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The 3-Modular Characters of the Twisted Chevalley Group ${}^2D_4(2)$ and ${}^2D_4(2).2$

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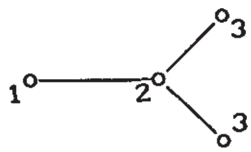
ABSTRACT. In this paper we calculate the 3-modular character table of the twisted Chevalley group ${}^2D_4(2)$ and its automorphism group ${}^2D_4(2).2$. The "Meat-Axe" package for calculating modular characters over finite fields [6] was used to calculate most of the characters. The method of "condensation", which was explained in [7] was used to determine the complete character table. All these methods are explained later in this paper.

INTRODUCTION

The twisted Chevalley group ${}^2D_4(2)$ is a simple group of order 197 406 720$=2^{12}$.$\underline{3}^4$ .5 .7 .17 . The Dynkin diagram and twisting automorphism of this group is:



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This group is isomorphic to the orthogonal group $O_8^-(2)$ which is the derived group of all 8×8 matrices over GF_2 preserving a quadratic form of Witt defect 1. Its automorphism group ${}^2D_4(2).2$ is the largest maximal subgroup of the symplectic group $S_8(2)$. The maximal subgroups for this group are as follows in "ATLAS" notation (see [2]):

Structure	Index
1) $2^6 : U_4(2)$	119
2) $S_6(2)$	136
3) $2^{3+6} : (L_3(2) \times 3)$	765
4) $2_+^{1+8} : (S_3 \times A_5)$	1071
5) $(3 \times A_8) : 2$	1632
6) $L_2(16) : 2$	24192
7) $S_3 \times S_3 \times A_5$	45696
8) $L_2(7)$	1175040

The 5-, 7- and 17-modular characters have been determined (see [4]). In this paper we determine and construct three 3-modular irreducible representations for ${}^2D_4(2)$ and ${}^2D_4(2).2$ in order to determine the 3-modular character tables of ${}^2D_4(2)$ and ${}^2D_4(2).2$.

THE 3-MODULAR CHARACTER TABLE OF ${}^2D_4(2)$ AND ${}^2D_4(2).2$

The 3-central characters of ${}^2D_4(2)$ give the following block distribution of the ordinary irreducible characters:

i) There are four blocks of defect zero:

$$B1 = \{2385_a\}\ ,\ B2 = \{2835_b\}\ ,\ B3 = \{2835_c\} \text{ and } B4 = \{2835_d\}\ .$$

Therefore 2835_a, 2835_b, 2835_c, and 2835_d are four 3-modular irreducible characters.

ii) There is a block $B5 = \{2295_a,\ 2295_b,\ 2295_c\}$ of defect 1. On the 3-regular classes the following relations hold:

$$2295_a = 2295_b = 2295_c$$

Therefore 2295 is a 3-modular irreducible character.

iii) There is a block B6 of defect 2 where

$B6 = \{1071_a,\ 1071_b,\ 1071_c,\ 1071_d,\ 2142_a,\ 2142_b,\ 2142_c,\ 2142_d,\ 4284\}$

On the 3-regular classes the following relations hold:

$$1071_a + 1071_c = 2142_c \qquad (i)$$
$$1071_a + 1071_d = 2142_d \qquad (ii)$$
$$1071_b + 1071_c = 2142_b \qquad (iii)$$
$$1071_b + 1071_d = 2142_a \qquad (iv)$$
$$2142_a + 2142_c = 2142_b + 2142_d = 4284 \qquad (v)$$

It follows that this block contains just four 3-modular characters, all of degree 1071. Hence 1071_a, 1071_b, 1071_c, 1071_d are four 3-modular irreducible representations for ${}^2D_4(2)$.

iv) The remaining twenty three ordinary characters are in the principal block Bo of defect 4.

The 3-regular classes are nineteen. Therefore there are nineteen 3-modular irreducibles for ${}^2D_4(2)$. All, except three of these 3-modular irreducibles were found by Parker (see [4]). In this part of this paper we constructed these three. We found that they are of degrees 511, 1036 and 2464. As for most of the incomplete character tables, we have to do again most of the calculations involving the other known irreducibles. The main method used here, was condensation of permutation modules, then we constructed the invariant subspaces which contains the required representations by spinning up the corresponding subspaces in the condensed module under group generators. This can be done using the (uncondense program) "UK". This method was explained in [7]. Here is a brief explanation of this method:

Condensation of permutation modules

Let G be a group, V be a kG-module where k is assumed to be a finite field of characteristic p. Let K be a subgroup such that $p \nmid |K|$.

Define the idempotent $e = \frac{1}{|K|} \sum_{h \in K} h$ of kG. Then $e.kG.e$ is a sub-algebra of kG known as a Hecke Algebra.

From any kG-module V, we obtain an $e.kG.e$-module Ve. We say that Ve is condensed from V since Ve consists of the fixed points of the action of K on V. In this way we get condensed module Ve such that dim $Ve = \frac{dim V}{|K|}$.

Accordingly, it should be much easier to apply the "Meat-Axe" to Ve rather than to V. Moreover, any information about Ve which we obtain with the "Meat-Axe" give rise to information about V. This can be seen using the following proposition:

Proposition. *Let $\chi_1, \chi_2, \ldots, \chi_r$ be the irreducible constituents of the module V, then the irreducible constituents of Ve are the non-zero members of the set $\{\chi_1 e, \chi_2 e, \ldots, \chi_r e\}$.* (See [7], [10]).

"UK" (Un-Condense program)

Given an invariant subspace Ve of a condensed module we can construct the corresponding invariant subspace V of the permutation module as follows:

(1) Embed Ve in V (which is what this program "UK" does. [Each basis vector for Ve is the sum of basis vectors in V over an orbit of K].

(2) Spin up under the group generators. That is multiply the vectors we have so far by the generating permutation (using a version of "MU" capable of multiply non-square matrices by permutations), together with Guassian elimination programs "EF" (echelon form), "CL" (clean) and "CE" (clean and extend) and put the whole collection of vectors into echelon form. Eventually, we obtain the subspace that is invariant under the group generators. This gives us degree of the representation. (i.e. dim (V)).

(3) If we want to construct the matrices for G in this representation, we can apply the generators to our invariant subspace (using "MU") and use "CL" to write the images of the basis vectors as linear combinations of the basis vectors.

1. THE PERMUTATION REPRESENTATIONS OF $^2D_4(2)$ AND $^2D_4(2).2$

We started with two 8-dimensional generators a and b for $^2D_4(2).2$ over GF_2 where

$$a = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad b = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

The permutation representations on 119, 136, 765, 1071, 1632, 3213, 13056 and 14280 points can be found using the MEAT-AXE. The first six permutation modules on 119, 136, 765, 1017, 1632, and 3213 were chopped up over GF_3, using the Meat-Axe. We find that:

$P1 = 119 = 1 + 2(34) + 50$

$P2 = 136 = 2(1) + 34 + 2(50)$

$P3 = 765 = 3(1) + 34 + 50 + 203 + 475$

$P4 = 1071 = 2(1) + 3(34) + 2(50) + 392 + 475$

$P5 = 1632 = 2(1) + 2(34) + 2(50) + 203 + 2(392) + 475$

$P6 = 3213 = 5(1) + 7(34) + 5(50) + 2(154) + 2(203) + 2(475) + 2(511)$

Then, using the Meat-Axe again we find that:

$\Lambda^2(34) = 154 + 203 + 203$

$\Lambda^2(50) = 154 + 1071_b$

$34 \otimes 50 = 511 + 511 + 475 + 203$

2. CONDENSATION OF PERMUTATION MODULES

To find the other 3-modular irreducible representations, we use the method of condensation which was explained. We take our generators a and b for $^2D_4(2).2$, and find words to get a suitable condensation subgroup K of order not divisible by 3. In this case we take the cyclic group K of order 17.

Although, we do not need to condense the permutation modules $P1$, $P2$, $P3$, $P4$, $P5$ and $P6$, but we did so to identify the irreducibles in the condensed modules corresponding to the 3-modular irreducibles.

Firstly, looking at the composition of the above permutation representations, we note that it is sufficient to consider the permutation representations of degrees 119, 765, 1071 and 3213, in order to identify the condensed irreducibles corresponding to the 3-modular irreducibles.

i) The permutation representation on 119 points

The condensation of the permutation module P1 over the cyclic subgroup K of order 17 gives a condensed module $M1$ of dimension 7. $M1$ is chopped up to irreducibles as follows:

$$7 = 1 + 2(2_a) + 2_b$$

and therefore, we have the following correspondence in table 2:

The permutation P1 on 119 points	1	34	50			
The 7-dimensional condensed module M1	1_a	2_a	2_b			

Table 2

ii) The permutation representation on 765 points

$P3$ is condensed, using the same subgroup K of order 17 to get a condensed module $M2$ of dimension 45, $M2$ is chopped up using the Meat-Axe to:

$$45 = 3(1) + 2_a + 2_b + 11 + 27$$

Therefore we get the following correspondence (table 3):

The permutation P3 on 765 points	1	34	50	203	475		
The 153-dimensional condensed module M2	1_a	2_a	2_b	11	27		

Table 3

iii) **The permutation representation on 1071 points**

Now we condense the permutation module $P4$ over the same cyclic subgroup K of order 17 to get a condensed module $M3$ of dimension 63. $M3$ is chopped up using the Meat-Axe to the following irreducibles:

$$63 = 2(1) + 3(2_a) + 2(2_b) + 24 + 27$$

Therefore we get the following correspondence as in table 4:

The permutation P4 on 1017 points	1	34	50	392	475		
The 63-dimensional condensed module M3	1	2_a	2_b	24	27		

Table 4

iv) **The permutation representation on 3123 points**

Using the same cyclic subgroup K of order 17 we condense $P5$ to get a condensed module $M4$ of dimension 141. The condensed module $M4$ is chopped up to the following irreducibles:

$$141 = 4(1) + 6(2_a) + 4(2_b) + 10 + 2(11) + 2(27) + 31$$

Thus we have the following correspondence as in table 5:

The permutation P5 on 3123 points	1	34	50	154	203	475	511			
The 141-dimensional condensed module M4	1	2_a	2_b	10	11	27	31			

Table 5

v) **The permutation representation on 14280 points**

This permutation module was condensed over the same cyclic group K of order 17 to get a condensed module $M5$ of dimension 840. The condensed module $M5$ is chopped up to the following irreducibles:

$$840 = 8(1) + 8(2_a) + 8(2_b) + 6(10) + 3(11) + 5(24)$$

$$+ 8(27) + 5(31) + 2(60) + 144$$

Therefore, we get two new 3-modular irreducibles corresponding to the condensed irreducibles 60 and 144. By spinning up the invariant subspaces containing 60 and 144 under group generators, one at time (using "UK"), we constructed the 3-modular irreducible representations of degree 1036 and 2464 respectively. Now we can write the correspondences between the condensed irreducibles and the 3-modular irreducibles as follows in table 6:

The permutation P5 on 8568 points	1	34	50	154	203	475	511	1036	2464	
The 504-dimensional condensed module M5	1	2_a	2_b	10	11	27	31	60	144	

Table 6

Therefore, the three missing representations of degrees 511, 1036 and 2464 are constructed and, hence we can find the character values on the

3-regular classes in order to complete the character table of ${}^2D_4(2)$ and ${}^2D_4(2).2$.

3. INDICATORS OF THE 3-MODULAR IRREDUCIBLES OF ${}^2D_4(2)$ AND ${}^2D_4(2).2$

To calculate the indicators of ${}^2D_4(2)$ and ${}^2D_4(2).2$, we use the symbols + and -. For each self dual representation we write + if there is an invariant quadratic form (or symmetric bilinear form), and if there is an invariant symplectic form (or skew-symetric form). To determine these indicators it is sufficient to use the following results:

Theorem. *If p is an odd prime, and ϕ is a p-modular self dual irreducible representation which occurs an odd number of times as a constituent of an ordinary self-dual irreducible representation χ, then the indicators of ϕ and χ are the same.* (See [11]).

4. CALCULATING THE CHARACTER VALUES

We find representatives for all the 3-regular classes of ${}^2D_4(2)$ and ${}^2D_4(2).2$ as words in a and b. Then we work out the character values on these classes using the program "EV" of the Meat-Axe which works out the eigenvalues of a matrix (see [7]).

Theorem. *Table 7 represents the complete 3-modular character table of ${}^2D_4(2)$ and ${}^2D_4(2).2$:*

;	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	@	;	;	@	@	@	@	@	@	@	@	@	@	@	@	@
		184	46	3	1	1																1451	4	3	2									
197	406720	320	080	072	536	536	192	64	180	21	32	32	60	60	20	17	17	17	17			520	608	840	304	384	384	128	96	32	8	30	7	10
p	power	A	A	A	A	A	B	C	A	A	A	B	AA	AA	AB	A	A	A	A			A	A	B	B	A	B	B	A	B	D	AD	AD	CE
p'	part	A	A	A	A	A	A	A	A	A	A	A	AA	AA	AB	A	A	A	A			A	A	A	A	A	A	A	A	A	A	AD	AD	AE
ind	1A	2A	2B	2C	4A	4B	4C	4D	5A	7A	8A	8B	10A	B*	10C	17A	B*2	C*3	D*6	fus	ind	2D	2E	4E	4F	4G	4H	4I	8C	8D	8E	10D	14A	20A
+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	:	++	1	1	1	1	1	1	1	1	1	1	1	1	1
+	34	2	10	2	-2	6	2	2	4	-1	0	0	2	2	0	0	0	0	0	:	++	20	4	0	8	0	4	0	-2	2	0	0	-1	0
+	50	18	10	2	6	-2	2	-2	0	1	0	0	-2	-2	0	-1	-1	-1	-1	:	++	22	6	10	2	2	-2	2	4	0	0	2	1	0
+	154	26	-6	-6	2	2	-2	2	-1	0	-2	2	1	1	-1	1	1	1	1	:	++	28	-4	16	0	-4	0	0	4	0	0	3	0	1
+	203	-21	19	-5	-1	7	-1	-1	3	0	1	-3	-1	-1	-1	-1	-1	-1	-1	:	++	77	-3	-11	13	1	1	-3	-1	-1	-1	-3	0	-1
+	392	-24	16	8	-4	4	-4	0	-3	0	-2	2	1	1	1	1	1	1	1	:	++	112	0	-16	8	-4	4	0	-2	-2	0	-3	0	-1
+	475	59	19	11	7	-1	-1	-1	-5	-1	1	-3	-1	-1	-1	-1	-1	-1	-1	:	++	96	17	9	1	5	5	1	-1	-1	-1	-3	-1	-1
+	511	-1	31	-1	-9	-9	3	-1	1	0	-1	3	-1	-1	1	1	1	1	1	:	++	133	5	1	1	-3	-7	1	-3	1	1	3	0	1
+	1036	44	-44	-4	0	8	0	0	1	0	2	-2	-1	-1	1	-1	-1	-1	-1	:	++	-98	14	-26	-2	2	2	-2	0	0	0	-3	0	-1
+	1071	-81	31	-1	7	-9	-1	3	1	0	-1	-1	-1	-1	1	0	0	0	0	:	++	189	-3	-19	-3	-3	-3	5	1	1	-1	-1	0	1
+	1071	111	31	-17	7	-9	-1	-1	1	0	-1	-1	1	1	1	0	0	0	0	:	++	189	-3	29	-3	-3	-3	-3	1	1	1	-1	0	-1
+	1071	15	-41	15	-5	3	-1	-1	1	0	1	1	r5	-r5	-1	0	0	0	0	.	+	0	0	0	0	0	0	0	0	0	0	0	0	0
+	1071	15	-41	15	-5	3	-1	-1	1	0	1	1	-r5	r5	-1	0	0	0	0	.														
+	2295	-105	15	-9	11	3	-1	-1	0	-1	1	1	0	0	0	0	0	0	0	:	++	405	-21	-15	9	9	-3	1	3	-1	1	0	-1	0
+	2464	-64	24	0	-4	-12	4	0	-1	0	2	-2	1	1	-1	-1	-1	-1	-1	:	++	308	4	-4	-12	0	0	-4	2	2	0	3	0	1
+	2835	-45	-45	3	3	3	3	-1	0	0	-1	-1	0	0	0	-d17	*2	*3	*6	.	+	0	0	0	0	0	0	0	0	0	0	0	0	0
+	2835	-45	-45	3	3	3	3	-1	0	0	-1	-1	0	0	0	*2	-d17	*6	*3	.														
+	2835	-45	-45	3	3	3	3	-1	0	0	-1	-1	0	0	0	*6	*3	-d17	*2	.	+	0	0	0	0	0	0	0	0	0	0	0	0	0
+	2835	-45	-45	3	3	3	3	-1	0	0	-1	-1	0	0	0	*3	*6	*2	-d17	.														

Table 7

References

[1] Alperin, J.: *Local representation theory.* Cambridge studies in advanced mathematics 11. Cambridge.

[2] Conway, J., Curtis, R., Norton, S., Parker, R. and Wilson, R.: *An ATLAS of finite groups*, Clarendon Press. Oxford 1985.

[3] Curtis, C. and Reiner, I.: *Representation theory of finite groups and associative algebra.* Interscience publishers, New York, 1962.

[4] Parker, R.A.: *Modular character tables*, (unpublished).

[5] Parker, R.A. and Wilson, R.A.: *The computer construction of matrix representations of finite groups over finite fields.* J. of Symb. Comp., Vol. 9, 1989.

[6] Ryba, A.J.E.: *Computer condensation of modular representations*, J. of Symbolic Comp. 9, 1990, pp. 591-600.

[7] Suleiman, I.A.: *Modular representations of finite simple groups*, Ph. D thesis, Birmingham University 1990.

[8] Suleiman, I.A. and Wilson, R.A.: *The 3- and 5-modular characters of the covering and the automorphism groups of the Higman Sims Group*, Journal of Algebra, Vol. 148, No. 1, May 1992.

[9] Suleiman, I.A. and Wilson, R.A.: *Computer construction of matrix representations of the covering group of the Higman Sims Group.* Journal of Algebra, Vol. 148, No. 1, May 1992.

[10] Suleiman, I.A. and Wilson, R.A.: *The 3- and 5-modular characters of Mclaughlin's group McL and its automorphism group.* (Accepted in the Proceedings of Durham Conferrence 1990).

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