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Exact controllability for the wave equation in domains with variable boundary.

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Abstract

This paper is concerned with the problem of exact boundary controllability for the equation:

$$u'' - \Delta u = 0 \text{ in } \widehat{Q}$$

where $\widehat{Q}$ is non cylindrical domain of $\mathbf{R}^{n+1}$. The result is obtained by transforming the problem in $\widehat{Q}$ in a problem defined in a cylindrical domain Q and the showing that these two problems are equivalent. The result in Q was studied by the author in an earlier paper applying the HUM of J. L. Lions.

1 Introduction

Let Ω be an open bounded set of $\mathbf{R}^n$ with boundary Γ of class C^2, which, without loss of generality, can be assumed containing the origin of $\mathbf{R}^n$, and $k : [0,\infty[\mapsto]0,\infty[$ a continuously differentiable function. Let us consider the subsets Ω_t of $\mathbf{R}^n$ given by

$$\Omega_t = \{x \in \mathbf{R}^n; x = k(t)y, y \in \Omega\}, \; 0 \le t \le T < \infty$$

whose boundaries are denoted by Γ_t, and $\widehat{Q}$ the non cylindrical domain of $\mathbf{R}^{n+1}$,

$$\widehat{Q} = \bigcup_{0<t<T} \Omega_t \times \{t\} \tag{1.1}$$

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with lateral boundary

$$\widehat{\Sigma} = \bigcup_{0<t<T} \Gamma_t \times \{t\}.$$

We have the following system:

$$\begin{aligned} &u'' - \Delta u = 0 \quad \text{in} \quad \widehat{Q} \\ &u = v \quad \text{on} \quad \widehat{\Sigma} \\ &u(0) = u^0, u'(0) = u^1 \quad \text{in} \quad \Omega_0 \end{aligned} \tag{*}$$

where u'' stands for $\frac{\partial^2 u}{\partial t^2}$ and $u(0), u'(0)$ denote, respectively, the functions $x \mapsto u(x,0)$, $x \mapsto u'(x,0)$. Here v is the control variable, that is, we act on the system (*) through the lateral boundary $\widehat{\Sigma}$.

The problem of exact controllability for system (*) states as follows: given $T > 0$ large enough, is it possible, for every initial data $\{u^0, u^1\}$ in an appropriate space to a find a control v driving the system to rest at time T, i.e., such that the solution $u(x,t)$ of (*) satisfies

$$u(T) = 0,\ u'(T) = 0? \tag{1.2}$$

In this paper we show that system (*) is exactly controllable. Our approach consists first in transforming (*), by using $k(t)$, in a system defined in the cylindrical domain $Q = \Omega \times]0,T[$. This system will have the following form:

$$\begin{aligned} &w'' - \frac{\partial}{\partial y_i}\left(a_{ij}(y,t)\frac{\partial w}{\partial y_j}\right) + b_i(y,t)\frac{\partial w'}{\partial y_i} + d_i(y,t)\frac{\partial w}{\partial y_i} = 0 \text{ in } Q \\ &w = g \text{ on } \Sigma = \Gamma \times]0,T[\\ &w(0) = w^0, w'(0) = w^1 \text{ in } \Omega. \end{aligned} \tag{**}$$

(Here and in what follows the summation convention of repeated indices is adopted). Then we show that the study of the exact controllability problem for (*) reduces to the study of the controllability for system (**). The second v will be expressed in function of a weak solution θ of the wave equation in the non cylindrical domain $\widehat{Q}$. For that, an appropriate change of variables is needed.

The exact controllability for system (**) was analised by the author in [14]. The Hilbert Uniqueness Method (HUM) of J. L. Lions [10], [11] is used in this analysis. The application of this method can be also found in C. Fabre and J. P. Puel [4], J. P. Puel [15], J. P.Puel and E. Zuazua [16], E. Zuazua [19], [20], [21], V. Komornik [7] and L. A. Medeiros [13].

One can find non cylindrical domains $\widehat{Q}$ like those we have considered in (*) in R. Dal Passo and M. Ughi [3] and in J. Límaco [8], both in the parabolic case and when Ω is the unit ball of $\mathbf{R}^n$. Other models in non cylindrical domains can be found in J. P. Zolesio [18].

The existence of solutions of the initial boundary value problem for the nonlinear wave equation in general non cylindrical domains $\widehat{Q}$ was studied among other author by J. L. Lions [9], L. A. Medeiros [12], when $\widehat{Q}$ is increasing and by C. Bardos and J. Cooper [2] when $\widehat{Q}$ is time like. A. Inoue [6] also analised this type of problems. The linear case was treated by J. Sikorav [17] when $\widehat{Q}$ is time like. He used tools of Differential Topology. The non cylindrical domain $\widehat{Q}$ that we have considered in (*) is time like but it is not necessarily increasing or decreasing. This occurs because the derivative $k'(t)$ does not have sign condition. $\widehat{Q}$ is named time like when the unit normal vector $\eta = (\eta_x, \eta_t)$ to $\widehat{\Sigma}$, directed towards the exterior of $\widehat{Q}$, satisfies $|\ \eta_t\ |<|\ \eta_x\ |$. The exact internal controllability problem for the wave equation in non cylindrical domains was treated by C. Bardos and G. Cheng [1]. They did not use HUM.

The paper is organized as follows:

2. Main result

3. Summary of Results on the Cylinder

4. Spaces of the Non Cylindrical Domain

5. Proof of the Main Result

2 Main result

Let us introduce some notations (cf. J. L. Lions [11]). Let $y^0 \in \mathbf{R}^n, m(y)$ the function $y - y^0$ and $\nu(y)$ the unit normal vector at $y \in \Gamma$, directed towards the exterior of Ω. We consider the sets

$$\Gamma(y^0) = \{y \in \Gamma; m(y) \cdot \nu(y) \geq 0\}, \quad \Sigma(y^0) = \Gamma(y^0) \times]0, T[$$

and the corresponding sets in the (x,t)-coordinates,

$$\Gamma_t(y^0) = \left\{x \in \Gamma_t; x = k(t)y, y \in \Gamma(y^0)\right\}, \quad 0 \le t \le T$$

$$\widehat{\Sigma}(y^0) = \bigcup_{0<t<T} \Gamma_t(y^0) \times \{t\}$$

In the definition of $\Gamma(y^0)$, $\cdot$ denotes the scalar product in $\mathbf{R}^n$. We represent by $\eta = (\eta_x, \eta_t)$ the unit normal vector to $\widehat{\Sigma}$, directed towards the exterior of $\widehat{Q}$ and by ν^* the vector $\eta_x / \mid \eta_x \mid$. Let

$$R(y^0) = \sup_{y\in\Omega} \mid m(y) \mid \qquad M = \sup_{y\in\Omega} \mid y \mid$$

and λ_1 the first eigenvalue of the spectral problem $-\Delta\varphi = \lambda\varphi, \varphi \in H_0^1(\Omega)$.

We make the following assumptions:

$$\text{The boundary } \Gamma \text{ of } \Omega \text{ is } C^2 \tag{$H1$}$$

and concerning the function k,

$$k \in W^{3,\infty}_{loc}(]0,\infty[) \tag{$H2$}$$

$$0 < k_0 = \inf_{t\ge 0} k(t), \quad \sup_{t\ge 0} k(t) = k_1 < \infty \tag{$H3$}$$

$$\sup_{t\ge 0} \mid k'(t) \mid = \tau < \frac{1}{M} \tag{$H4$}$$

$$\ell_1 = \int_0^\infty \mid k' \mid dt < \infty, \qquad \ell_2 = \int_0^\infty \mid k'' \mid dt < \infty \tag{$H5$}$$

Hypothesis (H4) implies that the non cylindrical domain $\widehat{Q}$ is time like. The unit outer normal vector $\eta(x,t)$ to $\widehat{\Sigma}$ is given in Remark 4.1.

All the scalar function considered in the paper will be real-valued.

In $\widehat{Q}, \widehat{Q}$ defined by (1.1), we have the following system:

$$\begin{aligned} & u'' - \Delta u = 0 \quad \text{in} \quad \widehat{Q} \\ & u = \begin{cases} v & \text{on} \quad \widehat{\Sigma}(y^0) \\ 0 & \text{on} \quad \widehat{\Sigma}\backslash\widehat{\Sigma}(y^0) \end{cases} \\ & u(0) = u^0, u'(0) = u^1 \end{aligned} \tag{2.1}$$

In (3.9) we will give an explicit value for the minimal controllability time T_0 depending on $n, R(y^0), \lambda_1$, the function k and on the geometry of Ω, and in (5.20), an isomorphism

$$\Lambda_1 : L^2(\Omega_0) \times H^{-1}(\Omega_0) \mapsto H_0^1(\Omega_0) \times L^2(\Omega_0), \quad \Lambda_1 \left\{u^0, u^1\right\} = \left\{\theta^0, \theta^1\right\}$$

which allows to compute the control v for the initial data $\{u^0, u^1\}$.

Now we state the main result of the paper.

Theorem 2.1 *We assume that the hypotheses (H1)-(H5) are satisfied. Let $T > T_0$. Then for each initial data $\{u^0, u^1\}$ belonging to $L^2(\Omega_0) \times H^{-1}(\Omega_0)$, there exists a control $v \in L^2(0,T; L^2(\Gamma_t(y^0))$ such that the solution u of system (2.1) satisfies the final condition (1.2). Moreover, the control v has the form $v = \partial\theta\backslash\partial v^*$ where θ is the weak solution of the problem*

$$\theta'' - \Delta\theta = 0 \quad \text{in} \quad \widehat{Q}$$

$$\theta = 0 \quad \text{on} \quad \widehat{\Sigma}$$

$$\theta(0) = \theta^0, \theta'(0) = \theta^1,$$

with $\{\theta^0, \theta^1\} = \Lambda_1 \{u^0, u^1\}$.

The next three section will be devoted to the proof of the above theorem.

3 Summary of results on the cylinder

In this section we list the results on the cylinder Q that we will use in Section 5. Its proofs can be found in [14].

We consider the operator

$$Lw = w'' - \frac{\partial}{\partial y_i}\left(a_{ij}(y,t)\frac{\partial w}{\partial y_j}\right) + b_i(y,t)\frac{\partial w'}{\partial y_i} + d_i(y,t)\frac{\partial w}{\partial y_i} \tag{3.1}$$

where

$$a_{ij}(y,t) = \left(\delta_{ij} - k'^2 y_i y_j\right) k^{-2},$$

$$b_i(y,t) = -2k'k^{-1}y_i, d_i(y,t) = [(1-n)k'^2 - k''k]k^{-2}y_i.$$

Then for z test function in Q, we have

$$\int_0^T \int_\Omega (Lw) z \, dy dt = \int_0^T \int_\Omega w \left[z'' - \frac{\partial}{\partial y_i}\left(a_{ij} \frac{\partial z}{\partial y_j} \right) + \right.$$

$$\left. + \quad \frac{\partial}{\partial y_i}(b_i z)' - \frac{\partial}{\partial y_i}(d_i z) \right] dy dt = \int_0^T \int_\Omega w L^* z \, dy dt.$$

We obtain

$$\frac{\partial}{\partial y_i}(b_i z)' = b_i \frac{\partial z'}{\partial y_i} - 2nk'k^{-1}z' + \left(2k'^2 - 2k''k\right) k^{-2} y_i \frac{\partial z}{\partial y_i}$$

$$+ \left(2nk'^2 - 2nk''k\right) k^{-2} z - \frac{\partial}{\partial y_i}(d_i z) = [k''k - (1-n)k'^2] R^{-2} y_i \frac{\partial z}{\partial y_i}$$

$$+ [nk''k - n(1-n)k'^2] k^{-2} z.$$

Thus L^*z, the formal adjoint of L, has the form

$$L^* z = z'' - \frac{\partial}{\partial y_i}\left(a_{ij}(y,t) \frac{\partial z}{\partial y_j} \right) + b_i(y,t) \frac{\partial z'}{\partial y_i} + P z \tag{3.2}$$

where

$$P z = -2nk'k^{-1}z' + [(n+1)k'^2 - k''k] k^{-2} y_i \frac{\partial z}{\partial y_i} + [n(n+1)k'^2 - nk''k] R^{-2} z.$$

Let us consider the problem

$$\begin{aligned} & L^* z = h \quad \text{in} \quad Q \\ & z = 0 \quad \text{on} \quad \Sigma \\ & z(0) = z^0, z'(0) = z^1 \quad \text{in} \quad \Omega \end{aligned} \tag{3.3}$$

with data

$$z^0 \in H_0^1(\Omega), \quad z^1 \in L^2(\Omega), \quad h \in L^1(0,T; L^2(\Omega)). \tag{3.4}$$

A function $z : Q \mapsto \mathbf{R}$ will be called a weak solution of Problem (3.3) if z belongs to the class

$$z \in L^\infty(0,T; H_0^1(\Omega)), \qquad z' \in L^\infty(0,T; L^2(\Omega));$$

satisfies the equation

$$-\int_0^T (z', \xi')dt + \int_0^T a(t, z, \xi)dt + \int_0^T \left\langle b_i \frac{\partial z'}{\partial y_i}, \xi \right\rangle dt$$

$$+ \int_0^T (Pz, \xi)dt = \int_0^T (h, \xi)dt$$

$$\forall \xi \in L^2(0, T; H_0^1(\Omega)), \xi' \in L^2(0, T; L^2(\Omega)), \xi(0) = \xi(T) = 0$$

and the initial conditions

$$z(0) = z^0, \qquad z'(0) = z^1.$$

Here $(\cdot,\cdot)$ denotes the inner product of $L^2(\Omega)$, $\langle \cdot,\cdot \rangle$ the duality pairing between F' and F, F being a generic space and F' it dual (these notations will be maintained throughout the paper) and

$$a(t, z, \xi) = \int_\Omega a_{ij}(y, t) \frac{\partial z}{\partial y_j} \frac{\partial \xi}{\partial y_i} dy.$$

We observe that if z is a weak solution of Problem (3.3) then z' is weakly continuous from $[0, T]$ with values in $L^2(\Omega)$. Therefore the above initial condition $z'(0)$ makes sense. The regularity of z' follows from $z' \in L^\infty(0, T; L^2(\Omega))$ and $z'' \in L^1(0, T; H^{-1}(\Omega))$. The second condition is obtained from the integral equation of the definition of weak solution.

Concerning to Problem (3.3) we have the following result:

Theorem 3.1 *For each data z^0, z^1, h in the class (3.4), there exists an unique weak solution z of Problem (3.3). This solution has the regularity:*

$$z \in C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$$

and

$$\frac{\partial z}{\partial \nu} \in L^2(0, T; L^2(\Gamma)). \tag{3.5}$$

From (3.5) it follows that $\frac{\partial z}{\partial \nu_A}$ belongs to $L^2(0, T; L^2(\Gamma))$ where

$$\frac{\partial z}{\partial \nu_A} = a_{ij}(y, t) \frac{\partial z}{\partial y_j} \nu_i.$$

We obtain all the above results if instead of Problem (3.3) we consider the backward problem:

$$
\begin{aligned}
&L^*z = h \quad \text{in} \quad Q \\
&z = 0 \quad \text{on} \quad \Sigma \\
&z(T) = z^0, z'(T) = z^1 \quad \text{in} \quad \Omega
\end{aligned}
\tag{3.6}
$$

Let us consider the problem

$$
\begin{aligned}
&Lw = 0 \quad \text{in} \quad Q \\
&w = g \quad \text{on} \quad \Sigma \\
&w(0) = w^0, w'(0) = w^1
\end{aligned}
\tag{3.7}
$$

with data

$$w^0 \in L^2(\Omega), \quad w^1 \in H^{-1}(\Omega), \quad g \in L^2(0,T;L^2(\Gamma)). \tag{3.8}$$

We say that $w \in L^\infty(0,T;L^2(\Omega))$ is a solution by transposition of Problem (3.7) if

$$\int_0^T (w,h)dt = \langle w^1, z(0)\rangle - (w^0, z'(0)) - \left\langle \frac{2k'(0)}{k(0)} y_i \frac{\partial w^0}{\partial y_i}, z(0) \right\rangle -$$

$$- \int_0^T \left(g, \frac{\partial z}{\partial \nu_A}\right)_{L^2(\Gamma)} dt$$

for every $h \in L^1(0,T;L^2(\Omega))$ where z is related to h by Problem (3.6) with $z^0 = z^1 = 0$.

We have the following result:

Theorem 3.2 *For each data w^0, w^1, g in the class (3.8), there exists an unique solution by transposition w of Problem (3.7). This solution has the regularity*

$$w \in C([0,T];L^2(\Omega)) \cap C^1([0,T];H^{-1}(\Omega)).$$

We can change the initial data at time $t = 0$ by final data at time $t = T$ in Problem (3.7) and obtain the same result above.

In the sequel we introduce some constants in order to state the main result of this section. By hypotheses (H3), (H4) of Section 2 one has that there exists a positive constant a_0 such that

$$a_{ij}(y,t)\xi_i\xi_j \geq a_0\xi_i\xi_i, \quad \forall\{y,t\} \in \Omega \times [0,\infty[, \quad \forall\xi \in \mathbf{R}^n.$$

With this and the notations of Section 2, we define:

$$\begin{aligned} C_0 &= 2\left(1+\tau k_1 M^2+\tau^2 M^2+n\,a_0 k_1^2\right)\left(a_0 k_0^3\right)^{-1}(\ell_1+\ell_2)+ \\ &\quad +2\left(\lambda_1^{1/2}M+n\right)(n\tau+\tau+k_1)\left(a_0^{1/2}k_0^2\lambda_1^{1/2}\right)^{-1}(\ell_1+\ell_2) \\ C_1 &= e^{-C_0},\ C_2 = e^{C_0}. \end{aligned}$$

The minimal controllability time T_0 is then defined by

$$T_0 = [2a_0^{-1/2}R(y^0)+K_1+K_2+K_3]C_2C_1^{-1} \tag{3.9}$$

where

$$\begin{aligned} &K_1 = 2\tau[(n-1)M+2R(y^0)+2\lambda_1^{1/2}M\,R(y^0)]/a_0k_0\lambda_1^{1/2} \\ &K_2 = 2\ell_1(n+1)R(y^0)[\tau M+a_0^{1/2}k_0]/a_0k_0^2 \\ &K_3 = \ell_1 n(n+1)[\tau M+a_0^{1/2}k_0]/a_0k_0^2\lambda_1^{1/2}. \end{aligned}$$

We consider the problem

$$\begin{aligned} &Lw = 0 \quad \text{in} \quad Q \\ &w = \begin{cases} g & \text{on} \quad \Sigma(y^0) \\ 0 & \text{on} \quad \Sigma\backslash\Sigma(y^0) \end{cases} \\ &w(0) = w^0, w'(0) = w^1 \end{aligned} \tag{3.10}$$

We have the following exact controllability result:

Theorem 3.3 *Let $T > T_0, T_0$ given by (3.9). Then for every $\{w^0, w^1\} \in L^2(\Omega) \times H^{-1}(\Omega)$ there exists a control $g \in L^2(\Sigma(y^0))$ such that the solution by transposition w of Problem (3.10) satisfies*

$$w(T) = 0, \qquad w'(T) = 0.$$

Remark 3.1 We observe that if $k(t) \equiv 1$ then $K_1 = K_2 = K_3 = 0, C_1 = C_2 = 1$ and $a_0 = 1$. Therefore $T_0 = 2R(y^0)$. Thus in this case T_0 coincides with the minimal controllability time obtained earlier by J. L. Lions [11] and V. Komornik [7] for the wave equation $u'' - \Delta u = 0$. Let φ be the weak solution of problem

$$\begin{aligned} &L^*\varphi = 0 \quad \text{in} \quad Q \\ &\varphi = 0 \quad \text{on} \quad \Sigma \\ &\varphi(0) = \varphi^0, \varphi'(0) = \varphi^1 \quad \text{in} \quad \Omega \end{aligned} \tag{3.11}$$

with $\{\varphi^0, \varphi^1\} \in H_0^1(\Omega) \times L^2(\Omega)$, and ψ the solution by transposition of the problem

$$\begin{aligned} &L\psi = 0 \quad \text{in} \quad Q \\ &\psi = \begin{cases} \frac{\partial\varphi}{\partial\nu} & \text{on} \quad \Sigma(y^0) \\ 0 & \text{on} \quad \Sigma\backslash\Sigma(y^0) \end{cases} \\ &\psi(T) = 0, \psi'(T) = 0 \end{aligned} \tag{3.12}$$

With these last two problems, we introduce the operator Λ,

$$\begin{aligned} &H_0^1 \times L^2(\Omega) \mapsto H^{-1}(\Omega) \times L^2(\Omega) \\ &\{\varphi^0, \varphi^1\} \mapsto \Lambda\{\varphi^0, \varphi^1\} = \left\{\psi'(0) - \tfrac{2k'(0)}{k(0)} y_i \tfrac{\partial\psi(0)}{\partial y_i}, -\psi(0)\right\} \end{aligned} \tag{3.13}$$

The proof of Theorem 3.3 is reduced to prove that the operator

Λ is an isomorphism from $H_0^1(\Omega) \times L^2(\Omega)$ onto $H^{-1}(\Omega) \times L^2(\Omega)$.

This is done by showing, by multiplier techniques, that the following observability inequality holds for $T > T_0$:

$$\frac{1}{2} \mid \varphi^1 \mid^2 + \frac{1}{2} a\left(0; \varphi^0, \varphi^0\right) \leq C \int_0^T \int_{\Gamma(y^0)} \left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\Gamma\, dt$$

where φ is the solution of problem (3.11). We refer to [14] for the technical details.

Remark 3.2 In system (3.12) we can consider $\frac{\partial\varphi}{\partial\nu_A}$ instead $\frac{\partial\varphi}{\partial\nu}$ and to obtain also the exact controllability for system (3.10). On the other side if $\varphi(y,t) = k^n(t)\theta(k(t)y,t), x = k(t)y$, *then*

$$\begin{aligned} \frac{\partial\varphi}{\partial\nu_A}(y,t) &= \left(\delta_{ij} - k'^2 y_i y_j\right) k^{-2} \frac{\partial\varphi}{\partial y_j}(y,t)\nu_i(y) = \\ &= \left(\delta_{ij} - k^{-2}k'^2 x_i x_j\right) k^{n-1} \frac{\partial\theta}{\partial x_j}(x,t)\nu_i^*(x,t) \end{aligned} \tag{3.14}$$

and

$$\frac{\partial \varphi}{\partial \nu}(y,t) = k^{n+1}\frac{\partial \theta}{\partial \nu^*}(x,t).$$

(For the calculations see (5.13)). We note that the second member of (3.14) is not a known derivative of the function θ. For this reason we consider $\frac{\partial \varphi}{\partial \nu}$ instead $\frac{\partial \varphi}{\partial \nu_A}$ in (3.12).

4 Spaces on the non cylindrical domain

Let $u : \widehat{Q} \mapsto \mathbf{R}$ be a function such that

$$u(x,t) = k^{-n}(t)\xi\left(\frac{x}{k(t)},t\right), \quad \xi \in L^p(0,T;W_0^{m,q}(\Omega)). \tag{4.1}$$

We then have $u(t) \in W_0^{m,q}(\Omega_t)$ a.e. t in $]0,T[$ and

$$\| u(t) \|_{W_0^{m,q}(\Omega_t)} = k^{\frac{n}{q}-m-n}(t) \| \xi(t) \|_{W_0^{m,q}(\Omega)} .$$

Therefore,

$$C_3 \| \xi(t) \|_{W_0^{m,q}(\Omega)} \leq \| u(t) \|_{W_0^{m,q}(\Omega_t)} \leq C_4 \| \xi(t) \|_{W_0^{m,q}(\Omega)} . \tag{4.2}$$

Here and in what follows C_3, C_4 will denote generic positive constants.

We denote by $L^p(0,T;W_0^{m,q}(\Omega_t))$ $(1 \leq p \leq \infty, 1 \leq q < \infty, m$ a non-negative integer) the space of (classes of) functions $u : \widehat{Q} \mapsto \mathbf{R}$ such that there exists $\xi \in L^p(0,T;W_0^{m,q}(\Omega))$ verifying (4.1), equipped with the norm

$$\| u \|_{L^p(0,T;W_0^{m,q}(\Omega_t))} = \left(\int_0^T \| u(t) \|^p_{W_0^{m,q}(\Omega_t)} dt \right)^{1/p}, 1 \leq p < \infty$$

$$\| u \|_{L^\infty(0,T;W_0^{m,q}(\Omega_t))} = \operatorname*{ess\,sup}_{t \in]0,T[} \| u(t) \|_{W_0^{m,q}(\Omega_t)} .$$

By (4.2), the space $X = L^p(0,T;W_0^{m,q}(\Omega_t))$ is a Banach space and the linear application

$$L^p(0;T;W_0^{m,q}(\Omega)) \mapsto X, \quad \xi \mapsto \mathcal{U}\xi = u \tag{4.3}$$

is an isomorphism.

We write $C([0,T];W_0^{m,q}(\Omega_t))$ to denote the closed subspace of $L^\infty(0,T;W_0^{m,q}(\Omega_t))$ constituted by functions u such that the corresponding ξ given by (4.1) belongs to $C([0,T];W_0^{m,q}(\Omega))$.

The dual space of $X = L^p(0,T;H_0^1(\Omega_t))$ $\left(1 \le p < \infty, \frac{1}{p} + \frac{1}{p'} = 1\right)$ will be identified with $L^{p'}(0,T;H^{-1}(\Omega_t))$. In what follows we characterize the vectors of this space. In fact, we have by the properties of $\mathcal{U}$ defined in (4.3), that if $S \in X'$ then there exists a unique $R \in L^{p'}(0,T;H^{-1}(\Omega))$ such that

$$\langle S,u\rangle = \langle R,\xi\rangle, \quad \xi = \mathcal{U}^{-1}u$$

and

$$C_3 \| R \| \le \| S \| \le C_4 \| R \| .$$

Tho show that, it is sufficient to take $R = \mathcal{U}^*S$ where $\mathcal{U}^*$ is the adjoint operator of $\mathcal{U}$. On the other side, with R we define the operator P:

$$\langle P(t),\alpha\rangle = \langle R(t),\beta\rangle, \quad \alpha \in H_0^1(\Omega_t)$$

where $\beta(y) = k^n(t)\alpha(k(t)y)$. Then

$$C_3 \| R(t) \|_{H^{-1}(\Omega)} \le \| P(t) \|_{H^{-1}(\Omega_t)} \le C_4 \| R(t) \|_{H^{-1}(\Omega)}$$

since

$$C_3 \| \beta \|_{H_0^1(\Omega)} \le \| \alpha \|_{H_0^1(\Omega_t)} \le C_4 \| \beta \|_{H_0^1(\Omega)} .$$

Thus, by identifying S with R and R with P, we obtain that the space $L^{p'}(0,T;H^{-1}(\Omega_t))$ is constituted by the functionals S such that

$$S :]0,T[\mapsto H^{-1}(\Omega_t), \; S \text{ measurable}$$

$$\exists R \in L^{p'}\left(0,T;H^{-1}(\Omega)\right) \text{ satisfying } \langle S(t),\alpha\rangle = \langle R(t),\beta\rangle$$
$$\text{a.e.} t \text{ in }]0,T[, \quad \beta(y) = k^n(t)\alpha(k(t)y)$$

and the norm is given by

$$\| S \|_{L^{p'}(0,T;H^{-1}(\Omega_t))} = \left(\int_0^T \| S(t) \|_{H^{-1}(\Omega_t)}^{p'} dt\right)^{1/p'}, 1 < p' < \infty$$
$$\| S \|_{L^\infty(0,T;H^{-1}(\Omega_t))} = \operatorname*{ess\,sup}_{t\in]0,T[} \| S(t) \|_{H^{-1}(\Omega_t)} .$$

The space $C([0,T];H^{-1}(\Omega_t))$ will be defined as the closed subspace of $L^\infty(0,T;H^{-1}(\Omega_t))$ constituted by the functionals S such that its corresponding R belongs to $C([0,T];H^{-1}(\Omega))$.

Let $u:\widehat{\mathbf{Q}}\mapsto\mathbf{R}$ be a function and

$$u(x,t)=w\left(\frac{x}{k(t)},t\right),\quad w:\mathbf{Q}\mapsto\mathbf{R}$$

then

$$u'(x,t)=-\frac{k'(t)}{k(t)}y_i\frac{\partial w}{\partial y_i}\left(\frac{x}{k(t)},t\right)+w'\left(\frac{x}{k(t)},t\right). \tag{4.4}$$

Let $u\in L^p(0,T;L^2(\Omega_t)), 1\le p\le\infty$, be such that ξ' belongs to $L^p(0,T;H^{-1}(\Omega))$, where ξ is so that $\mathcal{U}\xi=u$. Let $w=k^{-n}\xi$, that is,

$$u(x,t)=k^{-n}(t)\xi\left(\frac{x}{k(t)},t\right)=w\left(\frac{x}{k(t)},t\right).$$

Then $w\in L^p(0,T;L^2(\Omega))$ and $w'\in L^p(0,T;H^{-1}(\Omega))$. By (4.4) we have

$$\langle u'(t),\alpha\rangle=\left\langle-\frac{k'(t)}{k(t)}y_i\frac{\partial w}{\partial y_i}+w',\beta\right\rangle$$

where $\alpha\in H_0^1(\Omega_t)$ and $\beta(y)=k^n(t)\alpha(k(t)y)$.
Clearly, $u'\in L^p(0,T;H^{-1}(\Omega_t))$.
In particular if $u\in L^p(0,T;H_0^1(\Omega_t))$ and $w'\in L^p(0,T;L^2(\Omega))$ then

$$(u'(t),\alpha)_{L^2(\Omega_t)}=\left(-\frac{k'(t)}{k(t)}y_i\frac{\partial w}{\partial y_i}+w',\beta\right)_{L^2(\Omega)}$$

with $\alpha\in L^2(\Omega_t)$. Clearly $u'\in L^p(0,T;L^2(\Omega_t))$.

We denote by $L^2(0,T;L^2(\Gamma_t))$ the Hilbert space of functions $v:\widehat{\Sigma}\mapsto\mathbf{R}$ such that ther exists $g\in L^2(0,T;L^2(\Gamma))$ verifying

$$v(x,t)=k^{-n-1}(t)g\left(\frac{x}{k(t)},t\right),$$

equipped with the inner product

$$(v,\widehat{v})_{L^2(0,T;L^2(\Gamma_t))}=\int_0^T(v(t),\widehat{v}(t))_{L^2(\Gamma_t)}\,dt.$$

Remark 4.1 The unit normal vector $\eta(x,t)$ at $(x,t) \in \widehat{\Sigma}$, directed towards the exterior of $\widehat{Q}$, has the form

$$\eta(x,t) = \{\nu(y), -k'(t)(y,\nu(y))\}\,[1 + k'^2(t) \mid (y,\nu(y)) \mid^2]^{-1/2}, \quad y = \frac{x}{k(t)}.$$

In fact, fixe $(x,t) \in \widehat{\Sigma}$. Let $\varphi = 0$ be a parametrization of a part U of Γ, U containing $y = x/k(t)$. Then a parametrization of a parte V of $\widehat{\Sigma}, (x,t) \in V$, is $\psi(x,t) = \varphi(x/k(t)) = 0$. We have

$$\nabla\psi(x,t) = \frac{1}{k(t)}\left\{\nabla\varphi(y), -k'(t)(y,\nabla\varphi(y))\right\}.$$

From this and observing that $\nu(y) = \nabla\varphi(y)/ \mid \nabla\varphi(y) \mid$, the remark follows.

Let $\nu^*(x,t)$ be the x-component of $\eta(x,t)$, $\mid \nu^*(x,t) \mid = 1$. Then by Remark 4.1, one has

$$\nu^*(x,t) = \nu\left(\frac{x}{k(t)}\right). \tag{4.5}$$

5 Proof of the main result

5.1 Weak Solutions and Solutions by Transposition.

In order to motivate the definition of weak solutions and solutions by transposition of the wave equation in $\widehat{Q}$, we obtain some relations between functions. We consider

$$u(x,t) = w\left(\frac{x}{k(t)},t\right), \quad \theta(x,t) = k^{-n}(t)z\left(\frac{x}{k(t)},t\right)$$
$$v(x,t) = k^{-n-1}(t)g\left(\frac{x}{k(t)},t\right), \quad v : \widehat{\Sigma} \mapsto \mathbf{R}.$$

One has

$$u'(x,t) = -\frac{k'(t)}{k(t)}y_i\frac{\partial w}{\partial y_i}\left(\frac{x}{k(t)},t\right) + w'\left(\frac{x}{k(t)},t\right) \tag{5.1}$$

$$\begin{aligned} \theta'(x,t) = & - nk^{-n-1}(t)k'(t)z\left(\frac{x}{k(t)},t\right) - \\ & - k^{-n-1}(t)k'(t)y_i\frac{\partial z}{\partial y_i}\left(\frac{x}{k(t)},t\right) + k^{-n}(t)z'\left(\frac{x}{k(t)},t\right) \end{aligned} \tag{5.2}$$

and

$$u''(x,t) - \Delta u(x,t) = Lw\left(\frac{x}{k(t)}, t\right),$$

$$\theta''(x,t) - \Delta\theta(x,t) = k^{-n}(t)L^* z\left(\frac{x}{k(t)}, t\right)$$

where L and L^* were defined, respectively, in (3.1) and (3.2).

With the above functions we obtain formally the following results: The change of variable $x = k(t)y$ gives

$$\int_0^T \int_{\Omega_t} (u'' - \Delta u)\theta dx dt = \int_0^T \int_\Omega Lw\, z\, dy dt \tag{5.3}$$

$$\int_0^T \int_\Omega wL^*\, z\, dy dt = \int_0^T \int_{\Omega_t} u(\theta'' - \Delta\theta) dx dt \tag{5.4}$$

and by (4.5),

$$\int_0^T \int_\Gamma \left(\delta_{ij} - k'^2 y_i y_j\right) k^{-2} \frac{\partial z}{\partial y_j} \nu_i g\, d\Gamma dt = \tag{5.5}$$

$$= \int_0^T \int_{\Gamma_t} \left(\delta_{ij} - k'^2 k^{-2} x_i x_j\right) k^{n+1} \frac{\partial \theta}{\partial x_j} \nu_i^*\, v\, d\Gamma\, dt.$$

The Green's formula, the condition $z(t) = 0$ on Γ, the change of variable $x = k(t)y$ and the relations (5.1), (5.2) furnish the identity

$$\int_\Omega [w'(t)z(t) - w(t)z'(t)] dy - \int_\Omega \frac{2k'(t)}{k(t)} y_i \frac{\partial w}{\partial y_i}(t) z(t) dy = \tag{5.6}$$

$$= \int_{\Omega_t} [u'(t)\theta(t) - u(t)\theta'(t)] dx$$

The Green's formula, the integration by parts on [0,T] and the conditions $z(t) = 0$ on $\Gamma, w = g$ on Σ, yield

$$\int_0^T \int_\Omega Lw\, z\, dy dt = \int_0^T \int_\Omega w\, L^* z\, dy dt + N(T) - N(0) + J \tag{5.7}$$

where $N(t)$ denotes the left side of (5.6) and J, the left side of (5.5). Then from (5.3)-(5.7) we have

$$\int_0^T \int_{\Omega_t} (u'' - \Delta u)\theta dx dt = \int_{\Omega_T} [u'(T)\theta(T) - u(T)\theta'(T)] dx \tag{5.8}$$

$$
\begin{aligned}
&- \int_{\Omega_0} [u'(0)\theta(0) - u(0)\theta'(0)]dx + \\
&+ \int_0^T \int_{\Gamma_t} \left(\delta_{ij} - k'^2 k^{-2} x_i x_j\right) R^{n+1} \frac{\partial \theta}{\partial x_j} \nu_i^* \, v \, d\Gamma dt + \\
&+ \int_0^T \int_{\Omega_t} u(\theta'' - \Delta\theta) dx dt.
\end{aligned}
$$

Motivated by (5.8), we introduce the following problem:

$$
\begin{aligned}
&\theta'' - \Delta\theta = \widehat{h} \quad \text{in} \quad \widehat{Q} \\
&\theta = 0 \quad \text{on} \quad \widehat{\Sigma} \qquad (5.9)\\
&\theta(0) = \theta^0, \ \theta'(0) = 0^1 \quad \text{in} \quad \Omega_0
\end{aligned}
$$

with data

$$
\theta^0 \in H_0^1(\Omega_0), \quad \theta^1 \in L^2(\Omega_0) \quad \widehat{h} \in L^1(0,T; L^2(\Omega_t)). \qquad (5.10)
$$

We say that θ is a weak solution of Problem (5.9) if

$$
\theta \in C([0,T]; H_0^1(\Omega_t)), \quad \theta' \in C([0,T]; L^2(\Omega_t)
$$

and verifies

$$
- \int_0^T (\theta', \alpha')_{L^2(\Omega_t)} dt + \int_0^T ((\theta, \alpha))_{H_0^1(\Omega_t)} dt = \int_0^T (\widehat{h}, \alpha)_{L^2(\Omega_t)} dt
$$

$$
\forall \alpha \in L^2(0,T; H_0^1(\Omega_t)), \ \alpha' \in L^2(0,T; L^2(\Omega_t)), \ \alpha(0) = \alpha(T) = 0
$$

$$
\theta(0) = \theta^0, \ \theta'(0) = \theta^1
$$

Theorem 5.1 *Let $\theta(x,t) = k^{-n}(t) z(x/k(t), t)$. We have that if z is a weak solution of Problem (3.3) then θ is a weak solution of Problem (5.9) and reciprocally. The data $\left\{\theta^0, \theta^1, \widehat{h}\right\}$ and $\{z^0, z^1, h\}$ are related by*

$$
\theta^0(x) = k^{-n}(0) z^0 \left(\frac{x}{k(0)}\right) \qquad (5.11)
$$

$$\theta^1(x) = \; - \; nk^{-n-1}(0)k'(0)z^0\left(\frac{x}{k(0)}\right) - \tag{5.12}$$
$$- \; k^{-n-1}(0)k'(0)y_i\frac{\partial z^0}{\partial y_i}\left(\frac{x}{k(0)}\right) + k^{-n}(0)z^1\left(\frac{x}{k(0)}\right)$$

(see (5.1), (5.2)).

Theorem 5.1 is showed by relating integrals on Ω_t and Ω and using Theorem 3.1 and (5.2).

The uniqueness of solutions of Problem (5.9) is a consequence of Theorem 5.1. We also have that, since $\frac{\partial\theta}{\partial x_j} = k^{-n-1}\frac{\partial z}{\partial y_j}$,

$$\frac{\partial\theta}{\partial x_j}, \frac{\partial\theta}{\partial\nu^*} \in L^2(0,T;L^2(\Gamma_t)) \text{ and } \frac{\partial z}{\partial\nu}(y,t) = k^{n+1}(t)\frac{\partial\theta}{\partial\nu^*}(k(t)y,t) \tag{5.13}$$

Remark 5.1 Clearly we can change the data at time $t = 0$ by final data at $t = T$ in Problem (5.9) and obtain all the above results for the solution w of the respective backward problem.

In the sequel we introduce the solutions by transposition. Let us consider the problem

$$\begin{aligned} &u'' - \Delta u = 0 \quad \text{in} \quad \widehat{Q} \\ &u = v \quad \text{on} \quad \widehat{\Sigma} \\ &u(0) = u^0, u'(0) = u^1 \quad \text{in} \quad \Omega_0 \end{aligned} \tag{5.14}$$

with data

$$u^0 \in L^2(\Omega_0), \quad u^1 \in H^{-1}(\Omega_0), \quad v \in L^2(0,T;L^2(\Gamma_t)). \tag{5.15}$$

Motivated by (5.8) one introduces the following definition: We say that $u \in L^\infty(0,T;L^2(\Omega_t))$ is a solution by transposition of Problem (5.14) if u verifies

$$\int_0^T (u,\widehat{h})_{L^2(\Omega_t)} = \langle u^1,\theta(0)\rangle - \left(u^0,\theta'(0)\right)_{L^2(\Omega_0)} -$$
$$- \int_0^T\int_{\Gamma_t}\left(\delta_{ij} - k'^2k^{-2}x_ix_j\right)k^{n+1}\frac{\partial\theta}{\partial x_j}\nu_i^*\, v\, d\Gamma dt$$

$$\forall \widehat{h} \in L^1(0,T;L^2(\Omega_t))$$

(ν^* defined in (4.4)) where θ is the weak solution of the problem

$$\theta'' - \Delta\theta = \widehat{h} \quad \text{in} \quad \widehat{Q}$$

$$\theta = 0 \quad \text{on} \quad \widehat{\Sigma}$$

$$\theta(t) = 0,\ \theta'(t) = 0$$

Theorem 5.2 *Let $u(x,t) = w\left(\frac{x}{k(t)}, t\right)$. We have that if w is a solution by transposition of Problem (3.7) then u is a solution by transposition of Problem (5.14) and reciprocally. The data $\{u^0, u^1, v\}$ and $\{w^0, w^1, g\}$ are related by*

$$u^0(x) = w^0\left(\frac{x}{k(0)}\right) \tag{5.16}$$

$$\langle u^1, \alpha\rangle = \left\langle -\frac{k'(0)}{k(0)} y_i \frac{\partial w^0}{\partial y_i} + w^1, \beta\right\rangle, \quad \alpha \in H_0^1(\Omega_0), \tag{5.17}$$

$$\alpha(x) = k^{-n}(0)\beta\left(\frac{x}{k(0)}\right)$$

$$v(x,t) = k^{-n-1}(t) g\left(\frac{x}{k(t)}, t\right) \tag{5.18}$$

The proof of Theorem 5.2 is obtained by the same arguments used in the proof of (5.8). For the initial conditions one uses the following result:

Remark 5.2 Let $u^0 \in L^2(\Omega_t)$ and $w^0(y) = u^0(k(t)y)$. Then

$$\left\langle x_i \frac{\partial u^0}{\partial x_i}, \alpha\right\rangle = \left\langle y_i \frac{\partial w^0}{\partial y_i}, \beta\right\rangle, \quad \alpha \in H_0^1(\Omega_t), \quad \alpha(x) = k^{-n}(t)\beta\left(\frac{x}{k(t)}\right).$$

To see this it is enough to make the respective integrations.

From Theorem 5.2 the uniqueness of solutions of Problem (5.14) follows and by Theorem 3.2,

$$u \in C([0,T];L^2(\Omega_t)) \cap C^1([0,T];H^{-1}(\Omega_t)).$$

We observe that, in addition to (5.6), we have

$$\int_0^T \int_{\Gamma_t} \left(\delta_{ij} - k'^2 k^{-2} x_i x_j\right) k^{n+1} \frac{\partial \theta}{\partial x_j} \nu_i^* \frac{\partial \theta}{\partial \nu^*} d\Gamma dt =$$
$$= \int_0^T \int_{\Gamma} \left(\delta_{ij} - k'^2 y_i y_j\right) k^{-2} \frac{\partial z}{\partial y_j} \nu_i \frac{\partial z}{\partial \nu} d\Gamma\, dt.$$

5.2 Proof of Theorem 2.1.

Let us consider the system (2.1), that is,

$$\begin{aligned} & u'' - \Delta u = 0 \quad \text{in} \quad \widehat{Q} \\ & u = \begin{cases} v & \text{on} \quad \widehat{\Sigma}(y^0) \\ 0 & \text{on} \quad \widehat{\Sigma} \backslash \widehat{\Sigma}(y^0) \end{cases} \\ & u(0) = u^0,\ u'(0) = u^1 \quad \text{in} \quad \Omega_0 \end{aligned} \tag{5.19}$$

where $\widehat{Q}$ is constructed with $T > T_0, T_0$ given by (3.9). With (5.10)-(5.12) and (5.15)-(5.17), we determine, respectively, the isomorphisms

$$G_1\left\{z^0, z^1\right\} = \left\{\theta^0, \theta^1\right\} \text{ and } G_2\left\{w^0, w^1\right\} = \left\{u^0, u^1\right\}.$$

Consider the operators

$$\sigma\left\{w^0, w^1\right\} = \left\{w^1 - \frac{2k'(0)}{k(0)} y_i \frac{\partial w^0}{\partial y_i}, -w^0\right\},$$

$$\Lambda\left\{z^0, z^1\right\} = \left\{w'(0) - \frac{2k'(0)}{k(0)} y_i \frac{\partial w(0)}{\partial y_i}, -w(0)\right\}.$$

where Λ is the isomorphism defined in (3.13), that is, z is the weak solution of the problem

$$\begin{aligned} & L^* z = 0 \quad \text{in} \quad Q \\ & z = 0 \quad \text{on} \quad \Sigma \\ & z(0) = z^0, z'(0) = z^1 \quad \text{in} \quad \Omega \end{aligned} \tag{5.20}$$

and w the solution by transposition of the problem

$$\begin{aligned}
&Lw = 0 \quad \text{in} \quad Q\\
&w = \begin{cases} \frac{\partial z}{\partial \nu} & \text{on} \quad \Sigma(y^0)\\ 0 & \text{on} \quad \Sigma\backslash\Sigma(y^0)\end{cases}\\
&w(T) = 0,\ w'(T) = 0
\end{aligned} \tag{5.21}$$

Since Λ is an isomorphism we have that for each $\{w^1, w^0\} \in H^{-1}(\Omega) \times L^2(\Omega)$ there exists an unique $\{z^0, z^1\} \in H_0^1(\Omega) \times L^2(\Omega)$ such that

$$\Lambda\left\{z^0, z^1\right\} = \left\{w^1 - \frac{2k'(0)}{k(0)} y_i \frac{\partial w^0}{\partial y_i}, -w^0\right\}. \tag{5.22}$$

Thus if w is the solution of problem (5.21) constructed with $\{z^0, z^1\}$, we have

$$w(0) = w^0, \qquad w'(0) = w^1.$$

With the above operators we determine the isomorphism

$$\Lambda_1 = G_1 \Lambda^{-1} \sigma G_2^{-1}, \text{ that is },$$

$$\Lambda_1 : L^2(\Omega_0) \times H^{-1}(\Omega_0) \mapsto H_0^1(\Omega_0) \times L^2(\Omega_0),\ \Lambda_1\left\{u^0, u^1\right\} = \left\{\theta^0, \theta^1\right\}. \tag{5.23}$$

Let $\{u^0, u^1\} \in L^2(\Omega_0) \times H^{-1}(\Omega_0)$. Then by (5.23), we determine $\{\theta^0, \theta^1\}$. With this data we find the weak solution θ of the problem

$$\begin{aligned}
&\theta'' - \Delta\theta = 0 \quad \text{in} \quad \widehat{Q}\\
&\theta = 0 \quad \text{on} \quad \widehat{\Sigma}\\
&\theta(0) = \theta^0,\ \theta'(0) = \theta^1 \quad \text{in} \quad \Omega_0
\end{aligned} \tag{5.24}$$

and with $\{z^0, z^1\} = G_1^{-1}\{\theta^0, \theta^1\}$, the weak solution z of the problem

$$\begin{aligned}
&L^* z = 0 \quad \text{in} \quad Q\\
&z = 0 \quad \text{on} \quad \Sigma\\
&z(0) = z^0,\ z'(0) = z^1 \quad \text{in} \quad \Omega
\end{aligned}$$

Next, we determine the solution by transposition $\widehat{w}$ of the problem

$$\begin{aligned} &L\widehat{w} = 0 \quad \text{in} \quad Q \\ &\widehat{w} = \begin{cases} \frac{\partial z}{\partial \nu} & \text{on} \quad \Sigma(y^0) \\ 0 & \text{on} \quad \Sigma\backslash\Sigma(y^0) \end{cases} \quad \widehat{w}(0) = w^0,\ \widehat{w}'(0) = w^1 \end{aligned} \tag{5.25}$$

where $\{w^0, w^1\}$ and $\{z^0, z^1\}$ are related by (5.22). We have the uniqueness of solutions of problem (5.25) that $\widehat{w} = w$, w the solution of (5.21) constructed with $\{z^0, z^1\}$. Therefore

$$\widehat{w}(T) = 0, \qquad \widehat{w}'(T) = 0.$$

Finally, from Theorem 5.2, it follows that $u(x,t) = \widehat{w}\left(\frac{x}{k(t)}, t\right)$ is the solution by transposition of Problem (5.19) and u satisfies the final condition

$$u(T) = 0, \qquad u'(T) = 0.$$

By (5.13) and (5.14), we have that the control v has the form

$$v = \frac{\partial \theta}{\partial \nu^*}, \quad \theta \text{ weak solution of } (5.24).$$

Thus, the proof of Theorem 2.1 is concluded.

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