac{B}{\varphi(B)}\sum_{q\le L^D}\frac{\mu(q)}{\varphi(q)^2}f_{A,B}(q)c_q(n)-\mathfrak{S}(n)\right|^2\ll HL^{2-D}\ll HL^{-C}.$$

Thus, we conclude that (1) is proved.

Proof of (2). By the Cauchy-Schwarz inequality, Parseval's identity, the Brun-Titchmarsh inequality and the Lemma 3 we have that

$$\sum_{\mathfrak{m}_B}=\sum_{\substack{N<n\le N+H\\ n\in\mathcal{A}}}\int_{\mathfrak{m}_B}S_A(\xi)S_B(\xi)e(-n\xi)d\xi\int_{\mathfrak{m}_B}\overline{S_A(\alpha)S_B(\alpha)}e(n\alpha)d\alpha\ll$$

$$\int_{\mathfrak{m}_B}|S_A(\xi)S_B(\xi)|\Big(\int_{\mathfrak{m}_B}|S_A(\alpha)S_B(\alpha)|\times$$

$$\min\left(\frac{H}{\hat{\delta}AB},\frac{1}{\|\hat{\delta}AB(\alpha-\xi)\|}\right)d\alpha\Big)d\xi\ll$$

$$(YL)^{\frac{3}{2}}\sup_{\xi\in\mathfrak{m}_B}\left(\int_{\mathfrak{m}_B}|S_B(\alpha)|^2\min\left(\frac{H}{\hat{\delta}AB},\frac{1}{\|\hat{\delta}AB(\alpha-\xi)\|}\right)^2d\alpha\right)^{\frac{1}{2}}\ll$$

$$(YL)^{\frac{3}{2}}\sup_{\xi\in[0,1]}\left(\int_{\mathfrak{m}}|S(\beta)|^2\min\left(\frac{H}{\hat{\delta}AB},\frac{1}{\|\hat{\delta}A(\beta-\xi)\|}\right)^2d\beta\right)^{\frac{1}{2}}\ll$$

$$(YL)^{\frac{3}{2}}\sum_{s=1}^{A}\sup_{\xi\in[0,1]}\left(\int_{\mathfrak{m}_s}|S(\beta)|^2\min\left(\frac{H}{\hat{\delta}AB},\frac{1}{\|\hat{\delta}A(\beta-\xi)\|}\right)^2d\beta\right)^{\frac{1}{2}},$$

where $\mathfrak{m}_s=\mathfrak{m}\cap J_s$, with $J_s=\left[\frac{s-1}{A},\frac{s}{A}\right]$, $s=1,\dots,A$.

Hence, (2) is proved whenever

$$\int_{\mathfrak{m}_s\cap(\xi-\frac{1}{H},\xi+\frac{1}{H})} |S(\beta)|^2 d\beta \ll YL^{-2C-3}, s=1,...,A \qquad (8)$$

for $Y_o(C,\varepsilon) \leq Y$, $Y^{1/3+\varepsilon} \leq H$, uniformly for $\xi \in [0,1]$.
Since, for $\frac{a}{q} \neq \frac{a'}{q'}$ and $q, q' \leq Q$, we have that

$$\left|\frac{a}{q} - \frac{a'}{q'}\right| \geq \frac{1}{Q^2} = \frac{4}{H},$$

then there are at most two punctured arcs $I''_{q,a}$, with $q \leq Q$ and $(a,q) = 1$ (see Lemma 4) which intersect $(\xi - 1/H, \xi + 1/H)$. Then, in order to establish (8) it sufficies to show that

$$\max_{\substack{q\leq Q\\(a,q)=1}} \int_{I''_{q,a}} |S(\beta)|^2 d\beta \ll YL^{-2C-3}.$$

But this is given by Lemma 4. Hence, (2) is proved and the proof of Theorems 1 and 2 is complete.

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