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Denting point in the space of operator-valued continuous maps

Ryszard GRZĄŚLEWICZ
and
Samir B. HADID

Abstract

In the paper[1] we describe the geometric properties of the space of continuous functions with values in the space of operators acting on a Hilbert space. In particular we show that

$$\operatorname{dent} B(\mathcal{L}(H)) = \begin{cases} \operatorname{ext} B(\mathcal{L}(H)) & \text{if } \dim H < \infty \text{ and } \operatorname{card} K < \infty \\ \emptyset & \text{if } \dim H = \infty \text{ or } \operatorname{card} K = \infty \end{cases}$$

and x-ext $C(K,\mathcal{L}(H)) = \operatorname{ext} C(K,\mathcal{L}(H))$.

In this paper we consider the geometric properties of the space of continuous functions from a compact Hausdorff space K with values in the space of operators acting on a Hilbert space H. Namely, we deal with the unit ball. We consider such points of the unit ball as strongly extreme, and denting points.

For a Banach space E we denote by $\mathbf{B}(E)$ and $\mathbf{S}(E)$ respectively the unit ball and the unit sphere of E.

Recall that a point $\mathbf{q}$ of a convex set $Q \subset E$ is *strongly extreme* ($\mathbf{q} \in s\text{-ext}\, Q$) if $\| \frac{x_n+y_n}{2} - \mathbf{q} \| \to 0$ for $x_n, y_n \in Q$ implies $\| x_n - \mathbf{q} \| \to 0$ (or equivalently $\| x_n - y_n \| \to 0$, since $x_n - \mathbf{q} = \frac{x_n-y_n}{2} + \left(\frac{x_n+y_n}{2} - \mathbf{q}\right)$); *exposed* ($\mathbf{q} \in \exp Q$) if there exists $\xi \in Q^*$ such that $\xi(q) = \sup \xi(Q) >$

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$\xi(x)$ for all $x \in Q \setminus \{q\}$; *strongly exposed* ($q \in s\text{-}\exp Q$) if it is exposed and if $\xi(x_n) \to \xi(q)$ for $x_n \in Q$ then $\| x_n - q \| \to 0$; and *denting* ($q \in \operatorname{dent} Q$) if $q \notin \overline{\operatorname{conv}}(Q \setminus \{q + \varepsilon \mathbf{B}(E)\})$ for all $\varepsilon > 0$. Point out that $s\text{-}\exp Q \subset \operatorname{dent} Q \subset s\text{-}\operatorname{ext} Q \subset \operatorname{ext} Q$ and $s\text{-}\exp Q \subset \exp Q \subset \operatorname{ext} Q$. Moreover, if Q is compact then $\operatorname{dent} Q = s\text{-}\operatorname{ext} Q = \operatorname{ext} Q$ and $s\text{-}\exp Q = \exp Q$. Note that if $q \in \operatorname{ext} Q$ is a point of continuity for $Q(x_n \to q$ weakly , $x_n \in Q$, implies $x_n \to q$ in norm) then $q \in \operatorname{dent} Q$([10]).

For an operator $T : E \to E$ we denote by
$\operatorname{IsDom} T = \{x \in E : \| Tx \| = \| x \|\}$ its isometric domain.

Let H be a (real or complex) Hilbert space equipped with the inner product $< \cdot, \cdot >$. Obviously $\operatorname{ext} \mathbf{B}(H) = s\text{-}\exp \mathbf{B}(H) = \text{smooth } \mathbf{B}(H) = \mathbf{S}(H)$. By $\mathcal{L}(H)$ we denote the space of bounded operators from H into H. The space $\mathcal{L}(H)$ is equipped with the standard operator norm $\| T \| = \sup\{\| Tx \| : x \in \mathbf{B}(H)\}$. Note that **IsDom** T is a closed linear subspace for all $T \in \mathbf{B}\mathcal{L}(H))$. Moreover, $T(\{x\}^{\perp}) \subset (Tx)^{\perp}$ for $x \in$ **IsDom** T and $T\left((\mathbf{IsDom}T)^{\perp}\right) \perp T(\mathbf{IsDom}T), T \in \mathbf{B}(\mathcal{L}(H))$. An operator $T \in \mathcal{L}(H)$ is called *isometry* if **IsDom** $T = H$ and $T \in \mathcal{L}(H)$ is a *coisometry* if its adjoint $T^* \in \mathcal{L}(H)$ is an isometry.

For $y, z \in H$ we denote by $y \bigotimes z$ the one dimensional operator defined by $(y \bigotimes z)(x) = y < x, z >, x \in H$.

Let $C(K, E)$ denote the Banach space for all continuous functions from a compact Hausdorff space K into a Banach space E equipped with the supremum norm $\| f \| = \sup_{k \in K} \| f(k) \|_E$.

It is easy to see that for a convex set $Q \subset E$ if $f(K) \subset \operatorname{ext} Q$ then $f \in \operatorname{ext}\{f \in C(K, E) : f(K) \subset Q\}$. There is a natural question for which classes of convex sets Q the inverse implication is a characterization of extreme points. Negative example can be easy given for $Q \subset \mathbb{R}^3$ with non-closed $\operatorname{ext} Q$. More complicated negative example of closed symmetric subset Q of $\mathbb{R}^4$ was presented in [1]. In fact it is presented an example of $f \in \operatorname{ext} \mathbf{B}(C(K, E))$ with $f(k) \notin \operatorname{ext} \mathbf{B}(E)$ for all $k \in K$.

Using Michael's selection theorem ([11]) one can prove that $\operatorname{ext}\{f \in C(K, E) : f(K) \subset Q\} = \{f \in C(K, E) : f(K) \subset \operatorname{ext} Q\}$ for any stable convex subset Q of E. Recall that a convex set $Q \subset E$ is said to be *stable* if the barycenter map $Q \times Q \ni (x, y) \to \frac{x+y}{2} \in Q$ in open. Point out that in finite dimensional space a set is stable (see [12]) if and only if all m-skeletons ($m = 0, 1, \cdots, n$) of Q are closed (a m-skeleton od Q is a set of all $x \in Q$ such that the face generated by x in Q has dimension less than or equal to m).

For the space of operators on Hilbert space we have

$$\operatorname{ext} \mathbf{B}(\mathcal{L}(H)) = \{T \in \mathcal{L}(H) : T \text{ is isometry or coisometry }\} \qquad ([8],[4])$$

$$\exp \mathbf{B}(\mathcal{L}(H)) = \begin{cases} \operatorname{ext} \mathbf{B}(\mathcal{L}(H)) & \text{if } H \text{ is separable} \\ \emptyset & \text{if } H \text{ is not separable} \end{cases},$$

$$s - \exp \mathbf{B}(\mathcal{L}(H)) = \operatorname{dent} \mathbf{B}(\mathcal{L}(H)) = \begin{cases} \operatorname{ext} \mathbf{B}(\mathcal{L}(H)) & \text{if } \dim H < \infty \\ \emptyset & \text{if } \dim H = \infty \end{cases},$$

$$s - \operatorname{ext} \mathbf{B}(\mathcal{L}(H)) = \operatorname{ext} \mathbf{B}(\mathcal{L}(H)) \qquad ([5],[7]).$$

$$\mathbf{smooth}\mathbf{B}(\mathcal{L}(H)) = \{T \in \mathbf{S}(\mathcal{K}(H)) : \dim \ \operatorname{IsDom} T = 1$$

and $\operatorname{dist}(T, \mathcal{K}(H)) < 1\}$ ([9]).

For the space of continuous functions with values in $\mathcal{L}(H)$ we have ([6]) the following results

$$\operatorname{ext} \mathbf{B}(C(K, \mathcal{L}(H))\,) = \{f \in C(K, \mathcal{L}(H)) \ : \ f(K) \subset \operatorname{ext} \mathbf{B}(\mathcal{L}(H))\},$$

$$\exp \mathbf{B}(C(K, \mathcal{L}(H))\,) = \begin{cases} \operatorname{ext} \mathbf{B}(C(K, \mathcal{L}(H))\,) & \text{if } H \text{ is separable} \\ \emptyset & \text{if is not separable} \end{cases},$$

$$s - \exp \mathbf{B}(\mathcal{L}(H)) = \begin{cases} \operatorname{ext} \mathbf{B}(\mathcal{L}(H)) & \text{if } H \text{ is separable} \\ \emptyset & \text{if } H \text{ is not separable} \end{cases},$$

$$s - \exp \mathbf{B}(\mathcal{L}(H)) = \begin{cases} \operatorname{ext} \mathbf{B}(\mathcal{L}(H)) & \text{if } \dim H < \infty \\ \emptyset & \text{if } \dim H = \infty \end{cases}.$$

The aim of this paper is to continue investigation giving the characterizations of strongly extreme and denting points of the unit ball of $C(K, \mathcal{L}(H))$. Namely we show that strongly extreme points of $\mathbf{B}(C(K, \mathcal{L}(H))\,)$ coincides and

$$\operatorname{dent} \mathbf{B}(\mathcal{L}(H)) = \begin{cases} \operatorname{ext} \mathbf{B}(\mathcal{L}(H)) & \text{if } \dim H < \infty \text{ and } \operatorname{card} K < \infty \\ \emptyset & \text{if } otherwise \end{cases}.$$

Theorem 1 *For any Hilbert space H we have*

$$s - \operatorname{ext} \mathbf{B}(C(K, \mathcal{L}(H)) = \operatorname{ext} \mathbf{B}(C(K, \mathcal{L}(H)))$$

Proof. Let $f \in \operatorname{ext} \mathbf{B}(C(K, \mathcal{L}(H)))$. Fix $\varepsilon > 0$. We need to show that there exists $\delta > 0$ such that $\| \frac{g_n+h_n}{2} - f \| < \delta, x, y \in \mathbf{B}(H)$ implies $\| g_n - h_n \| < \varepsilon$.

From the uniform convexity of H ([2]) there exists δ_ε such that $\| \frac{x+y}{2} \| > 1 - \delta_\varepsilon$ implies $\| x - y \| < \varepsilon$.

Put $K_1 = \{k \in K : f(k)$ is an isometry $\}$ and $K_2 = \{k \in K : f(k)$ is an coisometry $\}$.

Suppose that $k \in K_1$. For $x \in \mathbf{S}(H)$ we have $\| \frac{g_n(k)x+h_n(k)x}{2} \| \geq \| f_n(k)x \| - \| \frac{g_n(k)x+h_n(k)x}{2} - f_n(k)x \| \geq 1 - \delta_\varepsilon$.
Thus $\| g_n(k)x - h_n(k)x \| \leq \varepsilon$. Hence $\| g_n(k) - h_n(k) \| \leq \varepsilon$ for all $k \in K_1$.

Now suppose that $k \in K_2$. Obviously $\| f^* \| = \| f \|$ and $f^*(k)$ is an isometry if and only if $f(k)$ is a coisometry. Now we consider f^* instead of f and we get $\| g_n^*(k) - h_n^*(k) \| = \| g_n(k) - h_n(k) \| \leq \varepsilon$ for all $k \in K_2$.

Therefore $\| g_n - h_n \| \leq \varepsilon$.□

Proposition 1 *If* $\dim H = \infty$ *or* $\operatorname{card} K = \infty$ *then*

$$\operatorname{dent} \mathbf{B}(C(K, \mathcal{L}(H))) = \emptyset.$$

Proof. At first consider the case $\dim H = \infty$.
Fix $f \in \operatorname{ext} \mathbf{B}(C(K, \mathcal{L}(H)))$. Obviously dim IsDom $f(k) = \infty$ for all $k \in K$.. Fix $k_o \in K$. Let $\{e_i\}_{i=1}^{\infty}$ be orthonormal system in IsDom $f(k_0)$. Let P_j be an orthogonal projection on $\{e_j\}^{\perp}$. Put $f_j = P_j f$. Obviously $\| f_j - f \| \geq \| f_j(k_0) - f(k_0) \| = \| (f(k_o)e_j) \otimes e_j \| = 1$. We have $\| I - \frac{1}{n}\sum_{i=1}^n P_i \| = \| \frac{1}{n}\sum_{i=1}^n e_i \otimes e_i \| = \frac{1}{n}$ and $\| f - \frac{1}{n}\sum_{i=1}^n f_i \| \leq \| I - \frac{1}{n}\sum_{i=1}^n P_i \| = \frac{1}{n}$, i.e. $f \notin \operatorname{dent} \mathbf{B}(C(K, \mathcal{L}(H)))$.

Now consider the case when $\operatorname{card} K = \infty$. Choose a sequence $\{k_n\}$ of distinct points of K such that $\lim_n k_n = k_o$. We choose the sequence of continuous functions $\gamma_n : K \to [0, 1]$ such that $\gamma_n(k_n) = 1$ and $\operatorname{supp} \gamma_{n_1} \cap \operatorname{supp} \gamma_{n_2} = \emptyset$ if $n_1 \neq n_2$. Put $f_j = (1-\gamma_j)f_o \in \mathbf{B}(C(K, \mathcal{L}(H)))$. It is easy to see that $\| f_j - f \| \geq f_j(k_j) - f(k_j) \| = 1$. We have $\| f - \frac{1}{n}\sum_{i=1}^n f_i \| \leq \| \frac{1}{n}\sum_{i=1}^n h_i \| = \frac{1}{n}$, i.e. $f \notin \operatorname{dent} \mathbf{B}(C(K, \mathcal{L}(H)))$.

■

As a consequence of Proposition 1 we get the following characterization of denting points.

Theorem 2 *For any Hilbert space H we have*

$$\operatorname{dent} \mathbf{B}(C(K, \mathcal{L}(H))) = \begin{cases} \operatorname{ext} \mathbf{B}(C(K, \mathcal{L}(H))) & if \dim H < \infty and \operatorname{card} K < \infty \\ \emptyset & if \dim H = \infty or \operatorname{card} K = \infty \end{cases}$$

Proof. If $\dim H < \infty$ and $\operatorname{card} K < \infty$ then $C(K, \mathcal{L}(H))$ is finite dimensional, so $\mathbf{B}(C(K, \mathcal{L}(H)))$ is compact. Hence exposed and strongly exposed are coincide. In view of Theorem 2 and 3 in [6] we get s-$\exp \mathbf{B}(C(K, \mathcal{L}(H))) = \operatorname{ext} \mathbf{B}(C(K, \mathcal{L}(H)))$, which finish the proof.

■

Question.

In [3] it is shown that the unit ball of $\mathcal{L}(H)$ is stable if $\dim H < \infty$. Is $\mathbf{B}(\mathcal{L}(H))$(and $\mathbf{B}(\mathcal{K}(H))$) stable for infinite dimensional H?

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Institute of Matematics
Politechnika
Wb. Wyspiańskiego 27
PL-50370 WROCLAW, Poland.
e-mail: GRZASLEW@im.pwr.wroc.pl
(in internet)
Institute of Mathematics
University of Mu'tah
P.O.Box 7
AL-KARAK, Jordan.

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