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Calderón weights and the real interpolation method.

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Dedicated to Professor Baltasar Rodríguez Salinas

Abstract

We introduce a class of weights for a which a rich theory of real interpolation can be developed. In particular it led us to extend the commutator theorems associated to this method.

Interpolation theory is a powerful tool to study operators in function spaces. In particular it provides methods to obtain new estimates from initial estimates. The theory originated with the convexity theorem of Riesz-Thorin (1926, 1939), which states that if an operator T is bounded on the complex spaces $L^{p_i} \to L^{q_i}$, $i = 0, 1$, then it is also bounded for all the range of intermediate values of p. Therefore, we may conclude that T is bounded from $L^p \to L^q$, whenever there exists $\theta \in (0,1)$ such that

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \tag{$*$}$$

Moreover,

$$\|T\|_{p\to q} \le \|T\|_{p_0\to q_0}^{1-\theta} \|T\|_{p_1\to q_1}^{\theta}$$

(see [R], [T]). The method developed by Thorin to prove this theorem led A. Calderón (see [C], 1963) to introduce the complex interpolation method.

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At the foundations of the theory also lies an important theorem obtained by Marcinkiewicz (1939), which is based on real methods. Suppose that T is a bounded operator from $L^{p_i} \to L^{q_i,\infty}$, $i = 0,1$. Then, if $p_0 \neq p_1$ and $p \leq q$, T is bounded from $L^p \to L^q$, whenever there exists $\theta \in (0,1)$ such that (*) holds. In this case $\|T\|_{p\to q} \leq C_\theta \|T\|^{1-\theta}_{p_0\to q_0} \|T\|^{\theta}_{p_1\to q_1}$ for some constant C_θ. This theorem is the model for the real method of interpolation for abstract spaces, developed by Lions and Peetre (see [LP]).

In this paper we shall be concerned with the real method of interpolation. We shall describe, without proofs, a recent extension of the real method obtained by the authors in [BMR1] and [BMR2].

Throughout the paper we shall follow the notation and terminology of [BL].

Let us start by recalling some definitions. Given a compatible couple of Banach spaces $\bar{A} = (A_0, A_1)$ (i.e. there exists a topological vector space V such that both A_0, A_1 are continuously embedded in V), and an element $a \in A_0 + A_1$, we define the K-functional of Peetre by $K(t,a;\bar{A}) = \inf \|a_0\|_{A_0} + t\|a_1\|_{A_1}$, $t > 0$, where the inf runs over all possible decompositions $a = a_0 + a_1$ with $a_i \in A_i$. If $a \in A_0 \cap A_1$ we define the J-functional by $J(t,a;\bar{A}) = \max\{\|a\|_{A_0}, t\|a\|_{A_1}\}$, $t > 0$.

Given $\theta \in (0,1)$, and $p \geq 1$, we let

$$A_{\theta,p;K} = \{a \in A_0 + A_1; \left(\int_0^\infty \left(\frac{K(t,a;\bar{A})}{t^\theta}\right)^p \frac{dt}{t}\right)^{1/p} < \infty\}.$$

Likewise, we define $\bar{A}_{\theta,p;J}$ as the class of elements $a \in A_0 + A_1$ for which there exists a representation $a = \displaystyle\int_0^\infty \frac{a(t)}{t} dt$ with $a(t) \in A_0 \cap A_1$ (convergence in the $A_0 + A_1$ norm) and satisfying $t^{-\theta} J(t, a(t); \bar{A}) \in L^p(dt/t)$. For this class we consider the corresponding norms

$$\|a\|_{\theta,p;J} = \inf \left(\int_0^\infty \left(\frac{J(t,a(t);\bar{A})}{t^\theta \;\; - }\right)^p \frac{dt}{t}\right)^{1/p}$$

where the inf runs over all possible representations af a.

The following facts follow readily from the definitions:

i) $A_0 \cap A_1 \hookrightarrow A_{\theta,p;K} \hookrightarrow A_0 + A_1$ and $A_0 \cap A_1 \hookrightarrow A_{\theta,p;J} \hookrightarrow A_0 + A_1$

ii) $A_{\theta,p;J}$ and $A_{\theta,p;K}$ are interpolation spaces, i.e. if T is a linear operator from $A_0 + A_1 \to B_0 + B_1$ such that is bounded from $: A_i \to B_i$, $i = 0, 1$ then $T : A_{\theta,p,K} \to B_{\theta,p,K}$, $T : A_{\theta,p,J} \to B_{\theta,p,J}$ and $\|T\| \leq \|T\|_0^{1-\theta}\|T\|_1^{\theta}$.

Since the duals of spaces obtained by the K-method are naturally described by the J-method it becomes an important problem in the theory to study the equivalence of these two methods.

The fundamental lemma of interpolation implies that $A_{\theta,p,K} \subseteq A_{\theta,p,J}$. For the converse, one uses classical Hardy's inequalities and the following well known fact, if $a \in A_0 + A_1$ is an element for which there exists a representation $a = \int_0^\infty \frac{a(t)}{t} dt$, with $a(t) \in A_0 \cap A_1$ then

$$\frac{K(t, a; \bar{A})}{t} \leq S\left(\frac{J(x, a(x); \bar{A})}{x}\right)(t),$$

where the operator S is the Calderón operator, defined for measurable functions by

$$\begin{aligned} Sf(t) &= \int_0^\infty \min\{1/x, 1/t\} f(x)dx \\ &= \frac{1}{t}\int_0^t f(x)dx + \int_t^\infty \frac{f(x)}{x}dx \\ &= P(f)(t) + Q(f)(t). \end{aligned}$$

Several different extensions of these interpolation functors have been considered before. The most studied classes of interpolation spaces are associated to the so called "functional parameter" and the "quasipower" case. The spaces are defined by a slighty more general class of weights than the powers $t^{-\theta}$ and the study of interpolation for these classes of "weights" was apparently initiated in [K] and continued in [G] and many other papers, cf. [BK].

Let w be a weight on $(0, \infty)$, i.e., $w > 0$ a.e. with respect to the Lebesgue measure. We recall that a weight is a *quasipower weight* (see [BK]) if there exists a positive constant C such that $C^{-1}w \leq Sw \leq Cw$, (briefly $Sw \sim w$). We say that a weight w is a *functional parameter or a Kalugina weight* if there exists a positive constant C and a C^1 positive function φ such that $C^{-1} \leq w\varphi \leq C$ and $\alpha\varphi(t) \leq t\varphi'(t) \leq \beta\varphi(t)$, for some $0 < \alpha < \beta < 1$ and for all $t > 0$. An integration by parts shows that a Kalugina weight is a quasipower weight.

In what follows we shall indicate a more general construction, which allows us to connect real interpolation with the theory of weighted norm inequalities initiated by Muckenhoupt and developed by many authors over the last 25 years.

Given a weight w, (i.e. a measurable function, $w > 0$ a.e.) we denote by $L^p(w)$, $1 \le p \le \infty$, the classes of Lebesgue measurable functions f defined on the interval $(0, \infty)$ such that

$$\|f\|_{L^p(w)} = \left(\int_0^\infty f(t)^p w(t)dt\right)^{1/p} < +\infty.$$

For $p = \infty$ the corresponding space will be $w^{-1}L^\infty$, that is, the space consisting of functions f such that

$$\|f\|_{L^\infty(w)} = \|fw\|_\infty < \infty.$$

Let w be a weight and let $1 \le p \le \infty$. We define $\bar{A}_{p,w,K}$ as the class of vectors $a \in A_0 + A_1$ for which the function $t^{-1}K(t, a; \bar{A}) \in L^p(w)$. For $a \in A_{p,w,K}$ we denote

$$\|a\|_{\bar{A}_{p,w,K}} = \left(\int_0^\infty \left(\frac{K(t, a; \bar{A})}{t}\right)^p w(t)dt\right)^{1/p}$$

If we consider the J-method of interpolation, we define $\bar{A}_{p,w,J}$ as the class of elements $a \in A_0 + A_1$ for which there exists a representation $a = \int_0^\infty \frac{a(t)}{t}dt$ with $a(t) \in A_0 \cap A_1$ satisfying $t^{-1}J(t, a(t); \bar{A}) \in L^p(w)$. For this class we consider the corresponding norms

$$\|a\|_{\bar{A}_{p,w,J}} = \inf \left(\int_0^\infty \left(\frac{J(t, a(t); \bar{A})}{t}\right)^p w(t)dt\right)^{1/p}$$

where the inf runs over all possible representations of a.

The classical scales of real interpolation spaces correspond to the power weights $w = t^{p-p\theta-1}$. It is not hard to convince oneself that the weights we have just introduced are more general than the Kalugina or quasi-power weights. Indeed, let

$$w(t) = \begin{cases} 1/\sqrt{t}, & \text{if } 0 < t \le 1\,; \\ 1/\sqrt{t-1}, & \text{if } 1 < t \end{cases}$$

then we compute

$$Sw(t)=\begin{cases}\pi-2+\frac{4}{\sqrt{t}}, & \text{if } 0<t\le 1\ ;\\ \frac{2}{t}+\frac{2\sqrt{t-1}}{t}+2\arctan\frac{1}{\sqrt{t-1}}, & \text{if } 1<t\end{cases}$$

and we see that w is not a quasipower weight and consequently it is not a Kalugina weight either (this example is suggested in [HS]). However, it is easy to verify that $Sw \le Cw$ and therefore, w verifies the $\mathcal{C}_1$ condition (cf. Definition 1 below).

In our work we show that the weights that control the Calderón operator can be used to develop a rich theory of interpolation and connects the subject with the theory of weighted norm inequalities for classical operators.

At the basis of our development are the classical results by Muckenhoupt (see [Mu], [Ma]), extending Hardy's inequalities. Recall from [Mu] that:
i) $Pf \in L^p(w)$ for all $f \in L^p(w)$ $(1 \le p < \infty)$ if and only if there exists a constant $C>0$ such that for almost all $t>0$

$$\left(\int_t^\infty \frac{w(x)}{x^p}dx\right)^{1/p}\left(\int_0^t w(x)^{-p'/p}dx\right)^{1/p'} \le C \qquad (M_p)$$

for $1<p<\infty$, or

$$\int_t^\infty \frac{w(x)}{x}dx \le Cw(t) \qquad (M_1)$$

for $p=1$, and ii) $Qf \in L^p(w)$ for all $f \in L^p(w)$ $(1 \le p < \infty)$ if and only if there exists a constant $C>0$ such that for almost all $t>0$

$$\left(\int_0^t w(x)dx\right)^{1/p}\left(\int_t^\infty \frac{w(x)^{-p'/p}}{x^{p'}}dx\right)^{1/p'} \le C. \qquad (M^p)$$

for $1<p<\infty$, or

$$\frac{1}{t}\int_0^t w(x)dx \le Cw(t). \qquad (M^1)$$

for $p=1$.

Definition 1 We say that a weight $w \in \mathcal{C}_p$ if it satisfies the conditions M_p and M^p simultaneously. We say that a weight $w \in \mathcal{C}_\infty$ if and only if $w^{-1} \in \mathcal{C}_1$, or equivalently, if S is bounded on $w^{-1}L^\infty$.

It is clear from our previous discussion that

$$A_{p,w,K} = A_{p,w,J} \qquad \Longleftrightarrow \qquad S : L^p(w) \to L^p(w).$$

The classes $\mathcal{C}_p$ enjoy nice properties which we collect in the following

Proposition 2 The following assertions are true:

i) S is self-adjoint linear and positive

ii) $w \in \mathcal{C}_p \Leftrightarrow w^{-p'/p} \in \mathcal{C}_{p'}$

iii) The class $\mathcal{C}_1$ is exactly the class of weights $w \geq 0$ such that $Sw \leq Cw$ for some positive constant C.

This properties lead us to apply the strong machinery developed by Rubio de Francia (see [GR]), and in particular to use Rubio de Francia's algorithm:

Proposition 3 (see [GR]). Let $1 \leq p < \infty$. Given $f \geq 0$, $f \in L^p(w)$ there exist a measurable function $g \geq 0$ and a positive constant C, such that

i) $f \leq g$

ii) $\|g\|_{L^p(w)} \leq 2\|f\|L^p(w)$

iii) $Sg \leq Cg$ ($C = 2\|S\|$ is enough).

As a consequence we show that the $\mathcal{C}_p$ classes of weights satisfy the following properties: factorization, a reverse Hölder's type inequalities and extrapolation.

Proposition 4 Let $1 \leq p < \infty$. A weight $w \in \mathcal{C}_p$ if and only if there exist two weights w_0, $w_1 \in \mathcal{C}_1$ such that $w = w_0 w_1^{1-p}$.

One implication is merely an application of Rubio de Francia's factorization of weights theorem. For the converse we only need to adjust a simple computation.

Proposition 5 If $w \in \mathcal{C}_p$ then there exists $\epsilon > 0$ such that $x^{-\epsilon p} w(x) \in M^p$ and $x^{\epsilon p} w(x) \in M_p$.

Due to factorization, the only thing we have to prove is the case $p = 1$ and in this case the corresponding result is that there exists $\epsilon > 0$ such that, for some constant $C > 0$, w satisfies

$$\int_0^t \left(\frac{t}{x}\right)^{\epsilon} w(x)dx + \int_t^{\infty} \left(\frac{t}{x}\right)^{1-\epsilon} w(x)dx \leq Ctw(t)$$

for all $t > 0$. This is easily proved by using the trivial facts

$$S = P + Q = P \circ Q = Q \circ P$$

and reiteration. Proposition 5 implies that the weights w support stronger integrability conditions in 0 and in ∞.

Proposition 6 Let T be a sublinear operator acting on functions defined on $(0, +\infty)$. Let $1 \leq r < +\infty$, $1 < p < \infty$. Suppose that T is bounded on $L^r(w)$ (respectively, T is of weak type (r,r)), for every weight $w \in \mathcal{C}_r$ with norm that depends only upon the $\mathcal{C}_r$-constant for w, then T is bounded on $L^p(w)$ (respectively, T is of weak type (p,p)), for all weights $w \in \mathcal{C}_p$ with a norm that depends only upon the $\mathcal{C}_p$-constant for w.

It is also possible to extrapolate in the case $r = \infty$, using the following extrapolation theorem by García-Cuerva ([GC]):

Proposition 7 (see [GC]). Let S be a positive sublinear operator and let T be a mapping satisfying the following condition: Every time that S is bounded in vL^{∞}, for some $v \geq 0$, T is also bounded in vL^{∞}, with norm depending only on that of S. Let $1 \leq p < \infty$ and $w \geq 0$. Suppose that S is bounded in $L^p(w)$. Then T is also bounded in $L^p(w)$ with norm depending only on that of S.

Applying the last result to the Calderón operator we obtain:

Corollary 8 If T is bounded in $v^{-1}L^{\infty}$, for any weight $v \in \mathcal{C}_{\infty}$, then T is bounded in $L^p(w)$, for all $w \in \mathcal{C}_p$ and for all $1 \leq p < \infty$.

Next we translate the corresponding properties to interpolation theory.

Proposition 9 The following assertions are true:

i) The spaces $\bar{A}_{p,w,K}$ and $\bar{A}_{p,w,J}$ are intermediate spaces, i.e., we have $A_0 \cap A_1 \hookrightarrow \bar{A}_{p,w,K} \hookrightarrow A_0 + A_1$ and $A_0 \cap A_1 \hookrightarrow \bar{A}_{p,w,J} \hookrightarrow A_0 + A_1$, $1 \leq p \leq \infty$.

ii) $A_0 \cap A_1$ is always dense in $\bar{A}_{p,w,J}$, $1 \leq p < \infty$.

iii) The spaces $\bar{A}_{p,w,K}$ and $\bar{A}_{p,w,J}$ are interpolation spaces, $1 \leq p \leq \infty$.

iv) $\bar{A}_{p,w,K} \hookrightarrow \bar{A}_{p,w,J}$, $1 \leq p \leq \infty$.

v) Let $1 \leq p \leq \infty$, then $\bar{A}_{p,w,J} \hookrightarrow \bar{A}_{p,w,K}$ for all compatible pairs of Banach spaces if and only if the weight w is in C_p.

Remark 10 In [BK] the authors showed the necessity of the C_p condition by means of considering the couple $(L^1(dt), L^1(dt/t))$ plus the technical conditions

$$L^p(w) \hookrightarrow L^1 + L^1(dt/t), \qquad L^p(w) \cap L^{\mathrm{loc}}(dt/t) \neq \{0\}$$

(see [BK], condition (3.4.2) and Lemma 3.4.4). It is instructive to see that one could use essentially any pair of rearrangement invariant spaces without any other assumptions. More precisely, let $w > 0$ a.e. and let $\bar{A} = (A_0, A_1)$ be a compatible pair of r.i. spaces such that the corresponding fundamental functions satisfy

$$\lim_{s\to 0} \frac{\phi_{A_0}(s)}{\phi_{A_1}(s)} = 0, \qquad \lim_{s\to\infty} \frac{\phi_{A_0}(s)}{\phi_{A_1}(s)} = \infty.$$

Suppose that for some $1 \leq p \leq \infty$, $A_0 \cap A_1 \hookrightarrow \bar{A}_{p,w,J} \hookrightarrow \bar{A}_{p,w,K}$, then the weight $w \in C_p$.

As a consequence we obtain

Corollary 11 Suppose that $w \in C_p$, and $A_0 \cap A_1$ is dense in A_0 and A_1, then,

$$(A_0, A_1)'_{p,w,K} \cong (A'_1, A'_0)_{p',w^{-p'/p},K} \qquad 1 < p < \infty,$$
$$(A_0, A_1)'_{1,w,K} \cong (A'_1, A'_0)_{\infty,w^{-1},K} \qquad p = \infty$$

Next we shall consider reiteration results. We shall begin by remarking that if $1 \leq p_0, p_1 \leq \infty$ and $w_0 \in C_{p_0}$, $w_1 \in C_{p_1}$, then, the couples

$$\overline{A}_{\overline{p},\overline{w}} = (\overline{A}_{p_0,w_0}, \overline{A}_{p_1,w_1}) \text{ and } L_{\overline{p}}(\overline{w}) = (L_{p_0}(w_0), L_{p_1}(w_1))$$

are almost "bl-pseudoretracts" of each other (in the sense indicated in [Cw], p.125]). (We recall that in case of $p_0 = \infty$ and/or $p_1 = \infty$ the spaces to be considered are $w_0^{-1}L^\infty$ and/or $w_1^{-1}L^\infty$).

As a consecuence we obtain that

$$K(t,a,\overline{A}_{\overline{p},\overline{w}}) \sim K(t, K(t,a)/t, L_{\overline{p}}(\overline{w})), \qquad a \in \Sigma(\overline{A}_{\overline{p},\overline{w}}). \qquad (\triangle)$$

and hence

Proposition 12 $\overline{A}_{\overline{p},\overline{w}}$ is a Calderón pair. That is, for any $a, b \in \Sigma(\overline{A}_{\overline{p},\overline{w}})$ with

$$K(t,b,\overline{A}_{\overline{p},\overline{w}}) \leq K(t,a,\overline{A}_{\overline{p},\overline{w}})$$

for all $t > 0$, there exists an operator U bounded in $\overline{A}_{\overline{p},\overline{w}}$ with $Ua = b$.

As an application of the factorization of $\mathcal{C}_p$ weights we prove the following.

Proposition 13 Let $1 < p < \infty$ and $w \in \mathcal{C}_p$. Then, there exist $w_0, w_1 \in \mathcal{C}_1$ such that

$$(\overline{A}_{1,w_0}, \overline{A}_{\infty,w_1^{-1}})_{1/p',p} = \overline{A}_{p,w}.$$

Moreover, $(\triangle)$ can be also used to prove reiteration results. An easy consequence of $(\triangle)$ is the following extension of the classical reiteration theorem:

$$(\overline{A}_{p_0,w_0}, \overline{A}_{p_1,w_1})_{p,w,K} = \overline{A}_{(L_{p_0}(w_0),L_{p_1}(w_1))_{p,w,K}} \qquad (\diamond)$$

where the space on the right hand side consists of the elements $a \in \Sigma(\overline{A})$ for which $K_a \in (L_{p_0}(w_0), L_{p_1}(w_1))_{p,w,K}$. Note that in order for $(\diamond)$ to be satisfied it is necessary that the weights w_0 and w_1 be in the $\mathcal{C}_{p_0}$ and $\mathcal{C}_{p_1}$ classes respectively, but it is not relevant whether the weight w belongs or not to the $\mathcal{C}_p$ class.

We also note that a corollary of $(\diamond)$ is the reiteration formula

$$(\overline{A}_{p_0,w_0}, \overline{A}_{p_1,w_1})_{\theta,p} = \overline{A}_{p,w}$$

where $1 \leq p_0, p_1 < \infty$, $w = w_0^{p(1-\theta)/p_0} w_1^{p\theta/p_1}$, $\dfrac{1}{p} = \dfrac{1-\theta}{p_0} + \dfrac{\theta}{p_1}$.

In particular we obtain the classical reiteration formula:

$$(A_{\theta_0,p_0}, A_{\theta_1,p_1})_{\theta,q} = (A_0, A_1)_{\eta,q}$$

where $\eta = (1-\theta)\theta_0 + \theta\theta_1$. Note that the second index is not important for reiteration. Thus, for power weights, that is, in the classical Lions-Peetre theory, "extrapolation" in the sense of Rubio de Francia is not interesting. In the setting of $\mathcal{C}_p$-weights, however, the following extrapolation result holds.

Proposition 14 Let $\bar{A}$, $\bar{B}$ be two compatible couples of Banach spaces, and let T be a linear operator bounded from $\bar{A}_{p,w,K}$ into $\bar{B}_{p,w,K}$ for some p, $1 \leq p < \infty$, and for all $w \in \mathcal{C}_p$ with norm that depends only upon the $\mathcal{C}_p$-constant for w. Then T is also bounded from $\bar{A}_{q,v,K}$ into $\bar{B}_{q,v,K}$ for any q, $1 < q < \infty$, and for all $v \in \mathcal{C}_q$ with norm that depends only upon the $\mathcal{C}_q$-constant for w.

We should mention here that in order to prove this proposition we have to extrapolate from the validity of the inequalities

$$\int_0^\infty \left(\frac{K(t,Ta;\bar{B})}{t}\right)^p w(t)dt \leq C \int_0^\infty \left(\frac{K(t,a;\bar{A})}{t}\right)^p w(t)dt$$

for any weight $w \in \mathcal{C}_p$ that the same estimates hold for any q and any $v \in \mathcal{C}_q$.

On the other hand, using the extrapolation theory developed in [JM], we show, under mild extra restrictions, a sharper version of the preceding proposition. In order to describe the results we introduce the following class of weights.

We shall say that a weight w belongs to the class $\mathcal{CB}_p$, $1 \leq p < \infty$, if there exists $C > 0$ such that for all $t > 0$ we have

$$\int_t^\infty (\frac{t}{x})^p w(x)dx \leq C \int_0^t w(x)dx.$$

and

$$\int_0^t Pw(x)dx \leq C \int_0^t w(x)dx.$$

It follows from [AM], [N1] and [N2] that a weight w belongs to the class $\mathcal{CB}_p$ if and only if $Sf \in L^p(w)$, $1 \leq p < \infty$, for all $f \in L^p(w)$, with f decreasing. For such weights we denote by $\|w\|_{\mathcal{CB}_p}$ the infimum of the constants C satisfying the inequalities above. Then, we have

Proposition 15 Let $\alpha > 0$, $1 < p < \infty$, and let $\bar{A}$ and $\bar{B}$ be Banach pairs. Suppose that T is a bounded linear operator mapping $T : \bar{A}_{w,p;K} \to \bar{B}_{w,p;K}$, for every $w \in \mathcal{CB}_p$, with

$$\|T\|_{\bar{A}_{w,p;K} \to \bar{B}_{w,p;K}} \leq C \|w\|^\alpha_{\mathcal{CB}_p},$$

then, for $1 < q < \infty$, and for all $w \in C_q$, we have

$$T : \bar{A}_{w,q;K} \to \bar{B}_{w,q;K}.$$

We give applications of reversed Hölder's type inequalities in the contex of the abstract theory of commutators. This theory was initiated by Rochberg and Weiss [RW] for the complex method and Jawerth, Rochberg and Weiss [JRW] for the real method of interpolation. It has been intensively developed over the last decade. A survey of the earlier work is given in [CJMR]. Let us recall that associated to the construction of intermediate spaces by the K and J methods, there certain mappings Ω_K, Ω_J, which are in general non-linear and unbounded and defined by using almost optimal decompositions. Let us review some of the basic facts. Suppose that $\bar{A} = (A_0, A_1)$ is a Banach couple, and let $a \in A_0 + A_1$, and $t > 0$. We shall say that a decomposition $a = a_0(t) + a_1(t)$ is almost optimal (for the K method) if

$$\|a_0(t)\|_0 + t\|a_1(t)\|_1 \leq cK(t, a; \bar{A})$$

(here c is a constant whose value is fixed during our discussion, for example $c = 2$). It is clear that we can always choose a measurable almost optimal decomposition, i.e., the function from $(0, \infty)$ to $A_0 + A_1$ given by $t \to a_0(t) + a_1(t)$ is measurable. We write $D_K(t, \bar{A})a = D_K(t)a = a_0(t)$. It is clear that the function $D_K(t)a$ can be choosen to be homogeneous in a. Suppose that $w \in C_p$, $1 < p < \infty$, then we can define the operator Ω_K for the elements of $\bar{A}_{p,w,K}$ as

$$\Omega_{K,\bar{A}}a = \Omega_K a = \int_0^1 D_K(t)a\frac{dt}{t} - \int_1^\infty (I - D_K(t)a)\frac{dt}{t} \in A_0 + A_1.$$

Although these operators are unbounded their commutators with bounded operators in the scale are bounded. Let us develop this point in more detail. Suppose that T is a bounded linear operator between the Banach couples $\bar{A}$ and $\bar{B}$. Given $a \in \bar{A}_{p,w,K}$, we can apply the operators $\Omega_{K,\bar{A}}$ and $\Omega_{K,\bar{B}}$ before and after applying T, respectively. This leads to the study of the commutator

$$[T, \Omega_K]a = (T\Omega_{K,\bar{A}} - \Omega_{K,\bar{B}}T)a.$$

We then have the following "commutator theorem".

Proposition 16 If the weight $w \in \mathcal{C}_p$, for $1 \leq p \leq \infty$, then the commutator is bounded from $\bar{A}_{p,w,K}$ into $\bar{B}_{p,w,K}$.

The previous result, applied to the identity operator, shows that if $\tilde{\Omega}_K$ is defined as before, but using a different almost optimal decomposition, then $\Omega_K - \tilde{\Omega}_K$ is also bounded from $\bar{A}_{p,w,K}$ into $\bar{A}_{p,w,J}$. If we define as usual $\text{Dom}\, \Omega_{K,A} = \{a \in \bar{A}_{p,w,K}; \Omega a \in \bar{A}_{p,w,K}\}$, with $\|a\|_D = \|a\|_{\bar{A}_{p,w,K}} + \|\Omega\|_{\bar{A}_{p,w,K}}$ then the space $\text{Dom}\, \Omega_{K,A}$ is well defined and independent of choice of Ω_K. In fact $\text{Dom}\, \Omega_{K,A}$ defines an interpolation method that can be explicitly computed and, in particular, is a special case of the spaces studied in our work (cf. [CJMR] and the references quoted therein).

A typical application of the theory is the following well known commutator theorem originally due to Coifman, Rochberg and Weiss (cf. [JRW] and the references quoted therein).

Proposition 17 Let $p > 1$. For $b \in BMO(\mathbf{R}^n)$ define $M_b f = bf$ and let T be a Calderón-Zygmund operator. Then, the commutator $[T, M_b]$ is bounded in $L^p(\mathbf{R}^n)$.

In a similar manner we can associate with the J method operators Ω_J which commute with bounded operators. We now review the basic definitions. Let a be an element in $\bar{A}_{p,w,J}$, then we select over all possible admissible decompositions $a = \int_0^\infty t^{-1} a(t) dt$ a choice of $a(t)$ for which the infimum is almost attained in the definition of $\|a\|_{\bar{A}_{p,w,J}}$. We call such decomposition an almost optimal J-decomposition and denote $a(t) = D_J(t, a; \bar{A})$. Thus, $a = \int_0^\infty t^{-1} D_J(t) a dt$ and

$$\left(\int_0^\infty \left(\frac{J(t, a(t); \bar{A})}{t}\right)^p w(t) dt\right)^{1/p} \leq c\|a\|_{\bar{A}_{p,w,J}}$$

We define the non-linear operator

$$\Omega_{J,\bar{A}} a = \Omega_J a = \int_0^\infty a(t) \log t \frac{dt}{t}.$$

Let $1 < p < \infty$, then the existence of $\Omega_J a$ for the elements of $\bar{A}_{p,w,J}$ follows from an application of a reverse Hölder's type inequality which insures that w satisfies

$$\int_0^\infty \min\left\{1, \frac{1}{t^{p'}}\right\} w^{-p'/p} |\log t|^n dt < \infty$$

It is readily seen that the operator Ω_J is bounded from $\bar{A}_{p,w,J}$ into $\bar{A}_{p,v,J}$, where v is the weight $v(t) = (1+|\log x|)^{-p}w(x)$. We remark here that by using a strong form of the fundamental lemma of interpolation it is not difficult to prove that $\int_0^t s^{-1}D_J(s)ads$ is an almost optimal decomposition for the K-method, i.e.,

$$\begin{aligned} a_0(t) &= \int_0^t s^{-1}D_J(s)ads \\ a_1(t) &= \int_t^\infty s^{-1}D_J(s)ads \end{aligned}$$

and consequently, $\Omega_{J,\bar{A}}a = -\Omega_{K,\bar{A}}a$.

Therefore commutator theorems for the J method can be deduced from the corresponding ones for the K method. For example we have.

Proposition 18 Let $1 < p < \infty$. If $T : \bar{A} \to \bar{B}$ and w is a weight in the class C_p then the commutator $[T, \Omega_J]$ is bounded from $\bar{A}_{p,w,J}$ into $\bar{B}_{p,w,J}$.

Higher order commutators can be also treated in this setting. For $n \in \mathbf{N}$ the operators $\Omega_{K,\bar{A};n}$ associated with the almost optimal decomposition $D_K(t)a$ are defined by

$$\Omega_{K,\bar{A};n}a = \frac{1}{(n-1)!}\left(\int_0^1 (\log t)^{n-1}D_K(t)a\frac{dt}{t} - \int_1^\infty (\log t)^{n-1}(I - D_K(t))a\frac{dt}{t}\right).$$

Using a reverse Hölder type inequality we see that these operators are well defined on the $\bar{A}_{p,w,K}$ spaces. The following result extends to weighted interpolation spaces a commutator theorem obtained by Rochberg and one of us in the unweighted case (cf. [MiR] and the references quoted therein).

Proposition 19 Let $1 < p < \infty$. Let w be in C_p. Let $T : \bar{A} \to \bar{B}$ a bounded linear operator. For $n = 0, 1, 2, \ldots$ define

$$C_n a = \begin{cases} T, & \text{if } n = 0 \\ [T, \Omega_{K;1}]a, & \text{if } n = 1 \\ \ldots & \\ [T, \Omega_{K;n}]a + \sum_{k=1}^{n-1} \Omega_{K;k}C_{n-k}a, & \text{if } n \geq 2 \end{cases}$$

Then the operator C_n is bounded from $\bar{A}_{p,w,K} \to \bar{B}_{p,w,K}$.

We can extend the results above to a more general setting, the framework of commutators of fractional order. In the sequel we shall only consider $\alpha > 0$ (obviously we are in a complex situation for $\alpha \notin \mathbf{N}$, but complex interpolation methods are not used at all in our context). We begin by defining the non linear operator Ω_α. Let $a \in A_0 + A_1$

$$\Omega_\alpha a = \frac{1}{\Gamma(\alpha)}\left[\int_0^1 \log(t)^{\alpha-1} D_K(t) a \frac{1}{t} dt - \int_1^\infty \log(t)^{\alpha-1}(I - D_K(t)) a \frac{1}{t} dt\right]$$

(we mean $\log(t)^{\alpha-1} = |\log(t)|^{\alpha-1} e^{i \text{ sig } \log(t)}$).

The existence of $\Omega_\alpha a$ for the elements of $\bar{A}_{p,w,K}$ as well as the estimates for the commutators are based in the following lemma.

Lemma 20 Let n be a natural number and let $0 < \theta < 1$. There exists a constant $C = C(n, \theta)$ such that for any $a \in \bar{A}_{p,w,K}$ and for all $t > 0$ we have:

i) $P^{(n}(|\log(x)|^{\theta-1} K(x, a; \bar{A}))(t) \leq C P^{(n}(K(x, a; \bar{A}))(t)$

ii) $Q^{(n}(|\log(x)|^{\theta-1} K(x, a; \bar{A}))(t) \leq C Q^{(n}(K(x, a; \bar{A}))(t)$

iii) If $a \in \bar{A}_{p,w,K}$ and $0 < \alpha < 1$, then $\Omega_\alpha a$ is well defined and moreover if $a = a_0(t) + a_1(t)$ is an almost optimal decomposition of a we have

$$\Omega_\alpha a + \frac{\log(t)^\alpha}{\Gamma(\alpha+1)} a = \int_0^t \frac{\log(x)^{\alpha-1}}{\Gamma(\alpha)} a_0(x) \frac{dx}{x} - \int_t^\infty \frac{\log(x)^{\alpha-1}}{\Gamma(\alpha)} a_1(x) \frac{dx}{x}.$$

iv) If $0 < \alpha < 1$, then the commutator $[T, \Omega_\alpha]$ is bounded from $\bar{A}_{p,w,K} \to \bar{B}_{p,w,K}$, for any bounded linear operator $T : \bar{A}_{p,w,K} \to \bar{B}_{p,w,K}$.

As a consequence we get.

Proposition 21 Let $1 < p < \infty$. Let w be a weight in C_p. Let $T : \bar{A} \to \bar{B}$ a bounded linear operator. For $\alpha \geq 0$ we define

$$C_\alpha a = \begin{cases} T, & \text{if } \alpha = 0 \\ [T, \Omega_\alpha] a, & \text{if } 0 < \alpha \leq 1 \\ [T, \Omega_\alpha] a + \frac{\Omega_1}{\alpha-1} C_{\alpha-1} a, & \text{if } 1 < \alpha \leq 2 \\ [T, \Omega_\alpha] a + \frac{2\Omega_1}{\alpha-1} C_{\alpha-1} a + \frac{2\Omega_2}{(\alpha-1)(\alpha-2)} C_{\alpha-2} a, & \text{if } 2 < \alpha \leq 3 \\ \dots & \\ [T, \Omega_\alpha] a + \sum_{k=1}^{n-1} \frac{\binom{n-1}{k}}{\binom{\alpha-1}{k}} \Omega_k C_{\alpha-k} a, & \text{if } 3 < \alpha \end{cases}$$

where $n-1<\alpha\leq n$. Then the operator C_α is bounded from $\bar{A}_{p,w,K}\to\bar{B}_{p,w,K}$.

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