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Stratifications Adapted to Finite Families of Differential 1-Forms (Pfaffian Geometry-Part One)

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ABSTRACT. A first part of a systematic presentation of Pfaffian geometry is given.

INTRODUCTION

In this paper we start a systematic presentation of Pfaffian geometry. The main interest is in the subpfaffian theory which can be considered as a generalization of the subanalytic case and can be applied in the theory of differential equations (e.g. singularities of the Pfaffian systems, dynamical systems in the plane etc.). In the present paper which is the first one of a series we introduce the technical tools of semipfaffian geometry which are necessary to build up the theory of subpfaffian sets. We use stratifications which are generalizations of normal decompositions, therefore in the first section we recall the previous notions

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of the theory developed by S. Lojasiewicz [L 2]. Following the idea of C.T.C. Wall [W] we define semivarieties. Semivarieties unify the concept of semianalytic sets in the real case and analytically constructible sets in the complex case. Going as well in the stratification theory towards possible unification between real and complex cases, we construct normal stratifications compatible with a finite family of semivarieties. More precisely let $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$ and M be a $\mathbf{K}$-analytic manifold; the proof of the fundamental theorem of Semianalytic Geometry on the existence of normal stratifications compatible with a finite family of semianalytic sets (cf. [L 2]) can be translated to the complex case. In Section I, we obtain

I.3.4. Theorem on the existence of normal stratifications (generalized version). *Let $E_1, \ldots, E_p$ be semivarieties of M and let $a \in M$. There exists a normal stratification $\mathcal{N}$ in a, of an arbitrarily small neighbourhood, which is compatible with $E_1, \ldots, E_p$.*

Following the same idea and the earlier works in the real case by R. Moussu and C. Roche ([M-R 1], [M-R 2]), in Section II, we define the stratification adapted to a finite family of analytic differential 1-forms $\Omega = \{\omega_1, \ldots, \omega_p\}$ and to a finite family of semivarieties $E_1, \ldots, E_p$. The normal stratification is a special case of it. Important results in this section are the existence theorem for adapted stratifications and its generalization:

II.2.4. Existence theorem for strongly adapted stratifications. *Let $\{E_\nu\}$ be a finite family of semivarieties of $\mathbf{K}^n$ and Ω a finite family of analytic differential 1-forms given in a neighbourhood of $0 \in \mathbf{K}^n$. There exists a stratification of a normal neighbourhood adapted to Ω, to $\{E_\nu\}$ and to each subfamily $\Omega' \subset \Omega$.*

To prove these theorems it has been convenient to introduce the concept of multinormal stratification which is obtained by making a normal stratification in each level. Although the multinormal and strongly adapted stratifications are quite technical and compound, they become extremely useful for decomposing Pfaffian sets into elementary objects which we call Pfaffian leaves. In Section III we use Pfaffian leaves to define basic semipfaffian sets. In the forthcoming work on subpfaffian geometry [H 2] we use projections of basic semipfaffian sets to define subpfaffian sets. The reason that it is necessary to reduce the family of semipfaffian sets to a smaller one relies on the observation that Pfaffian leaves inherit carpeting functions from the leaves of normal decompositions. Recall that in the subanalytic geometry the carpeting function is an analytic function defined in the neighbourhood of the closure of a stratum of a normal decomposition positive on the stratum and zero on its boundary. Existence of carpeting functions is essential in the proof of the fiber-cutting lemma (so called Lemma B cf.[L-Z], [D-L-S]). Another important advantage of the existence of strongly adapted stratifications is the fundamental observation (see Corollary III.3.6.1) that semipfaffian and basic semipfaffian sets have locally finite families of connected components. We close the present work with the Tangent Mapping Theorem which is proved in semipfaffian geometry but has important applications in the theory of subpfaffian sets.

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I. PREVIOUS CONCEPTS

The theory we are developing is local and so the ambient space M will be a $\mathbf{K}$-analytic manifold, with $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}\}$.

1. SEMIVARIETIES

1.1 Definition. *A subset $E \subset M$ is called semivariety if for each point $p \in M$ there exists an open neighbourhood U_p in which E is a finite union of finite intersections of sets of the form:*

$$A_{ij} = \{f_{ij} = 0\} \text{ or } A_{ij} = \{f_{ij} > 0\} \text{ if } \mathbf{K} = \mathbf{R} \tag{1}$$

$$A_{ij} = \{f_{ij} = 0\} \text{ or } A_{ij} = \{f_{ij} \neq 0\} \text{ if } \mathbf{K} = \mathbf{C} \tag{2}$$

where the functions f_{ij} are $\mathbf{K}$-analytic in U_p.

More precisely,

$$E \cap U_p = \cup_{i=1}^{r} \cap_{j=1}^{s} A_{ij}$$

for some $r, s \in \mathbf{N}$.

So the real semivarieties (if we set $\mathbf{K} = \mathbf{R}$) are the semianalytic sets (cf. [L-Z]) and the complex semivarieties (if we set $\mathbf{K} = \mathbf{C}$) are the analytically constructible sets (cf. [L 1]).

2. STRATIFICATIONS

2.1 Definition. *Let $\mathcal{A}$, $\mathcal{B}$ be two families of subsets of the space M. We say that the family $\mathcal{A}$ is compatible with the family $\mathcal{B}$ if for each $A \in \mathcal{A}$ and each $B \in \mathcal{B}$, either $A \subset B$ or $A \subset B^c$. In the case when $\mathcal{B} = \{B\}$, we say that the family $\mathcal{A}$ is compatible with the set B. If, in addition, $\mathcal{A} = \{A\}$, then we say that the set A is compatible with the set B.*

2.1.1 It is clear that the family $\mathcal{A}$ is compatible with the family $\mathcal{B}$ if and only if each set $A \in \mathcal{A}$ is compatible with each set $B \in \mathcal{B}$. Any reduction of sizes of the families preserves their compatibility. If the family $\mathcal{A}$ covers the set B and is compatible with this set, then the set B is the union of some sets of the family $\mathcal{A}$.

2.1.2 Let $\{f_i\}$ be a finite family of continuous functions in M. We say that the family $\mathcal{A}$ is compatible with $\{f_i\}$ if and only if for each $A \in \mathcal{A}$, $f_{i|A} \equiv 0$ or $f_i(x) \neq 0$ for each $x \in A$.

2.2 Definition. *A quasi-stratification $\mathcal{A}$ of M is a locally finite partition of M in disjoint subsets of M, such that each member $\Gamma \in \mathcal{A}$ of the partition is a connected $\mathbf{K}$- analytic submanifold of M.*

2.2.1 If a quasi-stratification is compatible with a finite family of sets, then it is compatible with each set obtained by the elementary operations of set theory, i.e., finite unions, finite intersections and the complement.

2.2.2 Let E be a semivariety in M and let $\{f_i\}_{i=1,\dots,k}$ be the analytic functions in U_p appearing in the definition of the A_{ij} such that $E \cap U_p = \cup_i \cap_j A_{ij}$. If $\mathcal{A}$ is a quasi-stratification of U_p compatible with the functions $\{f_i\}$, then $\mathcal{A}$ is compatible with $E \cap U_p$.

2.3 Definition. *A stratification is a quasi-stratification which moreover satisfies the boundary condition, i.e., the boundary of each element is a union of elements of smaller dimension.*

We have then that a stratification $\mathcal{N} = \{\Gamma_\nu^k\}_{k,\nu}$ of M is a locally finite partition of $M = \coprod_{k,\nu} \Gamma_\nu^k$ such that

(1) $\dim \Gamma_\nu^k = k$,

(2) Γ_ν^k *is a connected* $\mathbf{K}$*-analytic submanifold of* M,

(3) $\partial\Gamma_\nu^k = \cup_{i,\mu}\Gamma_\mu^i$, *where* $i < k$.

2.4 Examples of stratifications

2.4.1 Triangulation. Let $h : |\mathcal{K}| \to \mathcal{M}$ be a semianalytic triangulation (cf. [L-Z]). Then the family $\mathcal{A} = \{h(\Delta)\}_{\Delta \in \mathcal{K}}$ is a stratification of M.

2.4.2 Łojasiewicz's complex stratification (cf. [L 1], [Wh 3]). Given a locally finite family of analytic sets $\{W_j\}$ of M, there exists a complex stratification $\{\Gamma_\nu^k\}$ of M compatible with the family $\{W_j\}$.

The constructing process is the following (cf. [L 1] pag. 247): Given W, V analytic sets of M, we define $\tau_k(W,V) = \cup_i(W \cap V_i)$ where V_i are the irreducible components of V of dimension k such that V_i is not contained in W. We have $\dim \tau_k(W,V) < k$. The construction is made by descending induction:

$$M = Z_n \supset Z_{n-1} \supset \ldots \supset Z_{-1} = \emptyset\ , \quad \dim Z_i \le i.$$

Let us assume that we have constructed $Z_n \supset \ldots \supset Z_k$. Let $\{\Gamma_\nu^i\}, i = n, \ldots, k+1$, be the connected components of $Z_i \setminus Z_{i-1} \subset Z_i^{(i)}$ (regular points of dimension i). Then we define

$$Z_{k-1} = \bigcup_\mu Z_k^\mu \cup Z_k^* \cup \bigcup_{i>k} \tau_k(\overline{\Gamma}_\nu^i, Z_k) \cup \bigcup_j \tau(W_j, Z_k)$$

where

$\{Z_k^\mu\}$ = irreducible components of Z_k such that $\dim Z_k^\mu < k$,

Z_k^* is the set of singular points of Z_k,

$\{\Gamma_\nu^i\}$ = connected components of $Z_i \setminus Z_{i-1}$.

2.5 Definition. *Let $\mathcal{N}$ and $\mathcal{N}'$ be two stratifications of M. The stratification $\mathcal{N}'$ is said to be finer than $\mathcal{N}$ if every element of $\mathcal{N}'$ is contained in an element of $\mathcal{N}$.*

3. NORMAL STRATIFICATIONS

3.1 Definition. *A normal system in* $\mathbf{K}^n$ *is a family* $\mathcal{H} = \{H_l^k\}_{0 \leq k < l \leq n}$, *where* $H_l^k(X_1, \ldots, X_k; X_l)$ *are distinguished polynomials in* X_l *with analytic coefficients in* $X_1, \ldots, X_k$ *in a neighbourhood of the origin, with non identically zero discriminants* D_l^k *and whose holomorphic extensions*[1] *satisfy:*

(1) $H_k^{k-1}(Z_1, \ldots, Z_{k-1}; Z_k) = H_l^k(Z_1, \ldots, Z_k; Z_l) \Rightarrow$
$\Rightarrow H_l^{k-1}(Z_1, \ldots, Z_{k-1}; Z_l) = 0$

(2) $D_l^k(Z_1, \ldots, Z_k) = 0 \Rightarrow H_k^{k-1}(Z_1, \ldots, Z_{k-1}; Z_k) = 0$

in a neighbourhood of $0 \in \mathbf{C}^n$, *for* $1 \leq k < l \leq n$.

$$
\begin{array}{ccccccccc}
 & & & & & & & & H_n^{n-1} \\
 & & & & & & & \cdot & \vdots \\
 & & & & & & \cdot & & \vdots \\
 & & & & & \cdot & & & \vdots \\
 & & & H_{k+1}^{k} & \cdots & H_l^k & & \cdots & H_n^k \\
 & & H_k^{k-1} & H_{k+1}^{k-1} & \cdots & H_l^{k-1} & & \cdots & H_n^{k-1} \\
 & \cdot & & & & & & & \vdots \\
 \cdot & & & & & & & & \vdots \\
H_1^0 & \cdots & & & \cdots & & & \cdots & H_n^0
\end{array}
$$

Remarks

- Condition (2) can be changed by the following condition
(2') $H_l^k = \frac{\partial H_l^k}{\partial Z_l} = 0 \Rightarrow H_k^{k-1} = 0$.

---

[1] In the complex case, the holomorphic extensions of the distinguished polynomials coincide with these.

- For the polynomials $H^0_\nu(Z_\nu)$ there are just two possibilities $H^0_\nu(Z_\nu) = 1$ or $H^0_\nu(Z_\nu) = Z_\nu$.

3.2 Normal neighbourhood. A neighbourhood of $0 \in \mathbf{K}^n$, $Q = \{x \in \mathbf{K}^n : |x_i| < \delta_i\}$ is called *normal* for $\mathcal{S}$ if the distinguished polynomials H^k_l are holomorphic in $\widetilde{Q} = \{z \in \mathbf{C}^n : |x_i| \leq \delta_i\}$, satisfy (1) and (2) in its interior and moreover

(3) $|z_1| < \delta_1, \ldots, |z_k| < \delta_k$ and $H^k_l(z_1, \ldots, z_k; z_l) = 0 \Rightarrow |z_l| < \delta_l$ in $\widetilde{Q}$.

Remark. Normal neighbourhoods exist and can be taken arbitrarily small, i.e., each neighbourhood of the origin contains a normal neighbourhood (cf. [L 2]).

3.3 Definition of normal stratification

Let $\mathcal{S} = \{H^k_l\}_{0 \leq k < l \leq n}$ be a normal system in $\mathbf{K}^n$ and Q a normal neighbourhood. In Q we define a family of subsets $\{V^k\}_{k=0,\ldots,n}$ giving a partition, i.e., $Q = V^0 \cup \ldots \cup V^n$, where

$$V^k = \{H^{k-1}_k \neq 0, H^k_{k+1} = \ldots = H^{n-1}_n = 0\},$$

$$V^0 = \{H^0_1 = \ldots = H^{n-1}_n = 0\}, \quad V^n = \{H^{n-1}_n \neq 0\}.$$

Let us also define a family $\{W^k\}_{k=0,\ldots,n}$, where

$$W^k = \{H^{k-1}_k \neq 0, H^k_{k+1} = \ldots = H^k_n = 0\}.$$

For each k, V^k is contained in W^k, by condition (1) and W^k is a locally topographic $\mathbf{K}$-analytic submanifold of dimension k, by condition (2). As moreover V^k is open in W^k (cf. [L-Z]), then V^k is also a locally topographic $\mathbf{K}$-analytic submanifold of dimension k.

The connected components $\{\Gamma^k_\nu\}_\nu$ of V^k are then analytic submanifolds of dimension k and $\mathcal{N} = \{\Gamma^k_\nu\}_{k,\nu}$ is a partition of Q, i.e., $Q = \coprod_{k,\nu} \Gamma^k_\nu$, which is called *normal partition* of Q given by $\mathcal{S}$, and the $\{\Gamma^k_\nu\}$ are called *leaves* of the partition.

We observe that $\Gamma^0_\nu = \{0\}$ or $\emptyset$ and the $\{\Gamma^n_\nu\}$ are open in Q.

Remark. The following two basic properties appeare in [L 2], [L-Z]:

(1) A normal partition is finite and, when Q is small enough, the number of elements depends only on the normal system.

(2) Every normal partition of a normal neighbourhood Q is a stratification, i.e., $\partial\Gamma_\nu^k = (\overline{\Gamma}_\nu^k - \Gamma_\nu^k) \cap Q$ is a union of elements of dimension smaller than k.

3.3.1 Definition. *Let $\mathcal{N}_1$ be a normal stratification of Q_1 and $\mathcal{N}_2$ of $Q_2 \subset Q_1$. We say that $\mathcal{N}_2$ is a refinement of $\mathcal{N}_1$ and we write $\mathcal{N}_2 \prec \mathcal{N}_1$ if the stratification $\mathcal{N}_2$ is compatible with the stratification $\mathcal{N}_1$.*

3.4 The fundamental theorem of Semianalytic Geometry states the existence of a normal stratification compatible with a finite family of semianalytic sets (cf. [L-Z]). Here we generalize it in the following way.

Theorem on the existence of normal stratifications (generalized version). *Let $E_1, \ldots, E_p$ be semivarieties of M and let $a \in M$. There exists a normal stratification $\mathcal{N}$ in a, of an arbitrarily small neighbourhood, which is compatible with $E_1, \ldots, E_p$.*

Proof. Since the proof follows the same strategy like for semianalytic sets given in [L-Z] (cf. pags. 15-19), we just outline the main steps of the construction of the normal system for the stratification. Note that our proof provides an inductive procedure to construct the normal system.

3.4.1 Let $F(x,y)$ be an analytic function in a neighbourhood of 0 in $\mathbf{K}^m \times \mathbf{K}$ such that $F(0,y) \not\equiv 0$. Let us denote $P = H_1^{r_1} \ldots H_q^{r_q}$ the factorization in monic irreducible elements of $\mathcal{O}_m[Y]$ of the distinguished polynomial corresponding to F by the Weierstrass preparation theorem; then, the distinguished polynomial associated[2] to F is defined as the reduction of P, that is, $H = H_1 \ldots H_q$.

3.4.2 Let $h(u;x)$, $H(u,x;y)$ be distinguished polynomials in $\mathbf{K}^m, \mathbf{K}^{m+1}$, respectively. We define the distinguished polynomial:

$$\pi^h H(u,y) = \widetilde{H}(u,\xi_1;y) \ldots \widetilde{H}(u,\xi_p;y)$$

[2] Note the difference between "corresponding to" and "associated to".

where $\{\xi_1, \ldots, \xi_p\}$ is the complete sequence of the complex roots of $h(u;x) = 0$ and $\widetilde{H}$ is a holomorphic extension of H.

3.4.3 Let $F(u, x_1, \ldots, x_s)$ be an analytic function in a neighbourhood of $0 \in \mathbf{K}^m \times \mathbf{K}^s$ and $H_1(u;x_1), \ldots, H_s(u;x_s)$ distinguished polynomials of degrees $k_1, \ldots, k_s$, respectively. We define a function $\sigma^{(H_1,\ldots,H_s)}F(u)$, analytic in a neighbourhood of $0 \in \mathbf{K}^m$, in the following way. Let $\sigma_0 = 1, \sigma_1, \ldots, \sigma_k$ be the elementary symmetric polynomials in k variables, for $k = k_1 \ldots k_s$. We consider the functions

$$\eta_{\nu_1,\ldots,\nu_s} = \widetilde{F}(u, \xi_1^{(\nu_1)}, \ldots \xi_s^{(\nu_s)})$$

where $\widetilde{F}$ is a holomorphic extension of F and $\xi_i^{(\nu_i)}$ runs the complete sequence $\{\xi_i^1, \ldots \xi_i^{k_i}\}$ of roots of $H_i(u, x_i)$. Next we define the functions $\chi_j(u), 0 \leq j \leq k$ by the formula

$$\chi_j(u) = \sigma_j(\{\eta_{\nu_1,\ldots,\nu_s}\}).$$

Then $\sigma^{(H_1,\ldots,H_s)}F(u)$ is the last not identically zero function in the sequence: $\chi_0, \ldots, \chi_k$ (cf. [L-Z]).

3.4.4 Construction of the normal system. Let $f_1, \ldots, f_r$ be the analytic functions used in the local description of the semivarieties $E_1, \ldots, E_p$. We set $\varphi_n = f_1 \ldots f_r$. By means of a change of coordinates we can get $\varphi_n(0, x_n) \not\equiv 0$. Then, we define H_n^{n-1} as the distinguished polynomial associated to φ_n.

$$\begin{array}{ccc} & & H_n^{n-1} \\ & \cdot & \vdots \\ & \cdot & \vdots \\ & H_{k+1}^{k} \cdots\cdots\cdots & H_n^{k} \\ H_k^{k-1} & & \end{array}$$

Let us assume that we have defined $\{H_j^i\}$ for $k \leq i < j \leq n$ and we want to construct H_k^{k-1}. We consider

$$\varphi_k = \prod_{i=1}^{r} \sigma^{(H_{k+1}^k \dots H_n^k)} f_i \quad \text{and} \quad J_k = \varphi_k \prod_{j=k+1}^{n} D_j^k.$$

By means of a change of coordinates, we have $J_k(0, X_k) \not\equiv 0$. Then, H_k^{k-1} is the distinguished polynomial associated to J_k. For $j = k + 1, \dots, n$, H_j^{k-1} *is the distinguished polynomial associated to* $\pi^{H_k^{k-1}} H_j^k$. So we have constructed in this way a finite family of distinguished polynomials $\{H_j^i\}_{0 \leq i < j \leq n}$ which form a normal system (cf. [L-Z]).

3.4.5 Remark. The method described above implies that the construction of the normal system is generic, in the sense that the change of coordinates in $\mathbf{K}^n$ which is necessary to realize the construction can be chosen from a dense open subset of the space of linear automorphisms of $\mathbf{K}^n$.

3.4.6 Remark. Notice that it is possible to realize our construction of the normal system using only isometric change of coordinates.

3.5 Examples of normal stratifications compatible with semi-analytic sets

3.5.1 Normal stratification compatible with $E = \{x_3^2 + x_2^2 - x_1^3 = 0\}$

We have $H_3^2 = \varphi_3 = f_1 = x_3^2 + x_2^2 - x_1^3$. It is clear that $\varphi_2 = \sigma^{(H_3^2)} H_3^2 = 1$. Then $J_2 = \varphi_2 D_3^2 = D_3^2 = -4(x_2^2 - x_1^3)$, so $H_2^1 = x_2^2 - x_1^3$ because $J_2(0, x_2) = -4x_2^2 \not\equiv 0$.

The associated distinguished polynomial of $\pi^{H_2^1} H_3^2 = x_3^4$ is $H_3^1 = x_3$. We can check that $\varphi_1 = \sigma^{(H_2^1, H_3^1)} H_3^2 = 1$, then $J_1 = \varphi_1 D_2^1 D_3^1 = 4x_1^3$. The distinguished polynomial associated to J_1 is $H_1^0 = x_1$. Now, $\pi^{H_1^0} = x_2^2$, so $H_2^0 = x_2$ and $\pi^{H_1^0} H_3^1 = x_3 = H_3^0$ (see fig.1).

The normal system is then

$$\begin{array}{lll} & & H_3^2 = x_3^2 + x_2^2 - x_1^3 \\ & H_2^1 = x_2^2 - x_1^3 & H_3^1 = x_3 \\ H_1^0 = x_1 & H_2^0 = x_2 & H_3^0 = x_3 \end{array}$$

The normal stratification associated to this system has two elements of dimension 3 corresponding to the inner and the outer space limited by the surface (see fig. 2). There are two elements of dimension 2 defined by the points of the surface in which the discriminant is not zero (see fig. 3). The elements of dimension 1 are contained in the zeros of the discriminant and the only element of dimension 0 is the origin (see fig. 4).

3.5.2 Normal stratification compatible with the Whitney umbrella $W = \{x_3^2 - x_1^2x_2 = 0\}$

We have $H_3^2 = \varphi_3 = f_1 = x_3^2 - x_1^2x_2$, $\varphi_2 = \sigma^{(H_3^2)}H_3^2 = 1$, then $J_2 = \varphi_2 D_3^2 = 4x_1^2x_2$. With the change of coordinates $x_1 = \widetilde{x}_1 - \widetilde{x}_2$; $x_2 = \widetilde{x}_1 + \widetilde{x}_2$, we get $\widetilde{J}_2 = 4(\widetilde{x}_1 - \widetilde{x}_2)^2(\widetilde{x}_1 + \widetilde{x}_2)$ satisfying $\widetilde{J}_2(0, \widetilde{x}_2) \not\equiv 0$. So the distinguished polynomial associated to $\widetilde{J}_2$ is $H_2^1 = (\widetilde{x}_2 - \widetilde{x}_1)(\widetilde{x}_2 + \widetilde{x}_1) = \widetilde{x}_2^2 - \widetilde{x}_1^2$.

It is visible that $\pi^{H_2^1}H_3^2 = x_3^4$ and so $H_3^1 = x_3$. Now, $\varphi_1 = \sigma^{(H_2^1, H_3^1)}H_3^2 = 1$ and so $J_1 = \varphi_1 D_2^1 D_3^1 = 4\widetilde{x}_1^2$ and $H_1^0 = \widetilde{x}_1$. We have $\pi^{H_1^0}H_2^1 = \widetilde{x}_2^2$ which implies $H_2^0 = \widetilde{x}_2$ and finally $\pi^{H_1^0}H_3^1 = x_3 = H_3^0$.

The normal system in the coordinates $(\widetilde{x}_1, \widetilde{x}_2, x_3)$ is then

$$\begin{array}{lll} & & H_3^2 = x_3^2 - (\widetilde{x}_1 - \widetilde{x}_2)^2(\widetilde{x}_1 + \widetilde{x}_2) \\ & H_2^1 = \widetilde{x}_2^2 - \widetilde{x}_1^2 & H_3^1 = x_3 \\ H_1^0 = \widetilde{x}_1 & H_2^0 = \widetilde{x}_2 & H_3^0 = x_3 \end{array}$$

The normal stratification has three elements of dimension 3. There are 4 elements of dimension 2 defined by the points of the surface in which the discriminant is not zero. The elements of dimension 1 are contained in the zeros of the discriminant.

4. FOLIATIONS OF CODIMENSION ONE

Let $(M, \mathcal{O}_M)$ be the ringed space given by a **K**-analytic manifold of dimension n. The cotangent sheaf Ω_M is the sheaf of differential **K**-analytic 1-forms and is a locally free $\mathcal{O}_M$-module of rank n. More precisely, it is locally generated by $df_1, \ldots, df_n$, where $f_1, \ldots, f_n$ are local coordinates.

4.1 Definition. *An analytic foliation of codimension 1 on M is an $\mathcal{O}_M$-submodule $\mathcal{F} \subset \Omega_M$ which satisfies the following conditions:*

(1) $\mathcal{F}$ is locally free of rank 1, i.e., $\mathcal{F}$ is locally generated by a differential 1-form $\omega = \sum_{i=1}^n b_i df_i$ with analytic coefficients;

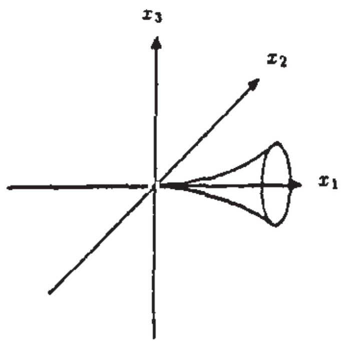

Figure 1.

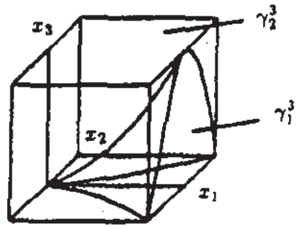

Figure 2.

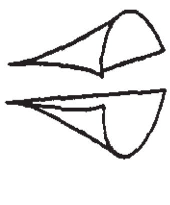

Figure 3.

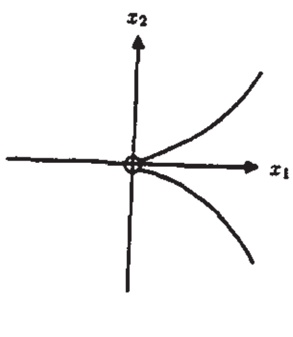

Figure 4.

(2) $\mathcal{F}$ is integrable : $\omega \wedge d\omega = 0$ for each local generator ω of $\mathcal{F}$;

(3) the quotient sheaf $\Omega_M/\mathcal{F}$ is torsion free: this is equivalent to the condition that the coefficients b_i satisfy locally $\gcd(b_1, \ldots, b_n) =$ $= 1$ (cf. [R]).

Remark. The notion of foliation is local, that is, if $U \subset M$ is open, then $\mathcal{F}_U$ is an analytic foliation of codimension 1 on U.

4.2 Definition. *The singular locus $Sing\mathcal{F}$ of the foliation $\mathcal{F}$ is the set of points in which the quotient sheaf $\Omega_M/\mathcal{F}$ is not free of rank $n-1$.*

Locally it is given by $Sing\ \mathcal{F} = \{b_i = 0\ ;\ i = 1, \ldots, n\}$.

II. ADAPTED STRATIFICATIONS

The aim of this section is to give a proof of the existence of adapted stratifications, valid in both real and complex cases.

1. DEFINITIONS

Let $\{\omega_1, \ldots, \omega_q\}$ be differential 1-forms given in a neighbourhood U of a point $a \in \mathbf{K}^n$; $\omega_j = \sum_{i=1}^n a_i^j dx_i$, where a_i^j are analytic functions in U, $j = 1, \ldots, q$; and $Y \subset U$ an analytic submanifold.

We set $\Omega = \{\omega_1, \ldots, \omega_q\}$ and, for each subset $J \subset \{1, \ldots, q\}$, $\Omega_J = \{\omega_j \in \Omega : j \in J\}$. We define $\Omega_J(x)$ as the subspace of $(\mathbf{K}^n)^*$ generated by $\{\omega_j(x) : j \in J\}$, Ker $\Omega_J(x)$ as the intersection $\cap_{j \in J}$ Ker $\omega_j(x)$ and $T_x^0(Y) \subset (\mathbf{K}^n)^*$ as the subspace of forms vanishing on T_xY.

1.1 Definition. *We say that Ω_J is tranversal to Y if and only if for each point $x \in Y$*

(1) $\dim\ (\mathrm{Ker}\ \Omega_J(x) \cap T_x(Y)) = \dim Y - \sharp J$.

If, in addition, for each $x \in Y$ we have

(2) $\Omega(x) + T_x^0(Y) = \Omega_J(x) + T_x^0(Y)$

or, equivalently,

(2') Ker $\Omega(x) \cap T_x(Y) =$ Ker $\Omega_J(x) \cap T_x(Y)$,

we say that Ω_J is a basis for Ω along Y (cf. [M-R 2]).

1.2 Definition. *Let $\mathcal{N}$ be a stratification of a neighbourhood of $a \in \mathbf{K}^n$ and let $\Omega = \{\omega_1, \ldots, \omega_q\}$. We say that $\mathcal{N}$ is adapted to the family Ω if and only if there exists a map $J : \mathcal{N} \to 2^{\{1,\ldots,q\}}$ such that for each*

$Y \in \mathcal{N}$, $\Omega_{J(Y)}$ is a basis of Ω along Y. If, in addition, $\mathcal{N}$ is compatible with a finite family of semivarieties $\{E_\nu\}$, we say that $\mathcal{N}$ is adapted to Ω and to $\{E_\nu\}$.

2. MAIN THEOREMS

2.1 Existence theorem for adapted stratifications. *Let $\{E_\nu\}$ be a finite family of semivarieties of $\mathbf{K}^n$ and Ω a finite family of analytic differential 1-forms given in a neighbourhood of $0 \in \mathbf{K}^n$. There exists a stratification of a normal neighbourhood adapted to Ω and to $\{E_\nu\}$.*

Proof. By the theorem of normal stratifications (generalized version) (cf. I.3.4) there exists a stratification $\mathcal{N}$ of a normal neighbourhood Q, compatible with the semivarieties $\{\{E_\nu\}, \text{Sing } \omega_1, \ldots, \text{Sing } \omega_q\}$. We shall prove that $\mathcal{N}$ has a refinement $\mathcal{N}'$ which admits a map $J : \mathcal{N}' \to 2^{\{1,2,\ldots,q\}}$ such that for each $\Gamma \in \mathcal{N}'$, $\Omega_{J(\Gamma)}$ is a basis for Ω along Γ. We will use descending induction on the dimension of the elements of the stratification.

First for each point x in the submanifold V^n we consider rk $M_n(x)$ where $M_n(x)$ is the $q \times n$ matrix whose rows $A_i(x) = (a_1^i(x), \ldots, a_n^i(x))$ are composed from the coefficients of the 1-forms $\omega_i(x), i = 1, \ldots, q$. Let $k = \max\{ \text{rk } M_n(x) : x \in Q\}$ and $\omega_{i_1}, \ldots, \omega_{i_k} \in \Omega$ be such that the matrix

$$M_n^k(x) = \begin{pmatrix} A_{i_1}(x) \\ \cdots \\ A_{i_k}(x) \end{pmatrix}$$

has rank k in an open and dense subset of Q.

V^n can be decomposed in the following way

$$V^n = \{x \in V^n : \text{ rk } M_n^k(x) = k\} \cup \{x \in V^n : \text{ rk } M_n^k(x) < k\}.$$

The second set has dimension smaller than n. Now we take a refinement $\mathcal{N}_1 \prec \mathcal{N}$ compatible with $\{x \in Q : \text{rk } M_n^k(x) < k\}$. For each $\Gamma^n \in \mathcal{N}_1$ the rank of M_n^k is maximal and we can take $\{\omega_{i_1}, \ldots, \omega_{i_k}\}$ as a basis. We denote $F^n = \cup\{\Gamma^n : \Gamma^n \in \mathcal{N}_1\}$.

Let us observe that in the new normal neighbourhood $Q_1 \subset Q$, we have a normal system $\mathcal{S}_1$:

$$
\begin{array}{cccccc}
 & & & & & {}^1H_n^{n-1} \\
 & & & & {}^1H_{n-1}^{n-2} & {}^1H_n^{n-2} \\
 & & & \cdot & & \\
 & & \cdot & & & \\
 & \cdot & & & & \\
{}^1H_1^0 & \cdot & \cdot & \cdot & \cdot & {}^1H_n^0
\end{array}
$$

which is giving the stratification $\mathcal{N}_1$. So in the second step we consider elements of dimension $n-1$, $\Gamma_\nu^{n-1} \subset V_1^{n-1}$ and the matrix:

$$
M_{n-1}(x) = \begin{pmatrix} A_1(x) \\ \cdots \\ A_q(x) \\ \text{grad } {}^1H_n^{n-1}(x) \end{pmatrix}.
$$

Let $l_\nu = \max\{ \text{ rk } M_{n-1}(x) : x \in \Gamma_\nu^{n-1}\} \in \{1, \ldots, \min\{q+1, n\}\}$,and $\{\omega_{i_1}, \ldots, \omega_{i_{k_\nu}}\}$ $(k_\nu = l_\nu - 1)$ be such that the matrix

$$
M_{n-1}^{l_\nu}(x) = \begin{pmatrix} A_{i_1}(x) \\ \cdots \\ A_{i_{k_\nu}}(x) \\ \text{grad } {}^1H_n^{n-1}(x) \end{pmatrix}
$$

has rank l_ν in an open and dense subset of Γ_ν^{n-1}. For each $\Gamma_\nu^{n-1} \subset V_1^{n-1}$, we have a semivariety $\{x \in \Gamma_\nu^{n-1} : \text{rk } M_{n-1}^{l_\nu}(x) < l_\nu\}$ and now again we take a refinement $\mathcal{N}_2 \prec \mathcal{N}_1$ compatible with all these semivarieties. Now, for each $\Gamma^{n-1} \in \mathcal{N}_2$ such that $\Gamma^{n-1} \subset \partial F^n$ we can choose some $\{\omega_{i_1}, \ldots, \omega_{i_{k_\nu}}\}$ as a basis. We denote $F^{n-1} = \cup\{\Gamma^{n-1} \in \mathcal{N}_2 : \Gamma^{n-1} \subset \partial F^n\}$.

Further, in the (i+1)-th step we will consider strata from $\mathcal{N}_i$ of dimension $n-i$ and the matrix

$$M_{n-i}(x) = \begin{pmatrix} A_1(x) \\ \dots \\ A_q(x) \\ \text{grad } {}^iH^{n-i}_{n-i+1}(x) \\ \dots \\ \text{grad } {}^iH^{n-i}_n(x) \end{pmatrix}.$$

In this way, we obtain a descending sequence of normal neighbourhoods $Q = Q_0 \supset Q_1 \supset \dots \supset Q_n$ and in each of them we obtain a normal system $\mathcal{S}_i$.

$$\begin{array}{ccccccccc}
 & Q_1 & & \supset & & Q_2 & & \dots & \supset Q_n \\
 & \mathcal{S}_1 & & & & \mathcal{S}_2 & & & \mathcal{S}_n \\
 & {}^1H^{n-1}_n & & & & {}^2H^{n-1}_n & & & {}^nH^{n-1}_n \\
 & \vdots & & & & \vdots & & & \vdots \\
{}^1H^0_1 \;\; . & . & {}^1H^0_n & {}^2H^0_1 \;\; . & . & . & {}^2H^0_n & {}^nH^0_1 \;\; . \;\; . \;\; . & {}^nH^0_n
\end{array}$$

Finally we obtain the sequence of submanifolds $F^n, F^{n-1}, \dots, F^0$. The stratification $\mathcal{N}'$ is constructed by taking the connected componets from all the sets: $F^n \cap Q_n, F^{n-1} \cap Q_n, \dots, F^0 \cap Q_n$. It is clear, by the construction process, that the partition obtained in this way satisfies the boundary condition (cf. I.2.3).

2.2 Note that the above stratification is obtained by doing a normal stratification in each step. We shall call it *multinormal stratification.*

2.3 We give now the following generalization of the preceding theorem. *Let $\{E_\nu\}$ be a finite family of semivarieties of* $\mathbf{K}^n$ *and* $\Omega_1, \dots, \Omega_r$ *finite families of analytic differential 1-forms given in a neighbourhood of* $0 \in \mathbf{K}^n$. *There exists a stratification of a normal neighbourhood adapted to each* Ω_i *and to* $\{E_\nu\}$.

Proof. We give the proof in the case when $r = 2$. We can deal with the general case in an analogous way. Let then $\Omega_1 = \{\omega_1, \dots, \omega_q\}$, $\Omega_2 = \{\widetilde{\omega_1}, \dots, \widetilde{\omega_p}\}$. We define a map

$$J : \mathcal{N} \to 2^{\{1,\ldots,q\}} \times 2^{\{1,\ldots,p\}} \quad , \quad J(\Gamma) = (J_1(\Gamma), J_2(\Gamma))$$

in such a way that $\Omega_{1J_1(\Gamma)}$ and $\Omega_{2J_2(\Gamma)}$ satisfy the basis conditions (cf. 1.1).

Let us consider as starting stratification the one in the proof of theorem 2.1. Let $k = \max\{\text{ rk } M_n(x)\}, \widetilde{k} = \max\{\text{ rk } \widetilde{M}_n(x)\}$. Then in $\{x \in V^n : \text{rk } M_n^k(x) = k\} \cap \{x \in V^n : \text{rk } \widetilde{M}_n^k(x) = \widetilde{k}\}$ we can assign a couple of bases $(\{w_i\}, \{\widetilde{w}_j\})$: the complement to this set is a semivariety.

Further, we proceed by induction as in theorem 2.1.

2.4 Existence theorem for strongly adapted stratifications. *With the hypothesis of theorem 2.1, there exists a stratification of a normal neighbourhood adapted not only to Ω and to $\{E_\nu\}$ but also to each subfamily $\Omega' \subset \Omega$.*

Proof. We obtain this theorem as a corollary of 2.3 by considering the sequence of all non-empty subfamilies $\Omega_i \subset \Omega$, $i = 1, \ldots, 2^q - 1$.

III. APPLICATION OF THE ADAPTED STRATIFICATIONS IN THE THEORY OF PFAFFIAN SETS

In this section we shall assume that the ambient space is $\mathbf{R}^n$ although the theory remains valid if we consider a real analytic variety of dimension n.

1. DEFINITIONS

1.1 A *Pfaffian hypersurface of $\mathbf{R}^n$ is a triple $(V, \mathcal{F}, M)$, where*

(1) M is an open semianalytic set of $\mathbf{R}^n$,

(2) $\mathcal{F}$ is a foliation of codimension 1 defined in an open neighbourhood of the closure of M,

(3) V is a leaf of $\mathcal{F}_M$, i.e., is a connected maximal integral submanifold of the restricted foliation,

(4) Sing $\mathcal{F} \cap M = \emptyset$

1.2 A Pfaffian hypersurface is called *separating* if $M \setminus V$ has two connected components and V is their common boundary in M. Each of these connected components is called a *Pfaffian block.*

1.3 A Pfaffian hypersurface has the *Rolle property* (i.e. is *Rollian*) if each analytic path $\gamma : [0,1] \to M$ such that $\gamma(0), \gamma(1)$ are points in V has at least one point $\gamma(t)$ such that the tangent vector $\gamma'(t)$ is tangent to the foliation $\mathcal{F}$ (i.e. if at this point $\mathcal{F}$ is determined by a local genarator ω then $\gamma'(t) \in \text{Ker } \omega$).

1.4 Remark. By the theorem of Hovanskii-Rolle, a separating hypersurface is Rollian but the converse is not true. Let, for example, $M = \mathbf{R}^2 \setminus \{(0,0)\}$ and $\omega = x^2dy - ydx$. We consider the curve $f(x) = c.e^{-1/x}, x > 0$ and $c \in \mathbf{R}$ (see fig. 5). It is clear that it defines a Rollian hypersurface but not separating because its complement $M \setminus \{c.e^{-1/x}, x > 0\}$ is connected.

1.5 Remark. If M is simply connected, every Pfaffian hypersurface is separating and, consequently, Rollian (cf. [M-R 2]).

1.6 Example. Let us see now an example of a Pfaffian hypersurface which is not Rollian. Let $M = \mathbf{R}^2 \setminus \{(0,0)\}$ and $\omega = (x+y)dx + (y-x)dy$. The solution V which is logarithmic spiral (see fig. 6) does not have the Rolle property because any half-line with its origin in the point $(0,0)$ cuts the logarithmic spiral with a constant angle.

2. FINITENESS THEOREMS

Theorem 1. Finiteness theorem of Hovanskii-Moussu-Roche. *Let M be an open semianalytic set in $\mathbf{R}^n$ and $X \subset M$ semianalytic and bounded in $\mathbf{R}^n$. For each finite collection of Pfaffian hypersurfaces $(V_1, \mathcal{F}_1, M), \ldots, (V_p, \mathcal{F}_p, M)$ which have the Rolle property for the paths in X, there exist a number $b_0 \in \mathbf{N}$ (depending only on $M, X, \mathcal{F}_1, \ldots, \mathcal{F}_p$) such that the number of connected components of $X \cap V_1 \cap \ldots \cap V_p$ is smaller than b_0.*

Proof. (cf. [M-R 1]).

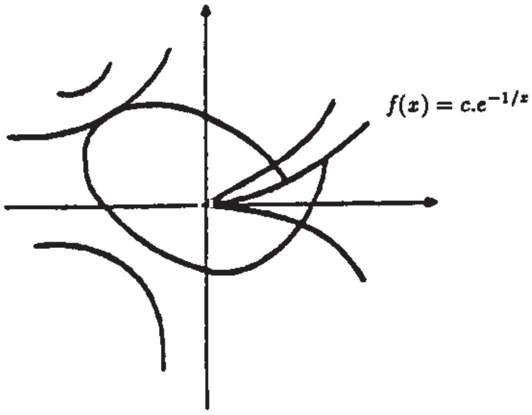

Figure 5.

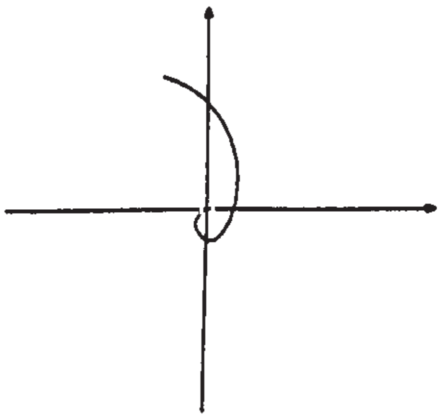

Figure 6.

Theorem 2. *Let M be an open semianalytic set in $\mathbf{R}^n$ and Ω a finite family of analytic differential 1-forms in a neighbourhood of a point $c \in \overline{M}$. Let $\{E_\nu\}$ be a finite family of semianalytic sets of $\mathbf{R}^n$.*
(i)There exists a basis $\mathcal{B}$ of neighbourhoods of c such that for each $U \in \mathcal{B}$ there exist a stratification $\mathcal{N}$ of U adapted to $M, \{E_\nu\}, \Omega$ and each $\Omega' \subset \Omega$, i.e., strongly adapted.
(ii)Moreover, if we consider a finite family of Pfaffian hypersurfaces: $\{(V_i, \omega_i, M)\}_{\omega_i \in \Omega'}$ which have the Rolle property for analytic paths contained in $Y \in \mathcal{N}$, then for a basis $\Omega'_J = \{\omega_{i_1}, \ldots, \omega_{i_p}\}$ for Ω' along Y, $\cap_{\omega_i \in \Omega'}(V_i \cap Y)$ is a union of connected components of $\cap_{j \in J}(V_j \cap Y)$.

Proof. (i) This is a special case of theorem II.2.2.4.
(ii) It is proved in [M-R 1] (see lemma 1).

3. ELEMENTS OF SEMIPFAFFIAN GEOMETRY

3.1 Proposition. *Let $\mathcal{N}$ be a stratification of a neighbourhood U of $c \in \mathbf{R}^n$, strongly adapted to the finite family of Pfaffian hypersurfaces $\mathcal{H} = \{(V_i, \omega_i, M_i)\}_{\omega_i \in \Omega}$ (i.e. strongly adapted to $\{M_i\}$ and Ω). Let Y be a leaf of $\mathcal{N}$ such that the family $\mathcal{H}$ has the Rolle property for the paths in Y. Then the connected components of the collection $\{\cap_{\omega_i \in \Omega'}(V_i \cap Y)\}_{\Omega' \subset \Omega}$ form a finite family $\mathcal{K}$ of analytic submanifolds of Y with normal crossings in Y.*

Proof. This proposition comes directly from theorems 1 and 2.

Let us denote $\mathcal{K}_Y$ the set of all the elements Γ of $\mathcal{K}$ with the property that $c \in \overline{\Gamma}$.

3.1.1 Remark. Let us observe that if, in addition, $\mathcal{N}$ is compatible with $\{c\}$, then $c \in \overline{\Gamma} \setminus \Gamma$ when $Y \neq \{c\}$.

3.2 Definition. *For $k = 0, 1, \ldots, p = \dim Y$ we define the skeleton varieties $L_k = \cup N_i, N_i \in \mathcal{K}_Y, \dim N_i \leq k$. The filtration $Y = L_p \supset \ldots \supset L_0$ gives a finite stratification $\{Y_\nu^i\}$ of Y in analytic submanifolds, where $\{Y_\nu^i\}$ is a family of connected components of $L_k \setminus L_{k-1}$ $(k = 1, \ldots, p)$. The stratification above is called stratification induced in Y by the family $\mathcal{K}_Y$.*

3.3 Definition. *An analytic submanifold Γ in a neighbourhood U of c is a semipfaffian leaf if there exist a stratification $\mathcal{N}$ of U and a finite family $\mathcal{H}$ of Pfaffian hypersurfaces such that Γ is an element of the stratification induced in some $Y \in \mathcal{N}$. If $\Gamma \in \mathcal{K}_Y$ for some $Y \in \mathcal{N}$, it is called Pfaffian leaf.*

3.4 Definition. *A subset $E \subset \mathbf{R}^n$ is a semipfaffian set (resp. basic semipfaffian set) if for each $c \in \mathbf{R}^n$ there exist*

1) a normal neighbourhood U of c,

2) a finite family of Pfaffian hypersurfaces $\mathcal{H}$ given by 1-forms defined in a neighbourhood of c,

3) a stratification $\mathcal{N}$ of U strongly adapted to $\mathcal{H}$ such that we have the following equality of germs $(E \cap U)_c = (\cup_{i,\nu}\Gamma^i_\nu)_c$, where $\{\Gamma^i_\nu\}$ is a finite collection of semipfaffian leaves (resp. Pfaffian leaves).

3.4.1 Corollary. *The families of basic semipfaffian sets and semipfaffian sets are closed on locally finite unions and intersections.*

Proof. Locally in each point $c \in \mathbf{R}^n$ we can take a stratification $\mathcal{N}$ of a neighbourhood Q_c strongly adapted to all Pfaffian hypersurfaces defining the sets at c. Considering the quasi-stratification of Q_c induced by the families $\mathcal{K}_Y$ for $Y \in \mathcal{N}$ we can conclude as in Section I.2.2.1.

3.5 Definition. *A Pfaffian set is a basic semipfaffian set of type $V_1 \cap \ldots \cap V_k \cap X$ where $\{V_i\}$ are Pfaffian hypersurfaces and $X \subset \mathbf{R}^n$ is semianalytic.*

3.6 Remark. In Semianalytic Geometry, semianalytic sets are defined as locally described by analytic functions and they are characterized by means of the normal stratifications. Every semianalytic set is semipfaffian. Stratifications strongly adapted to Pfaffian hypersurfaces are used to define semipfaffian sets. Let us observe that our definition of semipfaffian and basic semipfaffian sets implies that locally they have finitely many connected components and therefore we have the following topological consequence (cf. [L 1] Chapter B, Section 1, Proposition).

3.6.1 Corollary. *The family of connected components of a semipfaffian (resp. basic semipfaffian) set is locally finite and each connected component is semipfaffian (resp. basic semipfaffian).*

3.7 Remark. The principal obstruction to build a self-contained theory of semipfaffian sets is the lack of a theorem on the closure of a semipfaffian set (cf. [C-Li-M], [Li]). We close the theory with the Tangent Mapping Theorem which is useful in Subfaffian Geometry (cf. [H 1], [H 2], [R]).

Tangent Mapping Theorem. *Let Y be a semianalytic leaf in $\mathbf{R}^n$, i.e. an analytic submanifold which is also a semianalytic set. Let $\Omega = \{\omega_1, \ldots, \omega_p\}$ be a finite family of analytic differential 1-forms which are defined in the neighbourhood of $\overline{Y}$ and transversal to Y. Let $\dim Y = k + p$ and $\Gamma = (V_1 \cap \ldots \cap V_p) \cap Y$ where $\{(V_i, \omega_i, M_i)\}$ is a family of Pfaffian hypersurfaces which have the Rolle property in Y. Then*

(1) the graph of the tangent map $\tau_\Gamma = \{(x, T_x\Gamma) \in \Gamma \times G_k(\mathbf{R}^n)\}$ is a Pfaffian subset of $\mathbf{R}^n \times G_k(\mathbf{R}^n)$.

(2) For each semialgebraic set $E \subset G_k(\mathbf{R}^n)$, the inverse image $\tau^{-1}(E)$ by the map:

$$\tau : Y \ni x \mapsto \text{ Ker } \omega_1(x) \cap \ldots \cap \text{ Ker } \omega_p(x) \cap T_x(Y) \in G_k(\mathbf{R}^n)$$

is semianalytic in $\mathbf{R}^n$.

(3) For each semialgebraic set $E \subset G_k(\mathbf{R}^n)$ the inverse image $\tau_\Gamma^{-1}(E)$ is a Pfaffian set in $\mathbf{R}^n$.

Proof. First let us recall two auxilary facts from semialgebraic geometry (cf. [L-Z], Section 10).

(I) The mapping

$$T : (\mathbf{R}^{n*})^p \setminus \Sigma \ni (v_1, \ldots, v_p) \mapsto \cap_{i=1}^p \text{ Ker } v_i \in G_{n-p}(\mathbf{R}^n)$$

(where $\Sigma = \{v \in (\mathbf{R}^{n*})^p : v_1 \wedge \ldots \wedge v_p = 0\}$) has semialgebraic graph in $(\mathbf{R}^{n*})^p \times G_{n-p}(\mathbf{R}^n)$.

(II) The mapping

$$S : G_{n-p}(\mathbf{R}^n) \times G_{k+p}(\mathbf{R}^n) \setminus \Lambda \ni (U, V) \mapsto U \cap V \in G_k(\mathbf{R}^n)$$

(where $\Lambda = \{(U,V) : U + V \neq \mathbf{R}^n\}$) has semialgebraic graph in $G_{n-p}(\mathbf{R}^n)$
$\times G_{k+p}(\mathbf{R}^n) \times G_k(\mathbf{R}^n)$.

The following mapping has $(\mathbf{R}^{n*})^p \times G_k(\mathbf{R}^n)$ - semialgebraic graph over $\mathbf{R}^n$ (cf. [R])

$$G : Y \ni x \mapsto (\omega_1(x), \ldots, \omega_p(x), T_xY) \in (\mathbf{R}^{n*})^p \times G_{k+p}(\mathbf{R}^n).$$

Then the composition

$$S \circ (T \times id_{G_{k+p}(R^n)}) \circ G = \tau$$

has $G_k(\mathbf{R}^n)$ - semialgebraic graph over $\mathbf{R}^n$ and this implies (2).

$\tau \cap (\Gamma \times G_k(\mathbf{R}^n)) = \tau_\Gamma$, so we have (1).

The formula: $\tau_\Gamma^{-1}(E) = \Gamma \cap \tau^{-1}(E)$ implies (3).

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