REVISTA MATEMÁTICA de la
Universidad Complutense de Madrid
Volumen 8, número 2: 1995
http://dx.doi.org/10.5209/rev_REMA.1995.v8.n2.17647

Metabelian Groups Acting on Compact Riemann Surfaces

G. GROMADZKI[1]

ABSTRACT. A metabelian group G acting as automorphism group on a compact Riemann surface of genus $g \geq 2$ has order less than or equal to $16(g-1)$. We calculate for which values of g this bound is achieved and on these cases we calculate a presentation of the group G.

1. INTRODUCTION

Given a class of finite groups $\mathcal{F}$ denote by $N(g, \mathcal{F})$ the order of a largest group from $\mathcal{F}$ that can stand as a group of automorphisms of a compact Riemann surface of genus $g \geq 2$. The exact values of $N(g, \mathcal{F})$ for all g are known only for cyclic [9] and abelian [14] groups. However the bounds for $N(g, \mathcal{F})$ have been intensively studied in the literature. For the classes of all finite, nilpotent, supersoluble, soluble and metabelian groups, an upper bound for $N(g, \mathcal{F})$ as well as infinite

[1] Supported by Direction General de Invextigacion Cientifica y Tecnica (DGICYT) of the Spanish Ministery of Education and Sciences.

1991 Mathematics Subject Classification:

Servicio publicaciones Univ. Complutense. Madrid, 1995.

sequences of g for which the corresponding bounds are sharp were found in [10,3,4,5,6,11,12,13,15,16] among the vaste literature, [18,19], [7], [1,8] and [2] respectively. However the problem of finding for given $\mathcal{F}$ all g and all groups for which the corresponding bound is achieved is rather difficult. In this paper we solve it for the class of metabelian groups. Partial results have been obtained by Chetiya and Patra in [2] where they proved that a metabelian group of automorphisms of a compact Riemann surface of genus $g \geq 2$ ($g \neq 2, 3, 5$) has at most $16(g-1)$ elements. In this paper we go much further by describing exactly those values of g for which this bound is sharp, finding the presentations of all corresponding groups by means of defining generators and relations and finally characterizing in terms of these groups those surfaces which are symmetric i.e., admitting an anticonformal involution. More precisely we prove the following theorems.

Theorem 1.1. *A necessary and sufficient condition for the existence of a Riemann surface of genus $g \geq 2$ admitting a metabelian group of automorphisms of order $16(g-1)$ is that $g = 2$ or $g = k^2\beta + 1$ is odd, where k is an arbitrary positive integer and β is any integer such that -1 is a quadratic residue mod β (i.e. -1 is a square in the ring of integers modulo β).*

Theorem 1.2. *Let $g = 2$ or $g = k^2\beta + 1$ be an odd integer, where β is an integer dividing $1 + \alpha^2$ for some α. Then*

$G_1 = \langle x, y | x^2, y^4, (xy)^8, (xy)^4(yx)^4, [y, x]^{k\beta}, [x, y^{-1}]^k[y, x]^{k\alpha} \rangle$
for k even,

$G_2 = \langle x, y | x^2, y^4, (xy)^8, (xy)^4(yx)^4, [y, x]^{k\beta}(xy)^4, [x, y^{-1}]^k[y, x]^{k\alpha} \rangle$
for $k + \beta$ odd,

$G_3 = \langle x, y | x^2, y^4, (xy)^8, (xy)^4(yx)^4, [y, x]^{k\beta}(xy)^4, [x, y^{-1}]^k[y, x]^{k\alpha}(xy)^4 \rangle$
for $k + \beta$ and $\alpha + \beta$ odd,

$G_4 = \langle x, y | x^2, y^4, (xy)^8, (xy)^4(yx)^4, [y, x]^{k\beta}, [x, y^{-1}]^k[y, x]^{k\alpha}(xy)^4 \rangle$
for k and β even,

$G_5 = \langle x, y | x^2, y^4, y^2(xy)^4 \rangle$ *for $g = 2$*

are the metabelian groups of order $16(g-1)$ acting as groups of automorphisms on compact Riemann surfaces of genus g. Conversely, every such group G is isomorphic to some G_i defined above.

Theorem 1.3. *A Riemann surface of genus $g \geq 2$ having a metabelian group G of automorphisms of order $16(g-1)$ is symmetric if and only if G is one of the following groups: G_1, G_2, G_3 for $\beta = 1, 2$, G_4 for $\beta = 2$ or G_5.*

2. PROOFS

A finite group G is said to be a (k, l, m)-*group* if it can be generated by two elements of order k and l whose product has order m. From [2] it follows that the problem of describing metabelian groups that can occur as groups of automorphisms of order $16(g-1)$ of a compact Riemann surface of genus $g \geq 2$ is equivalent to the purely group theoretical problem of finding all finite metabelian $(2, 4, 8)$-groups.

We start with a series of elementary results concerining presentations of groups of small order. Throughout all the paper we shall use left hand notations for commutators and conjugations i.e. $[x, y] = xyx^{-1}y^{-1}$ and $x^y = yxy^{-1}$. We denote by G' the commutator subgroup of G and by $G_{ab} = G/G'$ its abelianization.

Lemma 2.1. *The group G with the presentation $\langle x, y | x^2, y^4, (xy)^4 y^2 \rangle$ is the only $(2, 4, 8)$-group of order 16.*

Proof. Let G be a group of order 16 generated by elements a and b of order 2 and 4 respectively whose product has order 8. The subgroup H generated by ab is normal in G as a subgroup of index 2. So $ba = a(ab)a = (ab)^i$, where $i = 1, 3, 5$ or 7. But if $i = 1$ then G is abelian which is impossible, whilst $ba = (ab)^7$ implies $b^2 = 1$ which is also impossible. Finally, $ba \neq (ab)^5$ since otherwise $b^2 = (ab)^6$ and so $(ab)^4 = b^4 = 1$. Therefore $ba = (ab)^3$ and so $b^2 = (ab)^4$. Thus we are done since the group from the lemma has clearly order 16.

Lemma 2.2. *Let G be a metabelian (2,4,8)-group. Then there exists a normal subgroup H of G order 2 such that G/H is a (2,4,4)-group and $G_{ab} \cong (G/H)_{ab}$.*

Proof. G is generated by elements a and b of order 2 and 4 respectively whose product has order 8. Now it is easy to check that

$$(ab)^4(ba)^4 = b^2[[b, a], [b^{-1}, a]]b^{-2}.$$

So since ab is an element of order 8 and G is metabelian we obtain $(ab)^4 = (ba)^4$. Therefore

$$((ab)^4)^a = (ba)^4 = (ab)^4, \quad ((ab)^4)^b = (ba)^4 = (ab)^4$$

and thus $H = \langle (ab)^4 \rangle$ is the subgroup we are looking for. ■

Lemma 2.3. *There are no (2,4,4)-groups of order 16 with the abelianization* $Z_2 \oplus Z_2$.

Proof. Assume to the contrary that G is such a group and let a and b be generators of order 2 and 4 respectively, whose product has order 4. Then $b^2, (ab)^2 \in G'$. Obviously $b^2 \neq (ab)^2$ and so they generate G'. Thus

$$ab^2a = b^2 \quad \text{or } ab^2a = (ab)^2 \quad \text{or } ab^2a = (ab)^2b^2.$$

However $ab^2a \neq b^2$ since otherwise $G = \langle a, b | a^2, b^4, (ab)^4, (ab^2)^2 \rangle$ as the last is a group of order 16 whilst on the other hand $G_{ab} = Z_2 \oplus Z_4$. Also $ab^2a \neq (ab)^2$ since in the other case G would be abelian. Finally in the last case $(ab)^2 = 1$ which is also implssible. ■

Lemma 2.4. *Let G be a metabelian (2,4,8)-group. Then G has order $16N$, where $N = 1$ or N is an even integer. Furthermore the abelianization G_{ab} of G is isomorphic to $Z_2 \oplus Z_4$ or to $Z_2 \oplus Z_2$.*

Proof. A (2,4,8)-group has order $16N$ by [2]. First we shall show that G contains a normal subgroup H of index 8. For, notice that since G' is abelian, $G' = K \oplus L$, where L is the Sylow 2-subgroup of G'. Clearly K and L are normal in G. Now since $\tilde{G} = G/K$ is a 2-group it contains a normal subgroup $\tilde{H}$ of index 8. So the subgroup H of G for which $\tilde{H} = H/K$ is a normal subgroup of G of index 8 as $G/H \simeq \tilde{G}/\tilde{H}$.

Now from the one hand G_{ab} is clearly a factor group of $Z_2 \oplus Z_4$ whilst from the other one it has $(G/H)_{ab}$, which is either $Z_2 \oplus Z_4$ or $Z_2 \oplus Z_2$, as a homomorphic image. Thus $G_{ab} = Z_2 \oplus Z_4$ or $G_{ab} = Z_2 \oplus Z_2$.

To finish the proof, suppose that N is odd. Then G/K has order 16 and is still a (2,4,8)-group. So by lemma 2.1 G/K has the presentation

$\langle x, y \mid x^2, y^4, (xy)^4y^2\rangle$ and therefore $Z_2 \oplus Z_2 = (G/K)_{ab} = G/G'K = G/G' = G_{ab}$ since $K \subsetneq G'$. Thus L is a group of order 4 and clearly G/L must be a (2,4,4)-group of order $4N$. A contradiction, since on the one side we have $(G/L)/(G'/L') \simeq G_{ab} \simeq Z_2 \oplus Z_2$ whilst on the other one this factor must still be a (2,4,4)-group as $G'/L \simeq K$ has odd order. ∎

Observe that from the above lemma follows that a necessary condition for the existence of a metabelian (2,4,8)-group of order $16N$ is that N is even in contrary to [2] where it is claimed that there exists such a group of order $16n^4$ for arbitrary positive integer n.

Lemma 2.5. *The only metabelian (2,4,4)-group with the abelianization $Z_2 \oplus Z_2$ is the quaternion group Q of order 8.*

Proof. Clearly, the quaternion group is the only nonabelian (2,4,4)-group of order 8. Let G be a group in question of the smallest possible order greater than 8 and let $G' = K \oplus L$, where L is the Sylow 2-subgroup of G'.

Now if $K = 1$ then G is a 2-group which by lemma 2.3 has order greater than 16. Let H be a normal subgroup of G of order 2. Clearly, $(G/H)_{ab} = Z_2 \oplus Z_2$ and so by the minimality of G, G/H is (2,2,2) or (2,2,4)-group. Thus G has order 8 or 16, a contradiction. So we see that Q is the only (2,4,4) metabelian 2-group with the abelianization $Z_2 \oplus Z_2$.

Finally if $K \neq 1$ then G/K is a metabelian 2-group being a (2,4,4)-group with abelianization $Z_2 \oplus Z_2$. But then $G/K = Q$ and thus in particular $L = Z_2$. So by the minimality of G, G/L is a (2,2,2) or (2,2,4)-group of order $4N$ where $N \neq 1$ is odd, a contradiction. ∎

Our task is to to find all metabelian (2,4,8)-groups. According to the second part of the Lemma 2.4 we divide our study into two parts. The following Proposition is an immediate consequence of lemmas 2.1, 2.2 and 2.5.

Proposition 2.6 *A metabelian (2,4,8)-group whose abelianization is isomorphic to $Z_2 \oplus Z_2$ has order 16 and the presentation*
$\langle x, y \mid x^2, y^4, (xy)^4y^2\rangle$.

The next Proposition together with the previous one gives the proof of Theorem 1.1.

Proposition 2.7. *There exists a metabelian (2,4,8)-group G with $G_{ab} \simeq Z_2 \oplus Z_4$ of order 16M if and only if $M = k^2\beta$ is even, where k is an arbitrary positive integer and β is any integer dividing $1 + \alpha^2$ for some α.*

Proof. Let $\Omega = \langle x, y \mid x^2, y^4 \rangle$ be the free product of two cyclic groups of order 2 and 4. Then $G = \Omega/K$ and $G' = N/K$ for some subgroups K and N of Ω. We first look at the group N. Notice that $G/G' = \Omega/N$. Thus $N = \Omega'$ and $A = [y, x]$, $B = [y^{-1}, x]$, $C = (y^2x)^2$ belong to N. On the other hand it is easy to check that

$$A^x = A^{-1},\ A^y = CA^{-1},$$

$$B^x = B^{-1},\ B^y = A^{-1}, \tag{2.1}$$

$$C^x = C^{-1},\ C^y = BA^{-1}.$$

Hence $N = \langle A, B, C\rangle$. By the Kurosh subgroup theorem N is a free group and it is not difficult to check that A, B and C generate it freely. Therefore we have to look for subgroups K of N that are normal in Ω (i.e. invariant with respect to the action (2.1)) and make N/K abelian.

First notice that, by (2.1), A, B and AC^{-1} represent in N/K elements of the same order and the same is true also for C and BA^{-1}. Moreover $y(xy)^4y^{-1} = AC^{-1}B$ and so $AC^{-1}B$ represents in N/K an element of order 2. Using (2.1) we see that it generates a normal subgroup H of G of order 2. Let $H = L/K$ for some subgroup L of N. Then $G'/H = N/L$ and $C = AB$ in this group. Thus N/L is generated by the images $\tilde{A}$ and $\tilde{B}$ of A and B and $\tilde{G} = \Omega/L$ acts on N/L subject to the following rules:

$$\begin{aligned} &\tilde{A}^{\tilde{x}} = \tilde{A}^{-1},\ \tilde{A}^{\tilde{y}} = \tilde{B} \\ &\tilde{B}^{\tilde{x}} = \tilde{B}^{-1},\ \tilde{B}^{\tilde{y}} = \tilde{A}^{-1} \end{aligned} \tag{2.2}$$

From (2.2) we see that $\tilde{A}$ and $\tilde{B}$ are elements of the same order n, say. By Lemma 2.4, n is even. So there exists k dividing n such that $\tilde{B}^k = (\tilde{A}^k)^\alpha$ for some α coprime with n/k. But by (2.2) $\tilde{A}^{-k} = (\tilde{B}^k)^\alpha$ and thus $\tilde{A}^{k(1+\alpha^2)} = 1$. Therefore n/k divides $1+\alpha^2$. Denote n/k by β and notice that if β divides $(1+\alpha^2)$ then in particular $(\beta,\alpha) = 1$. So we have shown that $\tilde{G} = G/H = \Omega/L$ has order $8k^2\beta$, where k is an arbitrary integer and β is an integer which divides $1+\alpha^2$ for some α. In particular G has order $16k^2\beta$ where k_β is even by Lemma 2.4.

In order to prove the converse, let us start with the scheme illustrating the subgroups of Ω involved in the proof up to now:

Let $n = k\beta$ be even, where k is an arbitrary integer and β is an arbitrary integer dividing $1+\alpha^2$ for some α. Then from the first part we know that L has to be chosen as the normal closure in N of $A^{k\beta}, B^{-k}A^{k\alpha}, C^{-1}AB, [A,B]$. Now as L/K is a group of order 2 generated by the image of $C^{-1}AB$ we see that the only candidates for K making N/K an abelian group of order $2k^2\beta$ are the normal closures K_i in N of the following sets:

$$X_1 = \{A^{k\beta}, B^{-k}A^{k\alpha}, (ABC^{-1})^2, [A,B], [A,C], [B,C]\},$$

$$X_2 = \{A^{k\beta}(ABC^{-1}), B^{-k}A^{k\alpha}, (ABC^{-1})^2, [A,B], [A,C], [B,C]\},$$

$$X_3 = \{A^{k\beta}(ABC^{-1}), B^{-k}A^{k\alpha}(ABC^{-1}), (ABC^{-1})^2, [A,B], [A,C], [B,C]\},$$

$$X_4 = \{A^{k\beta}, B^{-k}A^{k\alpha}(ABC^{-1}), (ABC^{-1})^2, [A,B], [A,C], [B,C]\}.$$

We claim that K_i is a normal subgroup of Ω if and only if

- k is even for $i = 1$,
- k, β have different parity for $i = 2$,
- k, β have different parity and k, α have the same parity for $i = 3$,
- k and β are even for $i = 4$.

Notice first that each K_i is invariant with respect to the conjugation by x. All equalities below are understood modulo K i.e. as equalities in N/K.

Case i=1. Since $k\beta$ is even, $(A^{k\beta})^y = C^{k\beta}A^{-k\beta} = B^{k\beta} = (A^{k\beta})^\alpha = 1$. Moreover $(B^{-k}A^{k\alpha})^y = C^{k\alpha}A^{k(1-\alpha)}$. Now if k is even then

$$C^{k\alpha}A^{k(1-\alpha)} = B^{k\alpha}A^k = Ak(\alpha^2+1) = 1.$$

So if k is even then K_1 is a normal subgroup of Ω such that Ω/K_1 is (2,4,8)-group of order $16k^2\beta$. However k cannot be odd since in such case

$$\begin{aligned} C^{k\alpha}A^{k(1-\alpha)} &= CA^{k\alpha-1}B^{k\alpha-1}A^{k(1-\alpha)} = CA^{k-1}B^{k\alpha-1} = \\ &(CA^{-1}B^{-1})A^{k(1+\alpha^2)} = CA^{-1}B^{-1} \neq 1. \end{aligned}$$

Case i=2. Here we must have $\#(A) = \#(B) = 2k\beta$. Thus $(\alpha, 2\beta) = 1$ and so α has to be odd. Hence

$$\begin{aligned} (A^{k\beta}(ABC^{-1}))^y = C^{k\beta}A^{-k\beta}(ABC^{-1}) &= B^{k\beta}(ABC^{-1}) = \\ &A^{k\beta}B^{k\beta} = A^{k\beta(\alpha+1)} = 1 \end{aligned}$$

as $\alpha+1$ is even. Now $(B^{-k}A^{k\alpha})^y = A^kC^{k\alpha}A^{-k\alpha} = A^{k(1-\alpha)}C^{k\alpha}$.

If k is even then $A^{k(1-\alpha)}C^{k\alpha} = A^kB^{k\alpha} = A^{k(1+\alpha^2)}$ whilst the last is equal to 1 if and only if 2β divides $(1+\alpha^2)$ what is equivalent to the fact that β is odd.

If k is odd then β is even. Conversely, if this is the case then in particular 2β does not divide $(1+\alpha^2)$ and so

$$A^{k(1-\alpha)}C^{k\alpha} = A^kB^{k\alpha}(A^{-1}B^{-1}C) = A^{k(1+\alpha^2)}(ABC^{-1})^{-1} = 1.$$

This completes the proof of the case $i=2$.

Case i=3. Here we have $A^{k(\alpha-\beta)} = B^k$ and $\#(A) = \#(B) = 2k\beta$. So $(\alpha-\beta, 2\beta) = 1$ and therefore α and β have different parity. In particular $B^{k\beta} = A^{k\beta}$.

Assume first that β is even. Then α is odd and 2β does not divide $(1+\alpha^2)$. Now

$$(A^{k\beta}(ABC^{-1}))^y = C^{k\beta}A^{-k\beta}(ABC^{-1}) = C^{k\beta} = A^{k\beta}B^{k\beta} = 1.$$

On the other hand

$$(B^{-k}A^{k\alpha}(ABC^{-1}))^y = A^kC^{k\alpha}A^{-k\alpha}(ABC^{-1}) = A^{k(1-\alpha)}C^{k\alpha}(ABC^{-1}).$$

Now if k is even then

$$A^{k(1-\alpha)}C^{k\alpha}(ABC^{-1}) = A^kB^{k\alpha}(ABC^{-1}) = A^k(B^k(ABC^{-1}))^\alpha =$$
$$= A^{k(1+\alpha^2)} \neq 1.$$

So if β is even then k and α have to be odd. However if this is the case then

$$A^{k(1-\alpha)}C^{k\alpha}(ABC^{-1}) = A^{k(1-\alpha)}ABA^{k\alpha-1}B^{k\alpha-1} = A^kB^{k\alpha} =$$

$$= A^kA^{k\alpha(\alpha-\beta)} = A^{k(1+\alpha^2)}A^{-k\alpha\beta} = A^{k(1+\alpha^2)}A^{k\beta} = 1$$

as 2β does not divide $1+\alpha^2$ and thus K_3 is a normal subgroup of Ω.

Now assume that β is odd. Then k and α are even. Conversely if this is the case then

$$(A^{k\beta}(ABC^{-1}))^y = C^{k\beta}A^{-k\beta}(ABC^{-1}) = C^{k\beta} = A^{k\beta}B^{k\beta} = 1.$$

Moreover,

$$(B^{-k}A^{k\alpha}(ABC^{-1}))^y = A^kC^{k\alpha}A^{-k\alpha}(ABC^{-1}) = A^kB^{k\alpha}(ABC^{-1}) =$$

$$= A^{k(1+\beta)}B^{k\alpha} = A^{k(1+\beta)}A^{k\alpha(\alpha-\beta)} = A^{k(1+\alpha^2)}A^{k\beta(1-\alpha)} = 1$$

as $\alpha - 1$ and $1+\alpha^2$ are odd. So also here K_3 is a normal subgroup of Ω.
Case i=4. Here we must have $\#(A) = \#(B) = k\beta$ and thus

$$(ABC^{-1})^\beta = (B^{-k}A^{k\alpha})^\beta = B^{-k\beta}A^{k\alpha\beta} = 1.$$

So β is even and therefore $(A^{k\beta})^y = C^{k\beta}A^{-k\beta} = B^{k\beta} = 1$. Now

$$(B^{-k}A^{k\alpha}(ABC^{-1}))^y = A^kC^{k\alpha}A^{-k\alpha}(ABC^{-1}) = C^{k\alpha}A^{k(1-\alpha)}(ABC^{-1}).$$

If k is even then since α is odd

$$\begin{aligned} C^{k\alpha}A^{k(1-\alpha)}(ABC^{-1}) = A^kB^{k\alpha}(ABC^{-1}) &= A^k(B^k(ABC^{-1}))^\alpha = \\ &= A^{k(1+\alpha^2)} = 1 \end{aligned}$$

and therefore K_4 is normal in Ω.

If k is odd then

$$C^{k\alpha}A^{k(1-\alpha)}(ABC^{-1}) = A^kB^{k\alpha}.$$

However if here $A^kB^{k\alpha} = 1$ then

$$1 = A^{k\alpha}B^{k\alpha^2} = A^{k\alpha}B^{k(1+\alpha^2)}B^{-k} = A^{k\alpha}B^{-k} = (ABC^{-1}),$$

which is absurd. This completes the proof of proposition 2.7. ∎

Now, rewriting the elements of X_i in terms of generators x and y of Ω and adding obtained in this way elements to the set $\{x^2, y^4\}$, we obtain all defining relations for all metabelian (2,4,8)-groups with abelianization $Z_2 \oplus Z_4$. Observe that the relations obtained from $[A, C]$

and $[B,C]$ are redundant as they obviously can be derived in Ω/K from $[A,B]$ in virtue of (2.1). In this way we have obtained the proof of theorem 1.2.

A Riemann surface X is said to be *symmetric* if it admits an anticonformal involution. Let X be a Riemann surface with (k,l,m)-group of automorphisms G generated by elements a and b of order k and l respectively whose product has order m. Then by the theorem of Singerman (Thm 2, [17]), X is symmetric if and only if there exists an automorphism ϕ of G for which

$$\phi(a) = a^{-1},\ \phi(b) = b^{-1} \quad \text{or} \quad \phi(a) = b^{-1},\ \phi(b) = a^{-1}.$$

Such automorphism clearly exists for the group from Proposition 2.6. Remaining metabelian groups of order $16(g-1)$ that act on Riemann surfaces of genus g as groups of automorphisms have been constructed in the proof of theorem 1.2 as quotients Ω/K_i for some subgroups K_i of N. Now an automorphism $\phi:\ \Omega/K_i \to \Omega/K_i$ in question exists if and only if for the automorphism of Ω given by $\varphi(x) = x,\ \varphi(y) = y^{-1}$, we have $\varphi(K_i) \subseteq K_i$. Observe, that φ induces an automorphism $\tilde{\varphi}$ of N satisfying $\tilde{\varphi}(A) = B,\ \tilde{\varphi}(B) = A,\ \tilde{\varphi}(C) = C$. So all this reduces to finding the additional conditions for the parameters defining X_i in order that K_i, its normal closure in N is invariant with respect to $\tilde{\varphi}$. We claim that the last is the case if and only if $\beta = 1, 2$ in the cases $i = 1, 2, 3$ and $\beta = 2$ in the case $i = 4$.

As an example let us consider the case $i = 3$ (the remaining cases being similar). As before all equalities below are understood modulo K i.e. as equalities in N/K.

Clearly, $\tilde{\varphi}(ABC^{-1}) = ABC^{-1}$. Now, if $\tilde{\varphi}(B^{-k}A^{k\alpha}(ABC^{-1})) = 1$ then $A^{k(\alpha^2-1)}(ABC^{-1})^{\alpha+1} = 1$. Hence if β is even then as α and k are odd we find that $A^{k(\alpha^2-1)} = 1$. Thus 2β has to divide $\alpha^2 - 1$. But as β also divides $1+\alpha^2$ we see that $\beta = 2$. Now assume that β is odd. Then α and k are even and thus $A^{k(\alpha^2-1)}(ABC^{-1}) = 1$. So $A^{k(\alpha^2-1+\beta)} = 1$ and therefore 2β divides $\alpha^2 - 1 + \beta$. But since β divides $1+\alpha^2$ we find that β divides $2-\beta$ and so $\beta = 1$. Conversely, if $\beta = 1$ or $\beta = 2$ then it is easy to check that $\tilde{\varphi}(K_3) = K_3$ and so the map φ induces an automorphism of G indeed.

References

[1] Chetiya, B.P., *Groups of automorphisms of compact Riemann surfaces*, Ph.D. thesis, Birmingham University (1971).

[2] Chetiya, B.P. and Patra, K., *On metabelian groups of automorphisms of compact Riemann surfaces*, J. London Math. Soc. **33**(2) 1986, 467-472.

[3] Cohen, J., *On Hurwitz extensions by $PSL_2(7)$*, Math. Proc. Cambridge Phil. Soc. **86** 1979, 395-400.

[4] Conder, M.D.E., *Generators for alternating and symmetric groups*, J. London Math. Soc (2) **22** 1980, 75-86.

[5] Conder, M.D.E., *A family of Hurwitz groups with non-trivial centres*, Australian Math. Soc. **18** 1986, 269-271.

[6] Conder, M.D.E., *Hurwitz groups: a brief survey*, Bull. Amer. Math. Soc. **23**(2) 1991, 369-370.

[7] Gromadzki, G. and Maclachlan, C., *Supersoluble groups of automorphisms of compact Riemann surfaces*, Glasgow Math. J. **31** 1989, 321-327.

[8] Gromadzki, G., *On soluble groups of automorphisms of Riemann surfaces*, Can. Math. Bull. **34**(1) 1991, 67-73.

[9] Harvey, W.J., *Cyclic groups of automorphisms of a compact Riemann surface*, Quar. J. Math. Oxford Ser. 2 **17** 1966, 86-98.

[10] Hurwitz, A., *Über algebraische Gelbilde mit eindeutigen Transformationen in sich*, Math. Ann **41** 1893, 403-442.

[11] Lehner, J. and Newman, M., *On Riemann surfaces with maximal automorphism groups*, Glasgow Math. J. **8** 1967, 102-112.

[12] Macbeath, A.M., *On a theorem of Hurwitz*, Proc. Glasgow Math. Assoc. **5**, 1961, 90-96.

[13] Macbeath, A.M., *Generators of the linear fractional groups*, Proc. Symp. Pure Math. **12** 1967, 14-32.

[14] Maclachlan, C., *Abelian groups of automorphisms of compact Riemann surfaces*, Proc. London Math Soc. (3) **15** 1965, 699-712.

[15] Sah, C.H., *Groups related to compact Riemann surfaces*, Acta Math. **123** 1969, 13-42.

[16] Singerman, D., *Automorphisms of compact non-orientable Riemann surfaces,* Glasgow Math J. **12** 1971, 50-59.

[17] Singerman, D., *Symmetries of Riemann surfaces with large automorphism group*, Math. Ann. **210** 1974, 17-32.

[18] Zomorrodian, R., *Nilpotent automorphism groups of Riemann surfaces*, Trans. Amer. Math. Soc. **288** 1985, 241-255.

[19] Zomorrodian, R., *Classification of p-groups of automorphisms of a Riemann surface and their lower series*, Glasgow Math. J. **29** 1987, 237-244.

Instytut Matematyki WSP,
Chodkiewicza 30,
85-064 Bydgoszcz,
Poland

Recibido: 8 de Junio de 1994
Revisado: 14 de Septiembre de 1994