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On Singular Cut-and-Pastes in the 3-Space with Applications to Link Theory

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ABSTRACT. In the study of surfaces in 3-manifolds, the so-called "cut-and-paste" of surfaces is frequently used. In this paper, we generalize this method, in a sense, to singular-surfaces, and as an application, we prove that two collections of singular-disks in the 3-space R^3 which span the same trivial link are link-homotopic in the upper-half 4-space $R^3[0,\infty)$ keeping the link fixed.

Throughout the paper, we work in the piecewise linear category, consisting of simplicial complexes and piecewise linear maps.

1. SINGULAR LOOPS IN A 2-CELL

We denote by ∂X and $^\circ X$, respectively, the boundary and the interior of a manifold X. For a subcomplex P in a complex M, by $N(P;M)$ we denote a regular neighborhood of P in M, that is, we construct the second derived of M and take the closed star of P, see [H], [RS].

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We shall say that a submanifold X of a manifold Y is *proper* iff $X \cap \partial Y = \partial X$.

By R^n, D^n and S^{n-1} we shall denote the Euclidean n-space, the standard n-cell and the standard $(n-1)$-sphere ∂D^n, respectively.

1.1. Definition. *(1) Let $f : D^1 \to M$ and $g : S^1 \to M$ be non-degenerate continuous maps into a manifold M. Then, the images $f(D^1) = A$ and $g(S^1) = J$ will be called a singular-arc (or simply an arc) and a singular-loop (or simply a loop), respectively. In particular, A and J will be called a simple arc and a simple loop, respectively, if f and g are embeddings. The boundary of an arc $f(D^1) = A$ is the image $f(\partial D^1)$ of the boundary ∂D^1, and we denote it by $\partial^* A$.*

(2) An arc A in a manifold M is said to be proper iff $A \cap \partial F = \partial^* A$. A loop J in a manifold M is said to be proper iff $J \subset {}^\circ F$.

(3) Let $\mathcal{B} = B_1 \cup \cdots \cup B_n$ be a finite union of proper arcs and proper loops in a 2-manifold F^2. A point p in $\mathcal{B}$ is said to be a singular-point of multiplicity k, iff the number of the preimages of p is k with $k \geq 2$.

We shall say that $\mathcal{B}$ is normal, iff

(i) $\mathcal{B}$ has only a finite number of singular-points of multiplicity 2, and

(ii) at every singular point of $\mathcal{B}$, $\mathcal{B}$ crosses transversally.

1.2. Lemma. *Let $\mathcal{J}_1 = J_{11} \cup \cdots \cup J_{1m(1)}$ and $\mathcal{J}_2 = J_{21} \cup \cdots \cup J_{2m(2)}$ be finite unions of proper loops in a simply connected 2-manifold F^2 such that $\mathcal{J}_1 \cap \mathcal{J}_2 = \emptyset$. Then, there exists $j \in \{1, \cdots, m(1)\}$ or $k \in \{1, \cdots, m(2)\}$ so that J_{1j} is contractible in $F^2 - \mathcal{J}_2$ or J_{2k} is contractible in $F^2 - \mathcal{J}_1$.*

Proof. We may assume that $\mathcal{J}_1 \cup \mathcal{J}_2$ is polygonal and normal.

Let $\mathcal{R} = \{R_1, \cdots, R_r\}$ be the set of regions of $F^2 - {}^\circ N(\mathcal{J}_1; F^2)$. It will be noticed that $R_1 \cup \cdots \cup R_r \supset \mathcal{J}_2$.

If there exist a loop, say J_{2k}, of $\mathcal{J}_2$, and a simply connected region, say R_h, of $\mathcal{R}$ with $J_{2k} \subset R_h$, then J_{2k} is contractible in $R_h \subset F^2 - \mathcal{J}_1$, and so the proof is complete.

So, we may assume that there exist some non-simply connected regions, say $Q_1, \cdots, Q_q$, of $\mathcal{R}$, so that $Q_1 \cup \cdots \cup Q_q \supset \mathcal{J}_2$. Let $C_1 \cup \cdots \cup C_s = \partial Q_1 \cup \cdots \cup \partial Q_q$ be the disjoint union of simple loops on F^2, and let Δ_h be the 2-cell on F^2 with $\partial \Delta_h = C_h (h = 1, \cdots, s)$. We choose an innermost 2-cell, say Δ_1, in $\{\Delta_1, \cdots, \Delta_n\}$, i.e. there is no other Δ_h in Δ_1. Since Δ_1 is not belong to $\mathcal{R}$ and $C_1 = \partial \Delta_1$ is the one of the boundary curves $\partial Q_1 \cup \cdots \cup \partial Q_q$, it holds that $\Delta_1 \cap \mathcal{J}_1 \neq \emptyset$, and since Δ_1 does not contain any $Q_1, \cdots, Q_q$ and $\mathcal{J}_2 \subset Q_1 \cup \cdots \cup Q_q$, it holds that $\Delta_1 \cap \mathcal{J}_2 = \emptyset$. Now, any J_{1j} of $\mathcal{J}_1$ with $J_{1j} \cap \Delta_1 \neq \emptyset$ is contractible in $\Delta_1 \subset F^2 - \mathcal{J}_2$, and so the proof is complete. ■

In the same way as that Lemma 1.2. we have the following:

1.3. Theorem. *Let $\mathcal{J}_i = J_1 \cup \cdots \cup J_{im(i)}$ be a finite union of proper loops in a simply connected 2-manifold F^2 for $i = 1, \cdots, \mu$, such that $\mathcal{J}_i \cap \mathcal{J}_h = \emptyset$ for $i \neq h$. Then, there exist $j \in \{1, \cdots, \mu\}$ and $k \in \{1, \cdots, m(j)\}$ so that J_{jk} is contractible in $F^2 - \bigcup_{i \neq j} \mathcal{J}_i$.*

Proof. We prove this by induction on the number μ of the classes $\mathcal{J}_i$. The case of $\mu = 1$ is trivial, and the case $\mu = 2$ is Lemma 1.2. So, we assume that $\mu \geq 3$ and Theorem is true for $\mu - 1$. We may assume that every $\mathcal{J}_i$ is polygonal and normal.

Let $\mathcal{R} = \{R_1, \cdots, R_r\}$ be the set of regions of $F^2 - {}^{\circ}N(\mathcal{J}_1; F^2)$. It will be noted that $R_1 \cup \cdots \cup R_r \supset \mathcal{J}_2 \cup \cdots \cup \mathcal{J}_\mu$.

If there exist a loop, say J_{jk}, of $\mathcal{J}_j$ and a simply connected region, say R_h, of $\mathcal{R}$ with $J_{jk} \subset R_h$, then $\mathcal{J}'_i = \mathcal{J}_i \cap R_h$ $(i = 2, \cdots, \mu)$ is a finite union of loops in the simply connected region R_h satisfying the conditions of Theorem. By induction hypothesis, we have a loop, say J_{jk}, of $\mathcal{J}'_j \subset \mathcal{J}_j$ so that J_{jk} is contractible in $R_h - \bigcup_{i \neq 1,j} \mathcal{J}'_i \subset F^2 - \bigcup_{i \neq j} \mathcal{J}_i$, and so the proof is complete.

So, we may assume that there exist some non-simply connected regions, say $Q_1, \cdots, Q_q$ of $\mathcal{R}$, so that $Q_1 \cup \cdots \cup Q_q \supset \mathcal{J}_2 \cup \cdots \cup \mathcal{J}_\mu$. Now, the proof of this case, which is omitted here, is the same as that of Lemma 1.2. ■

In general, we have the following:

1.4. Theorem. *Let $\mathcal{A}_i = A_{i1} \cup \cdots \cup A_{in(i)}$ be a finite union of proper arcs in a simply connected 2-manifold F^2 for $i = 1, \cdots, \mu$, and let $\mathcal{J}_i = J_{i1} \cup \cdots \cup J_{im(i)}$ be a finite union of proper loops in F^2, such that $(\mathcal{A}_i \cup \mathcal{J}_i) \cap (\mathcal{A}_h \cup \mathcal{J}_h) = \emptyset$ for $i \neq h$. Then, there exist $j \in \{1, \cdots, \mu\}$ and $k \in \{1, \cdots, m(j)\}$ so that J_{jk} is contractible in $F^2 - \bigcup_{i \neq j} (\mathcal{A}_i \cup \mathcal{J}_i)$.*

Proof. We may assume that every $\mathcal{A}_i \cup \mathcal{J}_i$ is polygonal and normal.

Since every region of $F^2 - {}^\circ N(\mathcal{A}_i; F^2)$ is simply connected, the proof of Theorem is similar to that of Theorem 1.3, and so it is omitted here. ■

2. SINGULAR SPHERES IN A 3-CELL

In this section, we will discuss singular 2-spheres in a 3-cell and prove similar theorems to those in the previous section.

First let us explain several well-known facts to be used in the sequel.

If a compact 3-manifold M is embeddable in the 3-sphere S^3, then there is a 1-complex G in S^3 such that the exterior $S^3 - {}^\circ N(G; S^3)$ is homeomorphic to M by Fox [F].

A 1-complex G in S^3 is said to be split, iff there exists a 2-sphere $S \subset S^3 - G$, such that both components of $S^3 - S$ contain points of G. If a 1-complex $G \subset S^3$ is not split, then the exterior $S^3 - {}^\circ N(G; S^3)$ is aspherical, i.e. the second homotopy group $\pi_2(S^3 - {}^\circ N(G; S^3)) = \{0\}$, by Papakyriakopoulos [P]. In particular, if $G \subset S^3$ is a connected 1-complex, then $S^3 - {}^\circ N(G; S^3)$ is aspherical.

We will call a compact 3-manifold M an *aspherical region*, iff M is embeddable in S^3 and aspherical.

It holds the following:

2.1. Proposition. *(i) If a compact 3-manifold M is embeddable in S^3 and ∂M is connected, then M is an aspherical region.*

(2) Let M be an aspherical region with connected boundary ∂M and let $F \subset {}^\circ M$ be a closed connected 2-manifold. Then, there exists an aspherical region R in M with $\partial R = F$. ■

The following corresponds to Definition 1.1.

2.2. Definition. *(1) Let $f : F^2 \to M$ be a non-degenerate continuous map of a compact 2-manifold F^2 into a manifold M. Then, the image $f(F^2) = F$ will be called a singular-surface. In particular, singular-surfaces $f(D^2) = D$ and $g(S^2) = S$ will be called a singular-disk and singular-sphere, respectively.*

The boundary of a singular-surface $f(F^2) = F$ is the image $f(\partial F^2)$, and we denote it by $\partial^* F$.

(2) A singular-surface F in a manifold M is said to be proper iff $F \cap \partial M = \partial^* F$.

(3) Let F be a proper singular-surface in a 3-manifold M. A point p in F is a singular-point of multiplicity k, iff the number of the preimages of p is k with $k \geq 2$.

We shall say that F is normal, iff

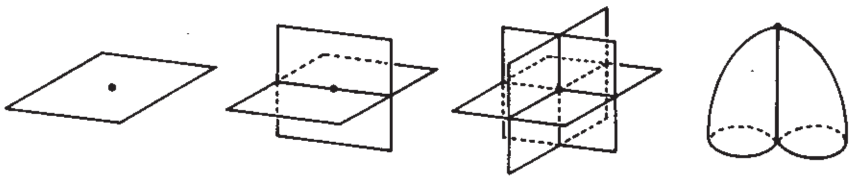
(i) F has only singular-points of multiplicity 2 and 3,

(ii) the set of singular-points of multiplicity 2 is a finite number of polygonal curves, that is, singular-arcs and singular-loops, which will be called double-lines,

(iii) the set of singular-points of multiplicity 3 consists of a finite number of points which are intersection points of the double-lines, which will be called triple-points, and

(iv) at every singular-point of multiplicity 2, F crosses transversally.

In fact, every singular-point p of F has one of the neighborhood described in Figure 1, and it is well known that every singular-surface may be ε-approximated by such a normal one.



regular point double point triple point branch point

Figure 1

2.3. Lemma. *Let* $\mathcal{S}_1 = S_{11} \cup \cdots \cup S_{1m(1)}$ *and* $\mathcal{S}_2 = S_{21} \cup \cdots \cup S_{2m(2)}$ *be finite unions of proper singular-spheres in an aspherical region* M *with connected boundary* ∂M *such that* $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$*. Then, there exists* $j \in \{1, \cdots, m(1)\}$ *or* $k \in \{1, \cdots, m(2)\}$ *so that* S_{1j} *is contractible in* $M - \mathcal{S}_2$ *or* S_{2k} *is contractible in* $M - \mathcal{S}_1$.

Proof. We may assume that $\mathcal{S}_1 \cup \mathcal{S}_2$ is normal. The proof of this Lemma is similar to that of Lemma 1.2.

Let $\mathcal{R} = \{R_1, \cdots, R_r\}$ be the set of regions of $M - {}^{\circ}N(\mathcal{S}_1; M)$. It will be noted that $R_1 \cup \cdots \cup R_r \supset \mathcal{S}_2$.

If there exist a singular-sphere, say S_{2k}, of $\mathcal{S}_2$ and an aspherical region, say R_h, of $\mathcal{R}$ with $S_{2k} \subset R_h$, then S_{2k} is contractible in $R_h \subset M - \mathcal{S}_1$, and we are finished.

So, we may assume that there exist some spherical regions, say $Q_1, \cdots, Q_q$, in $\mathcal{R}$, so that $Q_1 \cup \cdots \cup Q_q \supset \mathcal{S}_2$. Let $F_1 \cup \cdots \cup F_s = \partial Q_1 \cup \cdots \cup \partial Q_q$ be the disjoint union of closed connected 2-manifolds, and let M_h be the aspherical region in M with $\partial M_h = F_h$ for $h = 1, \cdots, s$, see Proposition 2.1 (2). We choose an innermost region, say M_1, in these aspherical regions, that is, there are no other M_h in M_1. Then, by the same way as the proof of Lemma 1.2, it is easily checked that $M_1 \cap \mathcal{S}_1 \neq \emptyset$ and $M_1 \cap \mathcal{S}_2 = \emptyset$. Now, any S_{1j} of $\mathcal{S}_1$ with $S_{1j} \cap M_1 \neq \emptyset$ is contractible in $M_1 \subset M - \mathcal{S}_2$, and completing the proof. ■

The following theorems correspond to Theorems 1.3 and 1.4, respectively.

2.4. Theorem. *Let $\mathcal{S}_i = S_{i1} \cup \cdots \cup S_{im(i)}$ be a finite union of proper singular-spheres in an aspherical region M with connected boundary ∂M for $i = 1, \cdots, \mu$, such that $\mathcal{S}_i \cap \mathcal{S}_h = \emptyset$ for $i \neq h$. Then, there exist $j \in \{1, \cdots, \mu\}$ and $k \in \{1, \cdots, m(j)\}$ so that S_{jk} is contractible in $M - \bigcup_{i \neq j} \mathcal{S}_i$.*

Proof. The proof is similar to that of Lemma 2.3, and is word for word that of Theorem 1.3. ■

2.5. Theorem. *Let M be an aspherical region with connected boundary ∂M. Let $\mathcal{D}_i = D_{i1} \cup \cdots \cup D_{in(i)}$ and $\mathcal{S}_i = S_{i1} \cup \cdots \cup S_{im(i)}$ be finite unions of proper singular-disks and proper singular-spheres in M, respectively, for $i = 1, \cdots, \mu$, such that $(\mathcal{D}_i \cup \mathcal{S}_i) \cap (\mathcal{D}_h \cup \mathcal{S}_h) = \emptyset$ for $i \neq h$. Then, there exist $j \in \{1, \cdots, \mu\}$ and $k \in \{1, \cdots, m(j)\}$, so that S_{jk} is contractible in $M - \bigcup_{i \neq j} (\mathcal{D}_i \cup \mathcal{S}_i)$.*

Proof. We may assume that every $\mathcal{D}_i \cup \mathcal{S}_i$ is normal. Since every region R of $M - {}^\circ N(\mathcal{D}_i; M)$ is an aspherical region, the proof of this Theorem is similar to that of Theorem 2.4, and is word for word that of Theorem 1.4. ■

3. SINGULAR CUT-AND-PASTES

3.1. Definition. *Let M^3 be a 3-manifold, and let E^2 be a compact 2-manifold in ${}^\circ M^3$. Let $f : F^2 \to M^3$ be a non-degenerate continuous map of a compact 2-manifold F^2 into M^3 such that*

(i) $f(F^2) = F$ is a normal singular-surface,

(ii) F intersects with E^2 transversally, and

(iii) no triple-point and no branch point of F lie on E^2.

Then, the intersection $F \cap E^2$ consists of a finite number of arcs and loops. Let J be a loop in $F \cap E^2$, and let J^* be the preimage of J in F^2 : J^* is a simple loop. We suppose that J^* is 2-sided on F^2, and let F'^2 be the 2-manifold obtained from F^2 by attaching a 2-handle along J^*. In fact, we define F'^2 as follows: We take a homeomorphism

$h^2 : \partial D^2 \times D^1 \to N(J^*; F^2)$ *with* $h^2(\partial D^2 \times 0) = J^*$, *and let* $F'^2 = (F^2 - {}^{\circ}N(J^*; F^2)) \cup h^2(D^2 \times \partial D^1)$.

Now, we suppose that J *is contractible on* E^2. *Then, we have a non-degenerate continuous map, say* g, *of* D^2 *into* $E^2 \subset M^3$ *such that* $g(\partial D^2) = J$. *Using the product structure* $N(E^2; M^3) \cong E^2 \times D^1$, *we define a non-degenerate continuous map* $f' : F'^2 \to M^3$ *as follows:*

$$f'|F'^2 - h^2(D^2 \times \partial D^1) = f|F^2 - h^2(\partial D^2 \times D^1),$$

$$f'(h^2(D^2 \times \partial D^1)) = g(D^2) \times \partial D^1 \subset E^2 \times D^1.$$

We say that $F' = f'(F'^2)$ *is obtained from* $F = f(F^2)$ *by a cut-and-paste along* $J \subset E^2$, *and we denote simply by* $F \to F'$.

It will be noticed that $F' \cap E^2 = F \cap E^2 - J$ and that $F'^2 = D^2 \amalg S^2$ (a disjoint union) provided that $F^2 = D^2$ and $F'^2 = S^2 \amalg S^2$ provided that $F^2 = S^2$.

3.2. Theorem. *Let* $\mathcal{O}_i = O_{i1} \cup \cdots \cup O_{in(i)}$ *be a trivial link in the 3-sphere* S^3 *(or the 3-space* R^3*) for* $i = 1, \cdots, \mu$, *such that* $\mathcal{O}_1 \cup \cdots \cup \mathcal{O}_\mu$ *is also a trivial link. Let* $\mathcal{D}_i = D_{i1} \cup \cdots \cup D_{in(i)}$ *be a finite union of normal singular-disks in* S^3 *for* $i = 1, \cdots, \mu$, *such that* $\partial^* D_{ij} = O_{ij}$ *for* $i = 1, \cdots, \mu$ *and* $j = 1, \cdots, n(i)$, *and* $\mathcal{D}_i \cap \mathcal{D}_h = \emptyset$ *for* $i \neq h$.

Let $\mathcal{D}_i^* = D_{i1}^* \cup \cdots \cup D_{in(i)}^*$ *be mutually disjoint 2-cells in* S^3 *(or* R^3*) for* $i = 1, \cdots, \mu$, *such that* $\partial D_{ij}^* = O_{ij}$ *for* $i = 1, \cdots, \mu$ *and* $j = 1, \cdots, n(i)$, *and* $\mathcal{D}_i^* \cap \mathcal{D}_h^* = \emptyset$ *for* $i \neq h$.

We suppose that $\mathcal{D}_1 \cup \cdots \cup \mathcal{D}_\mu$ *intersects with* $\mathcal{D}_1^* \cup \cdots \cup \mathcal{D}_\mu^*$ *transversally, and any triple-point and any branch-point of* $\mathcal{D}_1 \cup \cdots \cup \mathcal{D}_\mu$ *do not lie on* $\mathcal{D}_1^* \cup \cdots \cup \mathcal{D}_\mu^*$.

Then, there exists a finite sequence of cut-and-pastes

$$\mathcal{D}_1 \cup \cdots \cup \mathcal{D}_\mu = \mathcal{D}_1^{(0)} \cup \cdots \cup \mathcal{D}_\mu^{(0)} \to \mathcal{D}_1^{(1)} \cup \cdots \cup \mathcal{D}_\mu^{(1)} \to \cdots$$

$$\to \mathcal{D}_1^{(u)} \cup \cdots \cup \mathcal{D}_\mu^{(u)} \to \cdots \to \mathcal{D}_1^{(w)} \cup \cdots \cup \mathcal{D}_\mu^{(w)}$$

along $(\mathcal{D}_1 \cup \cdots \cup \mathcal{D}_\mu) \dot{\cap} (\mathcal{D}_1^* \cup \cdots \cup \mathcal{D}_\mu^*) \subset \mathcal{D}_1^* \cup \cdots \cup \mathcal{D}_\mu^*$ *such that*

(1) $\mathcal{D}_i^{(u)} = D_{i1}^{(u)} \cup \cdots \cup D_{in(i)}^{(u)} \cup S_{i1}^{(u)} \cup \cdots \cup S_{im(i)}^{(u)}$, *where* $D_{ij}^{(u)}$ *is a singular-disk with* $\partial^* D_{ij}^{(u)} = O_{ij}$ *and* $S_{is}^{(u)}$ *is a singular-sphere, for* $i = 1, \cdots, \mu$; $j = 1, \cdots, n(i)$; $u = 1, \cdots, w$; $s = 1, \cdots, m(i)$,

(2) $\mathcal{D}_i^{(u)} \cap \mathcal{D}_h^{(u)} = \emptyset$ *for* $i \neq h$, $u = 1, \cdots, w$, *and*

(3) $\mathcal{D}_i^{(w)} \cap \mathcal{D}_h^* = \emptyset$ *for* $i \neq h$, *and* $\mathcal{D}_i^{(w)} \cap \mathcal{D}_i^* = (D_{i1}^{(w)} \cup \cdots \cup D_{in(i)}^{(w)}) \cap \mathcal{D}_i^*$ *consists of a finite number of proper arcs in* $\mathcal{D}_i^*$.

Proof. From our hypothesis, $D_{ij} \cap D_{hk}^*$ consists of proper loops in D_{hk}^* provided that $i \neq h$, and $D_{ij} \cap D_{ik}^*$ consists of proper loops and proper arcs in D_{ik}^* for every i, j, k. Therefore, by the induction on the number $n = n(1) + \cdots + n(\mu)$ of 2-cells in $\mathcal{D}_1^* \cup \cdots \cup \mathcal{D}_\mu^*$, it suffices to show that there exists a finite sequence of cut-and-pastes of $\mathcal{D}_1 \cup \cdots \cup \mathcal{D}_\mu$ along proper loops $(\mathcal{D}_1 \cup \cdots \cup \mathcal{D}_\mu) \cap D_{11}^* \subset D_{11}^*$ so that $\mathcal{D}_1^{(u)} \cup \cdots \cup \mathcal{D}_\mu^{(u)}$ satisfies the conditions (1), (2) and

(3) $\mathcal{D}_i^{(w)} \cap D_{11}^* = \emptyset$ and $\mathcal{D}_i^{(w)} \cap D_{1j}^* = \mathcal{D}_i \cap D_{1j}^*$ for $i = 2, \cdots, t$ and $j = 2, \cdots, n(1)$, and $\mathcal{D}_1^{(w)} \cap D_{11}^*$ consists of a finite number of proper arcs in D_{11}^* and $\mathcal{D}_1^{(w)} \cap D_{1j}^* = \mathcal{D}_1 \cap D_{1j}^*$ for $j = 2, \cdots, n(1)$.

We consider $\mathcal{D}_1 \cup \cdots \cup \mathcal{D}_\mu$ and D_{11}^*. Let $\mathcal{A}_1 = A_{11} \cup \cdots \cup A_{1a(1)}$ be the collection of proper arcs in $\mathcal{D}_1 \cap D_{11}^*$ on D_{11}^*, and let $\mathcal{A}_i = \emptyset$ be the collection of proper arcs in $\mathcal{D}_i \cap D_{11}^*$ for $i = 2, \cdots, \mu$. Let $\mathcal{J}_i = J_{i1} \cup \cdots \cup J_{ib(i)}$ be a collection of proper loops in $\mathcal{D}_i \cap D_{11}^*$ on D_{11}^* for $i = 1, \cdots, \mu$. Then, $\mathcal{A}_i \cup \mathcal{J}_i$ satisfies the assumptions in Theorem 1.4, and so there exists a loop J_{jk} of some $\mathcal{J}_j$ such that J_{jk} is contractible in $D_{11}^* - \bigcup_{i \neq j} (\mathcal{A}_i \cup \mathcal{J}_i)$. We have a non-degenerate continuous map $g : D^2 \to D_{11}^*$ such that $g(D^2) \cap (\mathcal{A}_i \cup \mathcal{J}_i) = \emptyset$ for $i \neq j$. Using this g, we perform the first cut-and-paste for $\mathcal{D}_j \subset \mathcal{D}_1 \cup \cdots \cup \mathcal{D}_\mu = \mathcal{D}_1^{(0)} \cup \cdots \cup \mathcal{D}_\mu^{(0)}$ and obtain $\mathcal{D}_1^{(1)} \cup \cdots \cup \mathcal{D}_\mu^{(1)}$. Let w be the number of loops in $(\mathcal{D}_1 \cup \cdots \cup \mathcal{D}_\mu) \cap D_{11}^*$. By the repetition of the procedure w times, we can get rid of all loops in $(\mathcal{D}_1 \cup \cdots \cup \mathcal{D}_\mu) \cap D_{11}^*$, and it is easily checked that $\mathcal{D}_1^{(u)} \cup \cdots \cup \mathcal{D}_\mu^{(u)}$ satisfies the required conditions for $u = 1, \cdots, w$, and we complete the proof of Theorem. ∎

3.3. Remarks. (1) From the proof of Theorem 3.2, we know that w is the number of loops in $(\mathcal{D}_1 \cup \ldots \cup \mathcal{D}_\mu) \cap (\mathcal{D}_1^* \cup \ldots \cup \mathcal{D}_\mu^*)$

and $w = m(1) + \cdots + m(\mu)$, which is the number of singular-spheres in $\mathcal{D}_1^{(w)} \cup \cdots \cup \mathcal{D}_\mu^{(w)}$.

(2) Let D and D^* be a normal singular-disk and a 2-cell, respectively, in S^3 (or R^3) such that $\partial^* D = \partial D^* = O$ (a trivial knot). Let A be a proper arc of $D \cap D^*$ in D^* and let α be a simple arc in O with $\partial\alpha = \partial^* A$. Since $A \cup \alpha$ is contractible in D^*, we can formulate a cut-and-paste of D along $A \cup \alpha \subset D^*$ as the same way as Definition 3.1 except for obvious modifications, so that $D \to D' = D_1' \cup S_1'$, where S_1' is a singular-sphere and D_1' is a singular-disk with $\partial^* D_1' = O$.

Now, in the notation and assumptions of Theorem 3.2, we suppose that $D_{ij} \cap D_{ik}^*$ does not contain proper arcs on D_{ik}^* for $i = 1, \cdots, \mu$ and $j \neq k$. Then, we can remove proper arcs of $\mathcal{D}_i^{(w)} \cap \mathcal{D}_i^*$ by a finite sequence of the modified cut-and-pastes.

4. APPLICATIONS TO LINK THEORY

A continuous image of the 3-cell D^3 will be called a *singular-ball.* The *boundary* of a singular-ball B is the image of ∂D^3, and we denote it by $\partial^* B$.

We use here the same notation as that of Section 0 in [KSS].

The following is a generalization of Horibe-Yanagawa's Lemma [KSS, Lemma 1.6] in a sense.

4.1. Theorem. *In the notation and assumptions of Theorem 3.2, let $\Sigma_i = \Sigma_{i1} \cup \cdots \cup \Sigma_{in(i)}$ be a finite union of singular-spheres in $R^3[0,1]$ defined by*

$$\Sigma_{ij} = D_{ij}[0] \cup O_{ij} \times [0,1] \cup D_{ij}^*[1]$$

for $i = 1, \cdots, \mu$ and $j = 1, \cdots, n(i)$. Then, we can find a finite union of singular-balls $B_i = B_{i1} \cup \cdots \cup B_{in(i)}$ in $R^3[0, \infty)$ for $i = 1, \cdots, \mu$, such that $\partial^* B_{ij} = \Sigma_{ij}$ for every i and j, and $B_i \cap B_h = \emptyset$ for $i \neq h$.

Proof. The proof is similar to that of [KSS, Lemma 1.6]. We shall construct the required singular-balls $B_1 \cup \cdots \cup B_\mu$ by specifying the cross-sections $B_{ij} \cap R^3[t]$ for all i and j.

Under the notation of Theorem 3.2, we also use Theorem 3.2. Let $g_u : D^2 \to \mathcal{D}_1^* \cup \cdots \cup \mathcal{D}_\mu^*$ $(u = 1, \cdots, w)$ be a non-degenerate continuous map such that we perform the u-th cut-and-paste

$$\mathcal{D}_1^{(u-1)} \cup \cdots \cup \mathcal{D}_\mu^{(u-1)} \to \mathcal{D}_1^{(u)} \cup \cdots \cup \mathcal{D}_\mu^{(u)}$$

in Theorem 3.2 along the loop $g_u(\partial D^2)$ under g_u. We extend g_u to a continuous map

$$g_u^\# : h^2(D^2 \times D^1) \to N(\mathcal{D}_1^* \cup \cdots \cup \mathcal{D}_\mu^*; R^3) \cong (\mathcal{D}_1^* \cup \cdots \cup \mathcal{D}_\mu^*) \times D^1$$

of the 3-cell $h^2(D^2 \times D^1)$ naturally, and we denote the singular-ball $g_u^\#(h^2(D^2 \times D^1))$ by H_u. We divide the interval [0,1] into the subintervals $[0, t_1], [t_1, t_2], \cdots, [t_{w-1}, t_w], [t_w, 1]$, where $t_u = u/(w+1)$, $u = 1, \cdots, w$. Let

$$(\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_\mu) \cap R^3[t] = (\mathcal{D}_1 \cup \cdots \cup \mathcal{D}_\mu)[t] \text{ for } 0 \le t < t_1$$
$$(\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_\mu) \cap R^3[t_1] = (\mathcal{D}_1 \cup \cdots \cup \mathcal{D}_\mu \cup H_1)[t_1],$$
$$(\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_\mu) \cap R^3[t] = (\mathcal{D}_1^{(1)} \cup \cdots \cup \mathcal{D}_\mu^{(1)})[t] \text{ for } t_1 < t < t_2,$$

$$\cdots \quad \cdots \quad \cdots$$

$$(\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_\mu) \cap R^3[t] = (\mathcal{D}_1^{(u-1)} \cup \cdots \cup \mathcal{D}_\mu^{(u-1)})[t] \text{ for } t_{u-1} < t < t_u,$$
$$(\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_\mu) \cap R^3[t_u] = (\mathcal{D}_1^{(u-1)} \cup \cdots \cup \mathcal{D}_\mu^{(u-1)} \cup H_u)[t_u],$$
$$(\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_\mu) \cap R^3[t] = (\mathcal{D}_1^{(u)} \cup \cdots \cup \mathcal{D}_\mu^{(u)})[t] \text{ for } t_u < t < t_{u+1},$$

$$\cdots \quad \cdots \quad \cdots$$

$$(\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_\mu) \cap R^3[t_w] = (\mathcal{D}_1^{(w-1)} \cup \cdots \cup \mathcal{D}_\mu^{(w-1)} \cup H_w)[t_w],$$
$$(\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_\mu) \cap R^3[t] = (\mathcal{D}_1^{(w)} \cup \cdots \cup \mathcal{D}_\mu^{(w)})[t] \text{ for } t_w < t \le 1.$$

Thus, we constructed $(\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_\mu) \cap R^3[0,1]$ which consists of $n = n(1) + \cdots + n(\mu)$ singular-balls with $w = m(1) + \cdots + m(\mu)$ singular-balls removed.

Let $S_{ij}^{(w)} = D_{ij}^{(w)} \cup D_{ij}^{*}$ be the singular-sphere for $i = 1, \cdots, \mu$ and $j = m(i)+1, \ldots, m(i)+n(i)$, and let $\mathcal{S}_i = \mathcal{D}_i^{(w)} \cup \mathcal{D}_i^{*} = S_{i1}^{(w)} \cup \cdots \cup S_{im(i)+n(i)}^{(w)}$, which consists of $m(i)+n(i)$ singular-spheres in R^3. From Theorem 3.2(2) and (3), it is easy to see that $\mathcal{S}_i \cap \mathcal{S}_h = \emptyset$ for $i \neq h$, which is the assumption of Theorem 2.4.

We divide the interval [1,2] into the $n+w+1$ subintervals $[1, s_1]$, $[s_1, s_2], \cdots, [s_{n+w-1}, s_{n+w}], [s_{n+w}, 2]$, where $s_v = 1 + v/(n+w+1)$, $v = 1, \cdots, n+w$. From now on, we construct $(\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_\mu) \cap R^3[1,2]$ so that $(\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_\mu) \cap R^3[0,2]$ forms the required singular-balls. By Theorem 2.4, there exist $j \in \{1, \cdots, \mu\}$ and $k \in \{1, \cdots, m(j)+n(j)\}$ so that $S_{jk}^{(w)}$ is contractible in $R^3 - \bigcup_{i \neq j} \mathcal{S}_i$. Let $g_1 : D^3 \to R^3 - \bigcup_{i \neq j} \mathcal{S}_i$ be a continuous map such that $g_1(\partial D^3) = S_{jk}^{(w)}$, and we denote $g_1(D^3)$ by E_1. We set $\mathcal{S}_j^{(1)} = \mathcal{S}_j - S_{jk}^{(w)}$, and $\mathcal{S}_i^{(1)} = \mathcal{S}_i$ for $i \neq j$. Then, we define $(\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_\mu) \cap R^3[1, s_2)$ as follows:

$$(\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_\mu) \cap R^3[t] = (\mathcal{S}_1 \cup \cdots \cup \mathcal{S}_\mu)[t] \text{ for } 1 \leq t < s_1,$$
$$(\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_\mu) \cap R^3[s_1] = (\mathcal{S}_1 \cup \cdots \cup \mathcal{S}_\mu \cup E_1)[s_1],$$
$$(\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_\mu) \cap R^3[t] = (\mathcal{S}_1^{(1)} \cup \cdots \cup \mathcal{S}_\mu^{(1)})[t] \text{ for } s_1 < t < s_2.$$

By Theorem 2.4. there exist $j' \in \{1, \cdots, \mu\}$ and $k' \in \{1, \cdots, m(j')+n(j')\}$ so that $S_{j'k'}$ is contractible in $R^3 - \bigcup_{i \neq j} \mathcal{S}_i^{(1)}$. Let $g_2 : D^3 \to R^3 - \bigcup_{i \neq j} \mathcal{S}_i^{(1)}$ be a continuous map with $g_2(\partial D^3) = S_{j'k'}$, and we denote $g_2(D^3)$ by E_2. We set $\mathcal{S}_{j'}^{(2)} = \mathcal{S}_{j'}^{(1)} - S_{j'k'}^{(1)}$ and $\mathcal{S}_i^{(2)} = \mathcal{S}_i^{(1)}$ for $i \neq j'$. We define $(\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_\mu) \cap R^3[s_2, s_3)$ as follows:

$$(\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_\mu) \cap R^3[s_2] = (\mathcal{S}_1^{(1)} \cup \cdots \cup \mathcal{S}_\mu^{(1)} E_2)[s_2],$$
$$(\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_\mu) \cap R^3[t] = (\mathcal{S}_1^{(2)} \cup \cdots \cup \mathcal{S}_\mu^{(2)})[t] \text{ for } s_2 < t < s_3.$$

For $R^3[s_3, s_4), \cdots, R^3[s_{n+w-1}, s_{n+w}), R^3[s_{n+w}, 2)$, we repeat this process. It should be noticed that $\mathcal{S}_1^{(n+w-1)} \cup \cdots \cup \mathcal{S}_\mu^{(n+w-1)}$ consists of a single singular-sphere and $\mathcal{S}_1^{(n+w)} \cup \cdots \cup \mathcal{S}_\mu^{(n+w)} = \emptyset$. Therefore, $(\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_\mu) \cap R^3[s_{n+w}]$ consists of a singular-ball $E_{n+w}[s_{n+w}]$, and $(\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_\mu) \cap R^3[t] = \emptyset$ for $s_{n+w} < t < 2$.

Thus, we obtain a union of singular-balls $\mathcal{B}_i = B_{i1} \cup \cdots \cup B_{in(i)}$ in $R^3[0,\infty)$ for $i = 1, \cdots, \mu$ such that $\partial^* B_{ij} = \Sigma_{ij}$. From our construction, it is easily checked that $\mathcal{B}_i \cap \mathcal{B}_h = \emptyset$ for $i \neq h$, and this completes the proof of Theorem 4.1. ■

The relation of link-homotopy was introduced in classical link theory by Milnor [M], and studied higher dimensional links by Massey-Rolfsen [MR] and Koschorke [K], etc. We record a corollary to Theorem 4.1 on link-homotopy.

4.2. Definition. *Let $P_1, \cdots, P_\mu$ be polyhedra, and let $\mathcal{P} = P_1 \amalg \cdots \amalg P_\mu$ be their disjoint union, and let X be a manifold. A continuous map $f : \mathcal{P} \to X$ is said to be a link-map, iff $f(P_i) \cap f(P_h) = \emptyset$ for $i \neq h$. Two link-maps f_0 and f_1 of $\mathcal{P}$ into X will be called link-homotopic, iff there exists a homotopy $\{\eta_t\}_{t\in I} : \mathcal{P} \to X$ such that $\eta_0 = f_0$, $\eta_1 = f_1$, and $\eta_t(P_i) \cap \eta_t(P_h) = \emptyset$ for $i \neq h$ and each $t \in I = [0,1]$.*

4.3. Theorem. *Let $\mathcal{O}_i = O_{i1} \cup \cdots \cup O_{in(i)}$ be a trivial link in the 3-space $R^3 = R^3[0] \subset R^3[0,\infty)$ (or $S^3 \subset \partial D^4$) for $i = 1, \cdots, \mu$, such that $\mathcal{O}_1 \cup \cdots \cup \mathcal{O}_\mu$ is also a trivial link. Let $P_i = D_{i1}^2 \amalg \cdots \amalg D_{in(i)}^2$ be the disjoint union of $n(i)$ 2-cells for $i = 1, \cdots, \mu$, and we set $\mathcal{P} = P_1 \amalg \cdots \amalg P_\mu$. Let f and e be non-degenerate link-maps of $\mathcal{P}$ into R^3 (or S^3) such that $f(\partial D_{ij}^2) = O_{ij} = e(\partial D_{ij}^2)$ for $i = 1, \cdots, \mu$ and $j = 1, \cdots, n(i)$.*

Then, f and e are link-homotopic in $R^3[0,\infty)$ (or D^4) keeping $\mathcal{O}_1 \cup \cdots \cup \mathcal{O}_\mu$ fixed.

Proof. Let $f(D_{ij}^2) = D_{ij}$ and $\mathcal{D}_i = D_{i1} \cup \cdots \cup D_{in(i)}$ for $i = 1, \cdots, \mu$ and $j = 1, \cdots, n(i)$. Let $g : \mathcal{P} \to R^3$ be an embedding, and let $g(D_{ij}^2) = D_{ij}^*$ and $\mathcal{D}_i^* = D_{i1}^* \cup \cdots \cup D_{in(i)}^*$. In this notation, it suffices to show that f and g are link-homotopic in $R^3[0,\infty)$ keeping $\mathcal{O}_1 \cup \cdots \cup \mathcal{O}_\mu$.

In the notation of Theorem 4.1, we have a finite union of singular-balls $\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_\mu$, $\mathcal{B}_i = B_{i1} \cup \cdots \cup B_{in(i)}$ in $R^3[0,\infty)$ such that $\mathcal{B}_i \cap \mathcal{B}_h = \emptyset$ for $i \neq h$ and $\partial^* B_{ij} = \Sigma_{ij}$. Let $b_{ij} : D^2 \times I \to R^3[0,\infty)$ be a continuous map of the 3-cell $D^2 \times I$ such that $b_{ij}(D^2 \times I) = B_{ij}$. We may assume that $b_{ij}|D^2 \times 0 = f|D^2_{ij}$ and $b_{ij}|D^2 \times 1 = g|D^2_{ij}$. Then, associating with these b_{ij}, we have a link-homotopy $\{\eta_t\}_{t\in I} : \mathcal{P} \to R^3[0,\infty)$ defined by

$$\eta_t(D^2_{ij}) = b_{ij}(D^2 \times t)$$

for every $t \in I$. From the condition of the singular-balls $\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_\mu$ in Theorem 4.1, it is easily checked that this homotopy $\{\eta_t\}_{t\in I}$ between f and g satisfies our required condition, and completing the proof of Theorem 4.3. ■

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