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Orthonormal bases for spaces of continuous and continuously differentiable functions defined on a subset of $\mathbb{Z}_p$

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Abstract

Let K be a non-archimedean valued field which contains $\mathbb{Q}_p$, and suppose that K is complete for the valuation $|\cdot|$, which extends the p-adic valuation. V_q is the closure of the set $\{aq^n|n = 0, 1, 2, \ldots\}$ where a and q are two units of $\mathbb{Z}_p$, q not a root of unity. $C(V_q \to K)$ (resp. $C^1(V_q \to K)$) is the Banach space of continuous functions (resp. continuously differentiable functions) from V_q to K. Our aim is to find orthonormal bases for $C(V_q \to K)$ and $C^1(V_q \to K)$.

1 Introduction

The main aim of this paper is to find orthonormal bases for the spaces $C(V_q \to K)$ of continuous and $C^1(V_q \to K)$ of continuously differentiable functions. Therefore we start by recalling some definitions and some previous results. Let E be a non-archimedean Banach space over a non-archimedean valued field L, E equipped with the norm $||\cdot||$.Let $f_1, f_2, \ldots$ be a finite or infinite sequence of elements of E. We say that this sequence is orthogonal if $||\alpha_1 f_1 + \ldots + \alpha_k f_k|| = max_{1\leq i\leq k}\{||\alpha_i f_i||\}$ for all k in $\mathbb{N}$ (or for all k that do not exceed the length of the sequence) and for all $\alpha_1, \ldots, \alpha_k$ in L. An orthogonal sequence $f_1, f_2, \ldots$ is called orthonormal if $||f_i|| = 1$ for all i. A sequence $f_1, f_2, \ldots$ of elements of E is an orthonormal base of E if the sequence is orthonormal and also a base. If M is a non-empty compact subset of L whithout isolated points,

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then $C(M \to L)$ is the Banach space of continuous functions from M to L equipped with the supremum norm $\|\cdot\|_\infty$. Let f be a function from M to L. The first difference quotient $\phi_1 f$ of the function f is the function of two variables given by $\phi_1 f(x,y) = \frac{f(x)-f(y)}{x-y}$ defined on $M \times M \setminus \Delta$ where $\Delta = \{(x,x)|x \in M\}$.We say that f is continuously differentiable at a point $b \in M$ (f is C^1 at b) if $lim_{(x,y)\to(b,b)}\phi_1 f(x,y)$ exists. The function f is called continuously differentiable (f is a C^1 function) if f is continuously differentiable at b for all b in M. If f is a function from M to L then f is continuously differentiable if and only if the function $\phi_1 f$ can (uniquely) be extended to a continuous function on $M \times M$. The set of all C_1 -functions from M to L is denoted by $C^1(M \to L)$, and $C^1(M \to L) \subset C(M \to L)$. For $f : M \to L$ we set $\|f\|_1 = sup\{\|f\|_\infty, \|\phi_1 f\|_\infty\}$. The function $\|\cdot\|_1$ is a norm on $C^1(M \to L)$ making it into an L -Banach algebra. Since M is compact, $\|f\|_1 < \infty$ if f is an element of $C^1(M \to L)$ (these results concerning continuously differentiable functions can be found in [2] or [5], chapter 27).

Let $\mathbb{Z}_p$ be the ring of p-adic integers, $\mathbb{Q}_p$ the field of p-adic numbers, and K is a non-archimedean valued field, K containing $\mathbb{Q}_p$, and we suppose that K is complete for the valuation $|\cdot|$, which extends the p-adic valuation. $\mathbb{N}$ denotes the set of natural numbers, and $\mathbb{N}_0$ is the set of natural numbers without zero. Let a and q be two units of $\mathbb{Z}_p$, q not a root of unity. We define V_q to be the closure of the set $\{aq^n|n = 0,1,2,\ldots\}$. For a description of the set V_q we refer to [7], section 2 or to [8], section 3. In section 3 our aim is to find orthonormal bases for the Banach space $C(V_q \to K)$. The results in section 3 can be seen as a sequel to the results in [9] and [8], sections 4,5 and 6. In section 4 we give necessary and sufficient conditions for a function f in $C(V_q \to K)$ to be continuously differentiable, and we find an orthonormal base for the Banach space $C^1(V_q \to K)$.

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2 Preliminaries

Let us introduce the following :
$[n]! = [n][n-1]\ldots[1]$ and $[0]! = 1$, where $[n] = \frac{q^n-1}{q-1}$ if $n \geq 1$.

$[{n \atop k}] = \frac{[n]!}{[k]![n-k]!}$ if $n \geq k$, $[{n \atop k}] = 0$ if $n < k$.
$\{{x \atop k}\} = \frac{(x-a)(x-aq)\dots(x-aq^{k-1})}{(aq^k-a)(aq^k-aq)\dots(aq^k-aq^{k-1})}$ if $k \geq 1$, $\{{x \atop 0}\} = 1$.
The sequence $(\{{x \atop k}\})$ forms an orthonormal base for $C(V_q \to K)$ ([8], corollary to lemma 8), analogous to Mahler's base for $C(\mathbb{Z}_p \to K)$ ([4]). We also have $[{n \atop k}] = \{{x \atop k}\}$ if $x = aq^n$. If x is an element of $\mathbb{Q}_p$ with Henselde-velopment $x = \sum_{j=-\infty}^{+\infty} a_j p^j$, we then put $x_n = \sum_{j=-\infty}^{n-1} a_j p^j$ $(n \in \mathbb{N})$. We write $m \triangleleft x$, if m is one of the numbers $x_0, x_1, \dots$ and we say that "m is an initial part of x" or "x starts with m" (see [5], section 62). If n belongs to $\mathbb{N}_0$, $n = \sum_{j=0}^{s} a_j p^j$ where $a_s \neq 0$, then we put $n_- = \sum_{j=0}^{s-1} a_j p^j$. We remark that $n_- \triangleleft n$. Let us now define the sequence of functions $(e_k(x))$ in the following way : write $k \in \mathbb{N}$ in the form $k = i + mj$, $0 \leq i < m$ $(i, j \in \mathbb{N})$. Then e_k is defined by
$e_k(x) = e_{i+mj}(x) = 1$ if $x = aq^{i_x}(q^m)^{\alpha_x}$ where $i_x = i, j \triangleleft \alpha_x$, $e_k(x) = 0$ otherwise.
The functions $(e_k(x))$ form an orthonormal base for $C(V_q \to K)$ ([9]), analogous to van der Put's base for $C(Z_p \to K)$ (see [3] or [5], section 62). We remark that $\{{aq^j \atop i}\} = e_i(aq^j) = 0$ if $j < i$ and that $\{{aq^i \atop i}\} = e_i(aq^i) = 1$. We shall use this frequently in the sequel.
We shall construct new orthonormal bases for $C(V_q \to K)$ using the bases $(\{{x \atop k}\})$ and $(e_k(x))$. Therefore we introduce the following : For each $n \in \mathbb{N}$, let I_n be a subset of the set $\{0, 1, \dots, n\}$ (I_n can also be empty or can be equal to $\{0, 1, \dots, n\}$). Let $p(x)$ be a continuous function of the following type $p(x) = \sum_{i \in I_n} a_i \{{x \atop i}\} + \sum_{i \in \{0,1,\dots,n\} \setminus I_n} a_i e_i(x)$ where each $a_i \in K$. For example, if $I_n = \{0, 1, \dots, n\}$, then $p(x)$ is a polynomial. If I_n is the subset of $\{0, 1, \dots, n\}$ consisting of all the even numbers, and if $a_i = 1$ for all i, then $p(x) = \sum_{i \in \{0,1,\dots,n\}, i\, even} \{{x \atop i}\} + \sum_{i \in \{0,1,\dots,n\}, i\, odd} e_i(x)$ and one can think of several other examples. For functions of this type we can prove the following lemmas

Lemma 1. *Let* $p(x)$ *be a continuous function of the type* $p(x) = \sum_{i \in I_n} a_i \{{x \atop i}\} + \sum_{i \in \{0,1,\dots,n\} \setminus I_n} a_i e_i(x)$ $(a_i \in K)$. *Then the following are equivalent :*

1) $|p(aq^n)| = 1$ *and* $|p(aq^k)| < 1$ *if* $0 \leq k < n$.
2) $|a_n| = 1$ *and* $|a_k| < 1$ *if* $0 \leq k < n$.

Proof.
1) $\Rightarrow$ 2) will be shown by induction. If $|p(a)| < 1$ then $|a_0| < 1$. Now suppose that $|a_k| < 1$ if $0 \leq k < n-1$. Then $|\sum_{i\in I_n\cap\{0,1,\ldots k+1\}} a_i\{{aq^{k+1} \atop i}\} + \sum_{i\in\{0,1,\ldots,k+1\}\setminus I_n} a_i e_i(aq^{k+1})| = |p(aq^{k+1})| < 1$ and by the induction hypothesis it follows that $|a_{k+1}| < 1$ and we can conclude $|a_i| < 1$ for all $0 \leq i < n$. Since $|\sum_{i\in I_n} a_i\{{aq^n \atop i}\} + \sum_{i\in\{0,1,\ldots,n\}\setminus I_n} a_i e_i(aq^n)| = |p(aq^n)| = 1$ we have $|a_n| = 1$.
2) $\Rightarrow$ 1) is obvious.

Lemma 2. *Let* $p(x)$ *be a continuous function of the type* $p(x) = \sum_{i\in I_n} a_i\{{x \atop i}\} + \sum_{i\in\{0,1,\ldots,n\}\setminus I_n} a_i e_i(x)$ $(a_i \in K)$. *Then the following are equivalent :*
1) $||p||_\infty \leq 1$.
2) $|a_k| \leq 1$ *for all* k *with* $0 \leq k \leq n$.

Proof.
1) $\Rightarrow$ 2) can be shown analogous as 1) $\Rightarrow$ 2) of the previous lemma.
2) $\Rightarrow$ 1) is obvious.

Let m be the smallest integer such that $q^m \equiv 1$ (mod p) $(1 \leq m \leq p-1)$. There exists a k_0 such that $q^m \equiv 1 \pmod{p^{k_0}}$, $q^m \not\equiv 1 \pmod{p^{k_0+1}}$. If $(p, k_0) = (2, 1)$, i.e. $q \equiv 3 \pmod 4$, then there exists a natural number N such that $q = 1 + 2 + 2^2\varepsilon$, $\varepsilon = \varepsilon_0 + \varepsilon_1 2 + \varepsilon_2 2^2 + \ldots$, $\varepsilon_0 = \varepsilon_1 = \ldots = \varepsilon_{N-1} = 1$, $\varepsilon_N = 0$. Then we have

Lemma 3.
1) *Let* $q^m \equiv 1$ *(mod* p^{k_0}*)*, $q^m \not\equiv 1$ *(mod* p^{k_0+1}*) with* $(p, k_0) \neq (2, 1)$.
If $x, y \in V_q$, $|x - y| \leq p^{-(k_0+t)}$ *then* $e_n(x) = e_n(y)$ *if* $0 \leq n < mp^t$.
2) *Let* $q \equiv 3$ *(mod 4)*, $q = 1 + 2 + 2^2\varepsilon$, $\varepsilon = \varepsilon_0 + \varepsilon_1 2 + \varepsilon_2 2^2 + \ldots$, $\varepsilon_0 = \varepsilon_1 = \ldots = \varepsilon_{N-1} = 1$, $\varepsilon_N = 0$. *If* $x, y \in V_q$, $|x - y| \leq p^{-(N+2+t)}$ *then* $e_n(x) = e_n(y)$ *if* $0 \leq n < 2^t$ $(t \geq 1)$.

Proof. This follows immediately from [8], lemmas 2 and 3.

Lemma 4. *Suppose $p(x)$ is a continuous function with $||p||_\infty \leq 1$ of the following type :* $p(x) = \sum_{i\in I_n} a_i\{{x \atop i}\} + \sum_{i\in\{0,1,\ldots,n\}\setminus I_n} a_i e_i(x)$ $(a_i \in K)$.

1) *Let* $q^m \equiv 1 \ (mod\ p^{k_0})$, $q^m \not\equiv 1 \ (mod\ p^{k_0+1})$ *with* $(p, k_0) \neq (2,1)$. *If* $x, y \in V_q$, $|x-y| \leq p^{-(k_0+t)}$ *then if* $j \in I\!N$, $0 \leq n < mp^t$: $|p(x)^j - p(y)^j| \leq 1/p$ *and* $|x^j - y^j| \leq 1/p$.

2) *Let* $q \equiv 3 \ (mod\ 4)$, $q = 1 + 2 + 2^2\varepsilon$, $\varepsilon = \varepsilon_0 + \varepsilon_1 2 + \varepsilon_2 2^2 + \ldots$, $\varepsilon_0 = \varepsilon_1 = \ldots = \varepsilon_{N-1} = 1$, $\varepsilon_N = 0$. *If* $x, y \in V_q$, $|x-y| \leq p^{-(N+2+t)}$ *then if* $j \in I\!N$, $0 \leq n < 2^t$ $(t \geq 1)$: $|p(x)^j - p(y)^j| \leq 1/2$ *and* $|x^j - y^j| \leq 1/2$.

Proof. It is clear that $|a_s| \leq 1$ if $0 \leq s \leq n$ (lemma 2). Suppose that x, y and n are as in 1) (resp. 2)). Then $|p(x) - p(y)| \leq max_{s\in I_n}\{|a_s||\{{x \atop s}\} - \{{y \atop s}\}|\} \leq 1/p$ (resp. $\leq 1/2$) by lemma 3 and [8], lemmas 11 and 12. If $j > 1$ then $|p(x)^j - p(y)^j| = |p(x) - p(y)||\sum_{s=0}^{j-1} p(x)^s p(y)^{j-1-s}| \leq 1/p$ (resp. $\leq 1/2$). So the lemma holds for $j \in I\!N$ (the case $j = 0$ is trivial). Further, if $j > 1$ then $|x^j - y^j| \leq |x-y||\sum_{s=0}^{j-1} x^s y^{j-1-s}| \leq 1/p$ (resp. $\leq 1/2$) so $|x^j - y^j| \leq 1/p$ (resp. $\leq 1/2$) for all $j \in I\!N$.

Let for each $n \in I\!N$ J_n be a subset of the set $\{0, 1, \ldots, n\}$. Then we can prove

Lemma 5. *Let $p(x)$ and $q(x)$ be continuous functions with $||p||_\infty \leq 1$ and $||q||_\infty \leq 1$ of the form*

$$p(x) = \sum_{i\in I_n} a_i\{{x \atop i}\} + \sum_{i\in\{0,1,\ldots,n\}\setminus I_n} a_i e_i(x), \ (a_i \in K)$$

$$q(x) = \sum_{i\in J_n} b_i\{{x \atop i}\} + \sum_{i\in\{0,1,\ldots,n\}\setminus J_n} b_i e_i(x), \ (b_i \in K).$$

1) Let $q^m \equiv 1 \ (mod\ p^{k_0})$, $q^m \not\equiv 1 \ (mod\ p^{k_0+1})$ *with* $(p, k_0) \neq (2,1)$. *If* $x, y \in V_q$, $|x-y| \leq p^{-(k_0+t)}$ *then if* $i, j \in I\!N$, $0 \leq n < mp^t$: $|q(x)^i p(x)^j - q(y)^i p(y)^j| \leq 1/p$ *and* $|x^i p(x)^j - y^i p(x)^j| \leq 1/p$.

2) Let $q \equiv 3 \ (mod\ 4)$, $q = 1 + 2 + 2^2\varepsilon$, $\varepsilon = \varepsilon_0 + \varepsilon_1 2 + \varepsilon_2 2^2 + \ldots$, $\varepsilon_0 = \varepsilon_1 = \ldots = \varepsilon_{N-1} = 1$, $\varepsilon_N = 0$. *If* $x, y \in V_q$, $|x-y| \leq p^{-(N+2+t)}$ *then if* $i, j \in I\!N$, $0 \leq n < 2^t$ $(t \geq 1)$: $|q(x)^i p(x)^j - q(y)^i p(y)^j| \leq 1/2$ *and* $|x^i p(x)^j - y^i p(x)^j| \leq 1/2$.

Proof. Let x, y, n, i and j be as in 1) (resp. 2)) then

$|q(x)^i p(x)^j - q(y)^i p(y)^j| \leq max\{|q(x)^i p(x)^j - q(x)^i p(y)^j|, |q(x)^i p(y)^j - q(y)^i p(y)^j|\}$
$\leq max\{|q(x)^i||p(x)^j - p(y)^j|, |p(y)^j||q(x)^i - q(y)^i|\}$
$\leq 1/p$ (resp. $\leq 1/2$) by lemma 5 and analogous
$|x^i p(x)^j - y^i p(y)^j| \leq max\{|x^i p(x)^j - x^i p(y)^j|, |x^i p(y)^j - y^i p(y)^j|\}$
$\leq max\{|x^i||p(x)^j - p(y)^j|, |p(y)^j||x^i - y^i|\}$
$\leq 1/p$ (resp. $\leq 1/2$) by lemma 5

We shall need lemmas 6 and 7 for the construction of an orthonormal base for $C^1(V_q \to K)$:

Lemma 6.
$$\left[{i+j \atop n}\right] = \sum_{s=0}^{n} \left[{j \atop n-s}\right]\left[{i \atop s}\right] q^{-(n-s)(-i+s)}$$

Proof. This follows immediately from [8], lemma 10 by putting first $s = n - k$ and then interchanging i and j.

Definition. *We define the sequence (ρ_n) as follows :*
$\rho_n = (q^m)^{i-i-} - 1$ if $n = im + j$, $0 \leq j < m$ and $i > 0$, $\rho_n = 1$ if $n < m$.

Lemma 7.
$|\rho_n| = min_{1 \leq s \leq n}\{|q^s - 1|\}$. $(n \in \mathbb{N}_0)$.

Proof. This follows immediately from [8], lemmas 2 and 3.

3 Orthonormal bases for $C(V_q \to K)$

Using the lemmas 1-5 in section 2, we can make orthonormal bases for $C(V_q \to K)$ with the aid of the following theorem :

Theorem 1. *Let $(p_n(x))$ and $(q_n(x))$ be sequences of continuous functions of the following form :*
for each n $p_n(x)$ is of the form $p_n(x) = \sum_{i \in I_n} a_{n,1}\{{x \atop i}\} + \sum_{i \in \{0,1,\ldots,n\} \setminus I_n} a_{n,i} e_i(x)$ with $|a_{n,n}| = 1$ and with $|a_{n,i}| < 1$ if $0 \leq i < n$ $(a_{n,i} \in \mathbb{Q}_p)$, and for each n we have $q_n(x) = \sum_{i \in J_n} b_{n,i}\{{x \atop i}\} + \sum_{i \in \{0,1,\ldots,n\} \setminus J_n} b_{n,i} e_i(x)$ with $|q_n(aq^n)| = 1$ and

$|b_{n,i}| \leq 1$ if $0 \leq i \leq n$ ($b_{n,i} \in \mathbb{Q}_p$). If (j_n) is a sequence in $\mathbb{N}$ and if (k_n) is a sequence in $\mathbb{N}_0$, then the sequences $(q_n(x)^{j_n} p_n(x)^{k_n})$ and $(x^{j_n} p_n(x)^{k_n})$ form orthonormal bases for $C(V_q \to K)$.

Proof. This proof is analogous to the proof of [8], theorem 5. We remark that for all n we have $\|p_n\|_\infty \leq 1$ and $\|q_n\|_\infty \leq 1$ (lemma 2), and that $p_n(x)$ and $q_n(x)$ are elements of $C(V_q \to \mathbb{Q}_p)$. By [1], 3.4.1 or [6], p. 123-133 it suffices to prove that $(q_n(x)^{j_n} p_n(x)^{k_n})$ and $(x^{j_n} p_n(x)^{k_n})$ form orthonormal bases for $C(V_q \to \mathbb{Q}_p)$ and by [1] proposition 3.1.5 p. 82 it suffices to prove that $(\overline{q_n(x)^{j_n} p_n(x)^{k_n}})$ and $(\overline{x^{j_n} p_n(x)^{k_n}})$ form vectorial bases for $C(V_q \to \mathbb{F}_p)$ (where $\overline{f(x)}$ stands for the canonical projection on $C(V_q \to \mathbb{F}_p)$, if f is in $C(V_q \to \mathbb{Q}_p)$ with $\|f\|_\infty \leq 1$). We distinguish two cases.

1) Let $q^m \equiv 1 \pmod{p^{k_0}}$, $q^m \not\equiv 1 \pmod{p^{k_0+1}}$ with $(p, k_0) \neq (2, 1)$, define C_t the space of the functions from V_q to $\mathbb{F}_p$ constant on balls of the type $\{x \in \mathbb{Z}_p : |x - \alpha| \leq p^{-(k_0+t)}\}$, $\alpha \in V_q$. Since $C(V_q \to \mathbb{F}_p) = \cup_{t<0} C_t$ ([8], lemma 4 and its proof) it suffices to prove that $(\overline{q_n(x)^{j_n} p_n(x)^{k_n}} | n < mp^t)$ and $(\overline{x^{j_n} p_n(x)^{k_n}} | n < mp^t)$ form bases for C_t. By the proof of [8], lemma 4, we can write V_q as the union of mp^t disjoint balls with radius $p^{-(k_0+t)}$ and with centers $aq^r(q^m)^n$, $0 \leq r \leq m-1$, $0 \leq n < p^t$. Let χ_i be the characteristic function of the ball with center aq^i. Using lemma 5, we have

$$\begin{aligned}
\overline{q_n(x)^{j_n} p_n(x)^{k_n}} &= \sum_{i=0}^{mp^t-1} \chi_i(x) \overline{q_n(aq^i)^{j_n} p_n(aq^i)^{k_n}} \\
&= \sum_{i=n}^{mp^t-1} \chi_i(x) \overline{q_n(aq^i)^{j_n} p_n(aq^i)^{k_n}}
\end{aligned}$$

since $|q_n(aq^i)^{j_n} p_n(aq^i)^{k_n}| < 1$ if $i < n$ (lemma 1) and hence the transition matrix from $(\chi_n | n < mp^t)$ to $(\overline{q_n(x)^{j_n} p_n(x)^{k_n}} | n < mp^t)$ is triangular since $|q_n(aq^n)^{j_n} p_n(aq^n)^{k_n}| = 1$ (lemma 1), so $(\overline{q_n(x)^{j_n} p_n(x)^{k_n}} | n < mp^t)$ forms a base for C_t. The proof for $(\overline{x^{j_n} p_n(x)^{k_n}})$ is analogous.

2) Let $q^m \equiv 3 \pmod 4$, $q = 1 + 2 + 2^2\varepsilon$, $\varepsilon = \varepsilon_0 + \varepsilon_1 2 + \varepsilon_2 2^2 + \ldots$, $\varepsilon_0 = \varepsilon_1 = \ldots = \varepsilon_{N-1} = 1$, $\varepsilon_N = 0$, define C_t te space of the functions from V_q to $\mathbb{F}_2$ constant on balls of the type $\{x \in \mathbb{Z}_2 : |x - \alpha| \leq 2^{-(N+2+t)}\}$, $\alpha \in V_q$. Since $C(V_q \to \mathbb{F}_2) = \cup_{t \geq 1} C_t$ ([8], lemma 5 and its proof) it

suffices to prove that $(\overline{q_n(x)^{j_n}p_n(x)^{k_n}}|n < 2^t)$ and $(\overline{x^{j_n}p_n(x)^{k_n}}|n < 2^t)$ form bases for C_t. By the proof of [8], lemma 5, we can write V_q as the union of 2^t disjoint balls with radius $2^{-(N+2+t)}$ and with centers aq^n, $0 \leq n < 2^t$. From now on the proof is analogous to the proof of 1).

Some examples.

1) If $(p_n(x))$ is a sequence of polynomials with coefficients in $\mathbb{Q}_p$ such that for all n we have that the degree of p_n is n, $|p_n(aq^n)| = 1$ and $|p_n(aq^i)| < 1$ if $0 \leq i < n$, and if (k_n) is a sequence in $\mathbb{N}_0$, then $(p_n(x)^{k_n})$ forms an orthonormal base for $C(V_q \to K)$. This follows immediately from lemma 1 and theorem 1, by putting $j_n = 0$ and $I_n = \{0, 1, \dots n\}$ and this for all n. The case $k_n = 1$ for all n can also be found in [8], theorem 4.

2) If (k_n) is a sequence in $\mathbb{N}_0$, then $(\{{x \atop n}\}^{k_n})$ forms an orthonormal base for $C(V_q \to K)$. Put therefore $p_n(x) = \{{x \atop n}\}$ in 1). If f is an element of $C(V_q \to K)$, and if s is a natural number different from zero, there exists a uniformly convergent expansion $f(x) = \sum_{n=0}^{\infty} \beta_n^{(s)} \{{x \atop n}\}^s$ and we are able to give an expression for the coeffiecients $\beta_n^{(s)}$. This can be found in [8], proposition 1.

3) If $(p_n(x))$ is a sequence in $C(V_q \to \mathbb{Q}_p)$ such that for all n we have $p_n(x) = \sum_{i=0}^{n} a_{n,i}e_i(x)$ with $|p_n(aq^n)| = 1$ and $|p_n(aq^i)| < 1$ if $0 \leq i < n$, and if (k_n) is a sequence in $\mathbb{N}_0$, then $(p_n(x)^{k_n})$ forms an orthonormal base for $C(V_q \to K)$. This follows immediately from lemma 1 and theorem 1, by putting $j_n = 0$ and by putting I_n equal to the empty set. The case $k_n = 1$ for all n can also be found in [9], theorem 2.

Remark. We can make an analogous result for the space $C(\mathbb{Z}p \to K)$: if we replace the polynomials $(\{{x \atop i}\})$ by $(({x \atop i}))$ (Mahler's base) and the functions $(e_i(x))$ by van der Put's base, then we can prove the following (we shall denote van der Put's base by $(g_i(x))$:

Let $(p_n(x))$ and $(q_n(x))$ be sequences of continuous functions on $\mathbb{Z}_p$ of the following form: for each n $p_n(x)$ is of the form $p_n(x) = \sum_{i \in I_n} a_{n,i}({x \atop i}) + \sum_{i \in \{0,1,\dots,n\} \setminus I_n} a_{n,i}g_i(x)$ with $|a_{n,n}| = 1$ and with $|a_{n,i}| < 1$ if $0 \leq i < n$ $(a_{n,i} \in \mathbb{Q}_p)$, and for each n we have

$q_n(x) = \sum_{i \in J_n} b_{n,i}\binom{x}{i} + \sum_{i \in \{0,1,\ldots,n\} \setminus J_n} b_{n,i} g_i(x)$ with $|q_n(n)| = 1$ and $|b_{n,i}| \leq 1$ if $0 \leq i \leq n$ ($b_{n,i} \in \mathbb{Q}_p$). If (j_n) is a sequence in $\mathbb{N}$ and if (k_n) is a sequence in $\mathbb{N}_0$, then the sequence $(q_n(x)^{j_n} p_n(x)^{k_n})$ forms an orthonormal base for $C(\mathbb{Z}_p \to K)$.

4 Continuously differentiable functions on V_q

In this section we give necessary and sufficient conditions for a continuous function defined on V_q to be continuously differentiable, and we find an orthonormal base for the space $C^1(V_q \to K)$. The result we'll find is analogous to the result for continuously differentiable functions on $\mathbb{Z}_p$ ([5], theorem 53.5) where we replace Mahler's base by the base $(\{{x \atop n}\})$. We remark that there is a one-to-one correspondence between $(u, v) \in V_q \times V_q$ and $(\frac{qyx}{a}, x)$ with $(x, y) \in V_q \times V_q$ (see [7], section 2). We shall use this several times in this section. Let ρ_n be as defined in section 2, then we can prove the following :

Proposition 1. *Let f be an element of $C(V_q \to K)$ with uniformly convergent expansion $f(x) = \sum_{n=0}^{\infty} a_n \{{x \atop n}\}$. If $lim_{n\to\infty}|a_n(\rho_n)^{-1}| = 0$, then f is an element of $C^1(V_q \to K)$.*

Proof. Let f be in $C(V_q \to K)$ with uniformly convergent expansion $f(x) = \sum_{n=0}^{\infty} a_n \{{x \atop n}\}$. Analogous to [5], theorems 53.4 and 53.5, we want to find an expression for $\phi_1 f(u, v)$ for special values for u and v. Therefore, let x, y be in $\{aq^n | n = 0, 1, 2, \ldots\}$, $x = aq^i$, $y = aq^j$ and suppose $y \neq a$ (i.e. $j \neq 0$). Then

$$\phi_1 f(\tfrac{yx}{a}, x) = \phi_1 f(x, \tfrac{yx}{a}) = \frac{f(\frac{yx}{a}) - f(x)}{\frac{yx}{a} - x} = \sum_{n=1}^{\infty} \frac{a_n}{aq^i(q^j - 1)}\left(\left[{i+j \atop n}\right] - \left[{i \atop n}\right]\right)$$

$$= \sum_{n=1}^{\infty} \frac{a_n}{aq^i(q^j-1)}\left(\sum_{s=0}^{n} \left[{j \atop n-s}\right]\left[{i \atop s}\right] q^{-(n-s)(-i+s)} - \left[{i \atop n}\right]\right) \text{ (by lemma 6)}$$

$$= \sum_{n=1}^{\infty} \frac{a_n}{aq^i(q^j-1)} \sum_{s=0}^{n-1} \left[{j \atop n-s}\right]\left[{i \atop s}\right] q^{-(n-s)(-i+s)}$$

since $\frac{1}{q^j-1}\left[{j \atop n-s}\right] = \frac{1}{q^{n-s}-1}\left[{j-1 \atop n-s-1}\right]$, we find, by putting $n = s + k + 1$, that

$$\phi_1 f(\frac{yx}{a}, x) = \sum_{k=0}^{\infty}\sum_{s=0}^{\infty} \frac{a_{k+s+1}q^{-s(k+1)}}{a^{k+1}(q^{k+1}-1)} x^k \{{x \atop s}\}\{{y/q \atop k}\}$$

and replacing y by yq this gives us, for all x, y in $\{aq^n | n = 0, 1, 2, \dots\}$

$$\phi_1 f(\frac{qyx}{a}, x) = \sum_{k=0}^{\infty}\sum_{s=0}^{\infty} \frac{a_{k+s+1}q^{-s(k+1)}}{a^{k+1}(q^{k+1}-1)} x^k \{{x \atop s}\}\{{y \atop k}\} \ (*)$$

Now $sup_{k+s+1=n}|\frac{a_{k+s+1}}{q^{k+1}-1}| = |a_n| max_{1\le k\le n}|\frac{1}{q^k-1}| = |a_n(\rho_n)^{-1}|$ (lemma 7), so if $lim_{n\to\infty}|a_n(r_n)^{-1}| = 0$, then $lim_{k+s\to\infty}|\frac{a_{k+s+1}}{q^{k+1}-1}| = 0$ and it is clear that (*) can be extended to a continuous function ([5], exercise 23.B). So we conclude : if $lim_{n\to\infty}|a_n(r_n)^{-1}| = 0$, then $f \in C^1(V_q \to K)$. This finishes the proof.

Remark. It is easy to prove that the functions $(x^k\{{x \atop s}\}\{{y \atop k}\})$ are orthonormal in $C(V_q \times V_q \to K)$.

Let A be the subset of $C(V_q \to K)$ defined as follows : if f is an element of $C(V_q \to K)$ with uniformly convergent expansion $f(x) = \sum_{n=0}^{\infty} a_n\{{x \atop n}\}$, then f is an element of A if and only if $lim_{n\to\infty}|a_n(\rho_n)^{-1}| = 0$.

Proposition 2. *The set A satisfies the following properties :*
1) A is a subset of $C^1(V_q \to K)$ containing the polynomials 2) A is closed for $\|\cdot\|_1$ 3) A is a subalgebra of $C^1(V_q \to K)$

Proof.
1) From proposition 1 it follows that A is a subset of $C^1(V_q \to K)$. It is clear that A contains the polynomials.
2) Suppose $f = lim_{n\to\infty} f_n$ for the norm $\|\cdot\|_1$ where $f_n \in A$ for all n. Then f is clearly continuous. So there exists the following uniformly convergent expansions : $f(x) = \sum_{k=0}^{\infty} a_k\{{x \atop k}\}$, $f_n(x) = \sum_{k=0}^{\infty} a_{n,k}\{{x \atop k}\}$, with $lim_{k\to\infty} \mid a_k \mid= 0$, $lim_{k\to\infty} \mid a_{n,k} \mid= 0$ for all n, $lim_{k\to\infty} \mid a_{n,k}(\rho_k)^{-1} \mid= 0$ for all n. Suppose that $lim_{k\to\infty}|a_k(\rho_k)^{-1}| \neq 0$. This will lead to a contradiction. Since $lim_{k\to\infty} \mid a_k(\rho_k)^{-1} \mid\neq 0$ there exists an $\epsilon > 0$ such that for all $\eta \in \mathbb{N}$, there exists an $n > \eta$ such that $|a_n(\rho_n)^{-1}| > \epsilon$. Let I be the set defined as follows : $I = \{k \in \mathbb{N}_0 : |a_k(\rho_k)^{-1}| > \epsilon\}$. Then I is infinite. Let ϵ be as above. Then there exists a $J \in \mathbb{N}$, such that for all $n \ge J$ we have $\|f - f_n\|_1 < \epsilon$. In particular, $sup_{x\neq y}\{|\frac{(f-f_J)(x)-(f-f_J)(y)}{x-y}|\} < \epsilon$, and from the calculations in proposition 1 it follows that

$|\phi_1(f-f_J)(\frac{qyx}{a},x)| = |\sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\frac{(a_{k+s+1}-a_{J,k+s+1})q^{-s(k+1)}}{a^{k+1}(q^{k+1}-1)}x^k\{{x \atop s}\}\{{y \atop k}\}| \leq$ ϵ for all x,y in $\{aq^n|n=0,1,2,\ldots\}$. From this it is easy to see that $|\frac{a_{k+s+1}-a_{J,k+s+1}}{q^{k+1}-1}| \leq \epsilon$ for all k and s, so $sup_{k,s}\{|\frac{a_{k+s+1}-a_{J,k+s+1}}{q^{k+1}-1}|\} \leq \epsilon$ and thus $sup_n\{|(a_n-a_{J,n})(\rho_n)^{-1}|\} \leq \epsilon$. Then, if $n \in I$ we have $|a_{J,n}(\rho_n)^{-1}| = |(a_{J,n}-a_n)(\rho_n)^{-1}+a_n(\rho_n)^{-1}| > \epsilon$, and from this it follows that $lim_{k\to\infty}$ $|a_{J,k}(\rho_k)^{-1}| \neq 0$ since I is infinite. This is impossible and we conclude that A is closed.

3) If $f,g \in A$, $k,j \in K$, then we immediately have that $kf+jg \in A$, and if r and u are polynomials ($\in A$) then ru is a polynomial and also an element of A. From the Weierstrass-theorem for C^1-functions ([2], theorem 1.4) it follows that for each $f,g \in A$ we have $fg \in A$ since A is closed.

Theorem 2. *Let f be an element of $C(V_q \to K)$ with uniformly convergent expansion $f(x) = \sum_{n=0}^{\infty} a_n\{{x \atop n}\}$. Then f is an element of $C^1(V_q \to K)$ if and only if $lim_{n\to\infty}|a_n(\rho_n)^{-1}| = 0$.*
If f is an element of $C^1(V_q \to K)$ then $||f||_1 = max_{n\geq 0}\{|a_n(\rho_n)^{-1}|\}$ and the functions $(\rho_n\{{x \atop n}\})$ form an orthonormal base for $C^1(V_q \to K)$.

Proof. From proposition 2 and the Weierstrass-Stone theorem for C^1-functions ([2], theorem 2.10) it follows that $A = C^1(V_q \to K)$. So f is an element of $A = C^1(V_q \to K)$ if and only if $lim_{n\to\infty}|a_n(\rho_n)^{-1}| = 0$. Let us first remark the following : since $lim_{n\to\infty}|a_n(\rho_n)^{-1}| = 0$, we have $sup_{n\geq 1}\{|a_n(\rho_n)^{-1}|\} = max_{n\geq 1}\{|a_n(\rho_n)^{-1}|\}$ and since $sup_{k,s\geq 0}\{|\frac{a_{k+s+1}}{q^{k+1}-1}|\} = sup_{n\geq 1}\{|a_n(\rho_n)^{-1}|\}$ with $k+s+1=n$, we have $max_{k,s\geq 0}\{|\frac{a_{k+s+1}}{q^{k+1}-1}|\} = sup_{k,s\geq 0}\{|\frac{a_{k+s+1}}{q^{k+1}-1}|\} = max_{n\geq 1}\{|a_n(\rho_n)^{-1}|\}$. From (*) it follows that for all x,y in $\{aq^n|n=0,1,2,\ldots\}$
$\phi_1 f(\frac{qyx}{a},x) = \sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\frac{a_{k+s+1}q^{-s(k+1)}}{a^{k+1}(q^{k+1}-1)}x^k\{{x \atop s}\}\{{y \atop k}\}$ and by continuity it then follows that for all x,y in V_q with y different from aq^{-1} we have

$$\phi_1 f(\frac{qyx}{a},x) = \sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\frac{a_{k+s+1}q^{-s(k+1)}}{a^{k+1}(q^{k+1}-1)}x^k\{{x \atop s}\}\{{y \atop k}\}$$

Then we immediately have $|\phi_1 f(\frac{qyx}{a},x)| \leq max_{k,s\geq 0}\{|\frac{a_{k+s+1}}{q^{k+1}-1}|\}$ for all

x, y in V_q with $y \neq aq^{-1}$ and so we have $||\phi_1 f||_\infty \leq max_{k,s\geq 0}\{|\frac{a_{k+s+1}}{q^{k+1}-1}|\}$. If $max_{k,s\geq 0}\{|\frac{a_{k+s+1}}{q^{k+1}-1}|\} = 0$ it is clear that $||\phi_1 f||_\infty = max_{k,s\geq 0}\{|\frac{a_{k+s+1}}{q^{k+1}-1}|\}$. If $max_{k,s\geq 0}\{|\frac{a_{k+s+1}}{q^{k+1}-1}|\} > 0$, then put $I = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |\frac{a_{j+i+1}}{q^{j+1}-1}| = max_{k,s\geq 0}\{|\frac{a_{k+s+1}}{q^{k+1}-1}|\}\}$. Now let $S = min\{i \in \mathbb{N}$: there exists a $j \in \mathbb{N}$ such that $(i,j) \in I\}$ and $T = min\{t \in \mathbb{N} : (S,t) \in I\}$ then it is easy to see that $|\phi_1 f(\frac{q}{a} aq^S aq^T, aq^S)| = |\frac{a_{T+S+1}}{q^{T+1}-1}| = max_{k,s\geq 0}\{|\frac{a_{k+s+1}}{q^{k+1}-1}|\}$ and so we conclude $||\phi_1 f||_\infty = max_{k,s\geq 0}\{|\frac{a_{k+s+1}}{q^{k+1}-1}|\} = max_{n\geq 1}\{|a_n(\rho_n)^{-1}|\}$. Since $||f||_1 = max\{||f||_\infty, ||\phi_1 f||_\infty\} = max\{max_{n\geq 0}\{|a_n|\}, max_{n\geq 1}\{|a_n(\rho_n)^{-1}|\}\}$ and since $|(\rho_n)^{-1}| \geq 1$ for all n we conclude that $||f||_1 = max_{n\geq 0}\{|a_n(\rho_n)^{-1}|\}$. From this it follows that $||\{{x \atop n}\}||_1 = |(\rho_n)^{-1}|$ so $||\rho_n\{{x \atop n}\}||_1 = 1$. Furthermore, $f(x) = \sum_{n=0}^{\infty} a_n\{{x \atop n}\} = \sum_{n=0}^{\infty} \frac{a_n}{\rho_n}\rho_n\{{x \atop n}\}$ with $||f||_1 = max_{n\geq 0}\{|a_n(\rho_n)^{-1}|\} = max_{n\geq 0}\{||\frac{a_n}{\rho_n}\rho_n\{{x \atop n}\}||_1\}$ so the functions $(\rho_n\{{x \atop n}\})$ form an orthonormal base for $C^1(V_q \to K)$. This finishes the proof.

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