

*On The Locally Uniformly Weak Star Rotundity of Orlicz Spaces **

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ABSTRACT. In the paper, a sufficient and necessary condition is given for the locally uniformly weak star rotundity of Orlicz spaces with Orlicz norms.

A Banach space X is said to be locally uniformly rotund (LUR), locally weakly uniformly rotund ($LWUR$), locally uniformly weak star rotund (LW^*UR) provided that $\|x_n\| = 1$ ($n = 0, 1, 2, \dots$), $\|x_n + x_0\| \rightarrow 2$ imply $\|x_n - x_0\| \rightarrow 0$, $x_n - x_0 \xrightarrow{w} 0$, $x_n - x_0 \xrightarrow{w^*} 0$, respectively. X is said to be uniformly weak star rotund (W^*UR) provided that $\|x_n\| = \|y_n\| = 1$, $\|x_n + y_n\| \rightarrow 2$ imply $x_n - y_n \xrightarrow{w^*} 0$. At a glance we know that

$$LUR \Rightarrow LWUR \Rightarrow LW^*UR \Rightarrow R$$
$$W^*UR \Rightarrow LW^*UR$$

where ' R ' stands for the rotundity.

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In the sequel (G, Σ, μ) denotes a finite non-atomic measurable space, M and N denote a pair of complemented N -functions, p and q denote their right-hand derivatives, respectively. For a measurable function $x(t)$ we denote the modular of x by $R_M(x) = \int_G M(x(t)) d\mu$. $L_M(G, \Sigma, \mu)$ denotes an Orlicz space generated by M , that is

$$L_M(G, \Sigma, \mu) = \{x(t) : \text{for some } a > 0, R_M(ax) < \infty\}$$

and endowed with the Orlicz norm

$$\|x\| = \sup_{R_M(y) \leq 1} \int_G x(t)y(t) d\mu = \inf_{k > 0} \frac{1}{k} (1 + R_M(kx)).$$

$M \in \Delta_2$ stands for that M which satisfies the condition Δ_2 for large u , $M \in \nabla_2$ stands for $N \in \Delta_2$, $M \in SC$ stands for that M which is strictly convex on the whole axis i.e. for $0 < \lambda < 1, u, v, u \neq v$,

$$M(\lambda u + (1 - \lambda)v) < \lambda M(u) + (1 - \lambda)M(v).$$

(cf [1] and [3]).

In Orlicz spaces, for Luxemburg norm, it was obtained in [2] that $LUR \Leftrightarrow LWUR \Leftrightarrow LW^*UR \Leftrightarrow R \Leftrightarrow M \in SC \cap \Delta_2$; for the Orlicz norm, it is more complicated, for instance, $LUR \Leftrightarrow LWUR \Leftrightarrow M \in \Delta_2 \cap \nabla_2 \cap SC$ (cf[3]), $W^*UR \Leftrightarrow M \in SC \cap UC$ (cf[4]) and $R \Leftrightarrow M \in SC$ (cf[5]). But so far it has not been discussed for LW^*UR . The goal of this paper is to fill this gap, we will find a criterion for Orlicz space equipped with the Orlicz norm to be LW^*UR . For the sake of convenience, we first establish several lemmas.

Lemma 1. *For arbitrary $0 \leq \lambda, \delta, \lambda' < 1$, there exists $0 < \delta' \leq \delta$ such that for all $u, v > 0$ if $M(\lambda u + (1 - \lambda)v) \leq (1 - \delta)(\lambda M(u) + (1 - \lambda)M(v))$, then*

$$M(\lambda' u + (1 - \lambda')v) \leq (1 - \delta')(\lambda' M(u) + (1 - \lambda')M(v))$$

Proof. Without loss of generality, we assume that $\lambda' > \lambda$. Take $\delta' = \min\left\{1, \frac{\lambda(1-\lambda')}{\lambda'(1-\lambda)}\right\}\delta$.

Hence $0 < \delta' \leq \delta$ and

$$\begin{aligned}
M(\lambda'u + (1-\lambda')v) &= M\left[\frac{1-\lambda'}{1-\lambda}(\lambda u + (1-\lambda)v) + \frac{\lambda'-\lambda}{1-\lambda}u\right] \\
&\leq \frac{1-\lambda'}{1-\lambda}M(\lambda u + (1-\lambda)v) + \frac{\lambda'-\lambda}{1-\lambda}M(u) \\
&\leq \frac{1-\lambda'}{1-\lambda}(1-\delta)[\lambda M(u) + (1-\lambda)M(v)] + \frac{\lambda'-\lambda}{1-\lambda}M(u) \\
&= \lambda'M(u) - \frac{1-\lambda'}{1-\lambda}\lambda\delta M(u) + (1-\delta)(1-\lambda')M(v) \\
&= \left(1 - \frac{\lambda(1-\lambda')}{\lambda'(1-\lambda)}\delta\right)\lambda'M(u) + (1-\delta)(1-\lambda')M(v) \\
&\leq (1-\delta')\lambda'M(u) + (1-\delta')(1-\lambda')M(v) \\
&= (1-\delta')(\lambda'M(u) + (1-\lambda')M(v)). \quad \blacksquare
\end{aligned}$$

Remark. Notice that for fixed $\lambda, \delta, \delta'(\lambda') = \min\left\{1, \frac{\lambda(1-\lambda')}{\lambda'(1-\lambda)}\right\}\delta$ is continuous over the interval $(0, 1)$. We deduce that for any $[\alpha, \beta] \subset (0, 1)$, in Lemma 1, there is a common δ'_0 such that for all $\lambda' \in [\alpha, \beta]$, and $u, v > 0, u \neq v$,

$$M(\lambda'u + (1-\lambda')v) \leq (1-\delta'_0)(\lambda'M(u) + (1-\lambda')M(v)).$$

Lemma 2. For $x \in L_M$, if for some $k > 0$, $R_N(p(kx)) = \int_G N(p(kx(t))) d\mu \leq 1$ and for all $\lambda > 1$, $R_N(p(\lambda kx)) > 1$, then

$$\|x\| = \frac{1}{k} \left(1 + R_M(kx)\right). \quad (cf[3])$$

Lemma 3. For $x \in L_M$, there is $k > 0$ satisfying

$$\|x\| = \frac{1}{k} \left(1 + R_M(kx) \right). \quad (\text{cf}[3])$$

Lemma 4. If $M \in SC$, $\|x_n\| = \frac{1}{k_n}(1 + R_M(k_n x_n))$ ($n = 0, 1, 2, \dots$) with bounded $\{k_n\}_{n=0}^\infty$ and $\|x_n + x_0\| \rightarrow 2$, then

$$k_n x_n - k_0 x_0 \xrightarrow{\mu} 0. \quad (\text{cf}[3])$$

Lemma 5. Under the same assumption as in Lemma 4, let $y_n \in L_N$, $R_N(y_n) \leq 1$ with $\int_G (x_n(t) + x_0(t))y_n(t) d\mu \rightarrow 2$. Then for every $e_n \subset G$,

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{e_n} [k_n x_n(t)y_n(t) - M(k_n x_n(t)) - N(y_n(t))] d\mu &= 0, \\ \lim_{n \rightarrow \infty} \int_{e_n} [k_0 x_0(t)y_n(t) - M(k_0 x_0(t)) - N(y_n(t))] d\mu &= 0, \\ \lim_{n \rightarrow \infty} \int_{e_n} (k_n x_n(t) - k_0 x_0(t))y_n(t) d\mu &= \\ &= \lim_{n \rightarrow \infty} \int_{e_n} M(k_n x_n(t)) - M(k_0 x_0(t)) d\mu. \end{aligned}$$

As n tends to ∞ , the above limits hold uniformly with respect to subsets e_n .

Proof. We have the following

$$\begin{aligned} 1 &\leftarrow \frac{1}{k_n} \int_G k_n x_n(t)y_n(t) d\mu \leq \frac{1}{k_n} \left(R_N(y_n) + R_M(k_n x_n) \right) \\ &\leq \frac{1}{k_n} (1 + R_M(k_n x_n)) = \|x_n\| = 1. \end{aligned}$$

Hence

$$\int_G \left[M(k_n x_n(t)) + N(y_n(t)) - k_n x_n(t) y_n(t) \right] d\mu \rightarrow 0.$$

Since the integrand is nonnegative, we immediately get the first and the second identity. The third one is a simple consequence of the others. ■

Lemma 6. *In Orlicz space $L_M(G, \Sigma, \mu)$ endowed with Orlicz norm, the set*

$$A = \left\{ x(t) \in L_M : R_N(p(kx)) = 1 \text{ where } \|x\| = \frac{1}{k}(1 + R_M(kx)) \right\}$$

is dense in L_M .

Proof. It is enough to show that for any $x \in L_M$ with $R_N(p(kx)) > 1$ or < 1 where $\|x\| = \frac{1}{k}(1 + R_M(kx))$, and for any $\varepsilon > 0$, there is $x' \in A$, such that $\|x - x'\| < \varepsilon$ and $R_N(p(kx')) = 1$.

Let $R_N(p(kx)) > 1$.

Notice that for any $\varepsilon > 0$, $R_N(p((1 - \varepsilon)kx)) \leq 1$. When $R_N(p((1 - \varepsilon)kx)) = 1$, set $x'(t) = (1 - \varepsilon)x(t)$. Then $R_N(p(kx')) = 1$. Now by Theorem 10.5 in [1], we get that $\|x'\| = \frac{1}{k}(1 + R_M(kx'))$, i.e., $x' \in A$. Clearly $\|x - x'\| \leq \varepsilon$.

When $R_N(p((1 - \varepsilon)kx)) < 1$, since (G, Σ, μ) is nonatomic, there is $G' \subset G$

$$\int_{G'} N(p((1 - \varepsilon)kx(t))) d\mu + \int_{G \setminus G'} N(p(kx(t))) d\mu = 1$$

Setting $x'(t) = (1 - \varepsilon)x(t)\chi_{G'}(t) + x(t)\chi_{G \setminus G'}(t)$, we get $R_N(p(kx')) = 1$. Also by [1], $\|x'\| = \frac{1}{k}(1 + R_M(kx'))$. Clearly $\|x - x'\| \leq \varepsilon$.

The argument is analogous for the case $R_N(p(kx)) < 1$. ■

Theorem. *Endowed with the Orlicz norm, Orlicz space $L_M(G, \Sigma, \mu)$ is LW^*UR if and only if*

$$(i) M \in SC,$$

$$(ii) M \in \nabla_2,$$

(iii) for any $\varepsilon > 0$, there exist $\delta, a > 0$ such that for u, v satisfying $\varepsilon^2 \leq \varepsilon u \leq v < u$, $M(u) \geq \varepsilon up(u)$, and $M\left(\frac{u+v}{2}\right) > (1-\delta) \frac{M(u)+M(v)}{2}$, we have

$$p((1-\varepsilon)u) \leq ap((1-\delta)v).$$

Proof. Sufficiency. Suppose $\|x_n\| = \frac{1}{k_n}(1+R_M(k_n x_n)) = 1$ ($n = 0, 1, 2, \dots$) and $\|x_n + x_0\| \rightarrow 2$. By Lemma 6, assume $R_N(p(k_n x_n)) = 1$ ($n = 1, 2, \dots$).

In view of $M \in \nabla_2$, we know from [3] that $\{k_n\}_{n=1}^\infty$ is bounded. Denote $\bar{k} = \sup_n k_n$. In the following we shall show that $x_n \xrightarrow{w^*} x_0$, i.e., any subsequence of $\{x_n\}_{n=1}^\infty$ has its subsequence w^* -convergent to x_0 . So we can assume that $k_n \rightarrow k$. On the other hand, by Lemma 4, it yields $k_n x_n - k_0 x_0 \xrightarrow{\mu} 0$. Therefore by Theorem 14.6 in [1], $k_n x_n - k_0 x_0 \xrightarrow{E_N} 0$.

At first we claim that $k \geq k_0$. Indeed, for any $\eta > 0$, take $y \in E_N$, $R_N(y) \leq 1$ such that $\int_G x_0(t)y(t) d\mu > \|x_0\| - \eta = 1 - \eta$. Since $\int_G k_n x_n y d\mu \rightarrow \int_G k_0 x_0 y d\mu$, we get that for n large enough $\int_G k_n x_n y d\mu > \int_G k_0 x_0 y d\mu - \eta > k_0(1-\eta) - \eta$. So $k \leftarrow k_n = \|k_n x_n\| \geq \int_G k_n x_n y d\mu > \int_G k_0 x_0 y d\mu - \eta > k_0(1-\eta) - \eta$.

Now we only need to show that $k = k_0$, so $x_n - x_0 \xrightarrow{\mu} 0$. Then by Theorem 14.6 in [1], we get that $x_n - x_0 \xrightarrow{E_N} 0$ i.e., $x_n - x_0 \xrightarrow{w^*} 0$.

Take $y_n \in E_N$, $R_N(y_n) \leq 1$ satisfying $\int_G (x_n(t) + x_0(t))y_n(t) d\mu \rightarrow 2$. Then $\int_G x_n(t)y_n(t) d\mu \rightarrow 1$, and $\int_G x_0(t)y_n(t) d\mu \rightarrow 1$. Therefore we have

$$k - k_0 = \lim_{n \rightarrow \infty} \int_G (k_n x_n(t) - k_0 x_0(t))y_n(t) d\mu \quad (1)$$

Let $\varepsilon > 0$ be arbitrary. By $M \in \nabla_2$, there exists $\varepsilon \geq \eta'(\varepsilon) > 0$ (cf[6,3]) such that for all $|u| \geq \varepsilon$, and for all λ , $\frac{1}{1+k} \leq \lambda \leq \frac{2\bar{k}+k_0}{2(k_0+k)}$, it holds

$$M(\lambda u) \leq (1-\eta')\lambda M(u) \quad (2)$$

Denote $m = 1 + \bar{k}$. For $\eta = \frac{\eta'}{m}$, by (iii), there exist $\delta, a > 0$ such that for $u, v, 0 < \eta^2 \leq \eta u \leq v < u$, if $M(u) \geq \eta u p(u)$, and $M(\frac{u+v}{2}) > (1 - \delta) \frac{M(u)+M(v)}{2}$ then

$$p((1 - \eta)u) \leq ap((1 - \delta)v) \tag{3}$$

For such δ and $[\alpha, \beta] = [\frac{1}{1+\bar{k}}, \frac{\bar{k}}{1+\bar{k}}]$, by the remark after Lemma 1, it follows that there exists δ' such that if $M(\frac{u+v}{2}) \leq (1 - \delta) \frac{M(u)+M(v)}{2}$, and $\lambda' \in [\frac{1}{1+\bar{k}}, \frac{\bar{k}}{1+\bar{k}}]$, then

$$M(\lambda'u + (1 - \lambda')v) \leq (1 - \delta')(\lambda'M(u) + (1 - \lambda')M(v)). \tag{4}$$

Since $\int_G |k_0 x_0(t)| p((1 - \delta)k_0 x_0(t)) d\mu \leq R_M(k_0 x_0) + R_N(p((1 - \delta)k_0 x_0)) \leq k_0$ we can find $G_0 \subset G$ such that $\mu(G \setminus G_0)$ is small enough to get the following

$$\int_{G \setminus G_0} |k_0 x_0(t)| p((1 - \delta)k_0 x_0(t)) d\mu < \frac{\eta \varepsilon}{a}$$

$$\int_{G \setminus G_0} M(k_0 x_0(t)) d\mu < \varepsilon \tag{5}$$

and

$$k_n x_n(t) - k_0 x_0(t) \xrightarrow{U} 0$$

uniformly over G_0 .

For each n , we split $G \setminus G_0$ into the five parts:

$$A_n = \{t \in G \setminus G_0 : |k_n x_n(t)| < |k_0 x_0(t)|\}$$

$$B_n = \{t \in G \setminus G_0 \setminus A_n : \max(|k_n x_n(t)|, |k_0 x_0(t)|) < \varepsilon\}$$

$$C_n = \{t \in G \setminus G_0 \setminus A_n \setminus B_n : M(k_n x_n(t)) < \eta |k_n x_n(t)| p(|k_n x_n(t)|)\},$$

$$D_n = \left\{ t \in G \setminus G_0 \setminus A_n \setminus B_n \setminus C_n : |k_0 x_0(t)| < \eta |k_n x_n(t)|; \text{ or } x_n(t)x_0(t) < 0; \right. \\ \left. \text{or } M\left(\frac{k_n x_n(t) + k_0 x_0(t)}{2}\right) \leq (1 - \delta) \frac{M(k_n x_n(t)) + M(k_0 x_0(t))}{2} \right\},$$

$$E_n = G \setminus G_0 \setminus A_n \setminus B_n \setminus C_n \setminus D_n \\ = \left\{ t \in G \setminus G_0 : x_n(t)x_0(t) \geq 0; \eta \varepsilon \leq \eta |k_n x_n(t)| \leq |k_0 x_0(t)| \leq |k_n x_n(t)| \right.$$

$$M(k_n x_n(t)) \geq \eta k_n |x_n(t)| p(k_n |x_n(t)|); \text{ and}$$

$$\left. M\left(\frac{k_n x_n(t) + k_0 x_0(t)}{2}\right) \geq (1 - \delta) \frac{M(k_n x_n(t)) + M(k_0 x_0(t))}{2} \right\}.$$

In the following, one by one, we estimate the integrals of the integrand $(k_n x_n - k_0 x_0)y_n$ over G_0 , A_n , B_n , C_n , D_n , and E_n .

From (6), for n large enough

$$\left| \int_{G_0} (k_n x_n - k_0 x_0)y_n \, d\mu \right| < \varepsilon \|y_n\|_{(N)} \quad (7)$$

From the structure of A_n , by Lemma 5, it follows that for n large enough

$$\int_{A_n} (k_n x_n - k_0 x_0)y_n \, d\mu \leq \int_{A_n} (M(k_n x_n(t)) - M(k_0 x_0(t))) \, d\mu + \varepsilon \leq \varepsilon. \quad (8)$$

From the structure of B_n ,

$$\left| \int_{B_n} (k_n x_n - k_0 x_0)y_n \, d\mu \right| \leq 2\varepsilon \|y_n\|_{(N)} \leq 2\varepsilon. \quad (9)$$

Since $\|x_n\| = 1$, $R_M(x_n) \leq 1$. From $R_N(p(k_n x_n)) = 1$, it yields that for n large enough

$$\begin{aligned} \int_{C_n} (k_n x_n - k_0 x_0) y_n \, d\mu &\leq \int_{C_n} \left(M(k_n x_n(t)) - M(k_0 x_0(t)) \right) d\mu + \varepsilon \\ &\leq \int_{C_n} M(k_n x_n(t)) \, d\mu + 2\varepsilon \leq \eta \int_{C_n} |k_n x_n(t)| p(k_n x_n(t)) \, d\mu + 2\varepsilon \\ &\leq \eta \bar{k} \left(\int_G M(x_n(t)) \, d\mu + \int_G N(p(k_n x_n(t))) \, d\mu \right) + 2\varepsilon \\ &\leq 2\bar{k}\eta + 2\varepsilon \leq 2(1 + \bar{k})\varepsilon. \end{aligned}$$

When $t \in D_n$, $|k_0 x_0(t)| < \eta |k_n x_n(t)|$, since $t \notin A_n \cup B_n$. So $|k_n x_n(t)| > \varepsilon$, and from (2) it follows

$$\begin{aligned} M\left(\frac{k_n k_0}{k_n + k_0} (x_n(t) + x_0(t))\right) &\leq M\left(\frac{k_0 + \eta k_n}{k_n + k_0} k_n x_n(t)\right) \\ &\leq (1 - m\eta) \frac{k_0 + \eta k_n}{k_n + k_0} M(k_n x_n(t)) = (1 - m\eta) \frac{k_0 + \eta k_n}{k_0} \frac{k_0}{k_n + k_0} M(k_n x_n(t)) \\ &\leq (1 - \eta) \frac{k_0}{k_n + k_0} M(k_n x_n(t)) \\ &\leq (1 - \eta) \left[\frac{k_0}{k_n + k_0} M(k_n x_n(t)) + \frac{k_n}{k_n + k_0} M(k_0 x_0(t)) \right] \quad (*) \end{aligned}$$

When $t \in D_n$, $x_n(t)x_0(t) < 0$, since $t \notin A_n \cup B_n$. While $|x_n(t)| \geq |x_0(t)|$,

$$\begin{aligned} M\left(\frac{k_n k_0}{k_n + k_0} (x_n(t) + x_0(t))\right) &\leq M\left(\frac{k_n k_0}{k_n + k_0} x_n(t)\right) \\ &\leq (1 - \eta') \frac{k_0}{k_n + k_0} M(k_n x_n(t)) \\ &\leq (1 - \eta') \left[\frac{k_0}{k_n + k_0} M(k_n x_n(t)) + \frac{k_n}{k_n + k_0} M(k_0 x_0(t)) \right] \quad (**) \end{aligned}$$

While $|x_n(t)| < |x_0(t)|$,

$$\begin{aligned} M\left(\frac{k_n k_0}{k_n + k_0} (x_n(t) + x_0(t))\right) &\leq M\left(\frac{k_n k_0}{k_n + k_0} x_0(t)\right) \\ &\leq \frac{k_n}{k_n + k_0} M(k_0 x_0(t)) \end{aligned}$$

$$\leq \left(1 - \frac{k_0}{k_n + k_0}\right) \left[\frac{k_0}{k_n + k_0} M(k_n x_n(t)) + \frac{k_n}{k_n + k_0} M(k_0 x_0(t)) \right]. \quad (***)$$

Taking $\delta'' = \min(\delta', \eta, \frac{1}{1+k})$, and applying (4), we have
 $0 \leftarrow 2 - \|x_n + x_0\|$

$$\begin{aligned} &\geq \frac{1}{k_n} (1 + R_M(k_n x_n)) + \frac{1}{k_0} (1 + R_M(k_0 x_0)) - \\ &\quad - \frac{k_n + k_0}{k_n k_0} (1 + R_M(\frac{k_n k_0}{k_n + k_0} (x_n + x_0))) \\ &= \frac{k_n + k_0}{k_n k_0} \int_G \left[\frac{k_0}{k_n + k_0} M(k_n x_n(t)) + \frac{k_n}{k_n + k_0} M(k_0 x_0(t)) - \right. \\ &\quad \left. - M\left(\frac{k_n k_0}{k_n + k_0} (x_n + x_0)(t)\right) \right] d\mu \\ &\geq \frac{k_n + k_0}{k_n k_0} \int_{D_n} \left[\frac{k_0}{k_n + k_0} M(k_n x_n(t)) + \frac{k_n}{k_n + k_0} M(k_0 x_0(t)) - \right. \\ &\quad \left. - M\left(\frac{k_n k_0}{k_n + k_0} (x_n + x_0)(t)\right) \right] d\mu \\ &\geq \frac{k_n + k_0}{k_n k_0} \int_{D_n} \delta'' \left[\frac{k_0}{k_n + k_0} M(k_n x_n(t)) + \frac{k_n}{k_n + k_0} M(k_0 x_0(t)) \right] d\mu \\ &\geq \frac{\delta''}{k} \int_{D_n} (M(k_n x_n(t)) + M(k_0 x_0(t))) d\mu. \end{aligned}$$

Obviously, for n large enough

$$\int_{D_n} (k_n x_n - k_0 x_0) y_n d\mu \leq \int_{D_n} M(k_n x_n(t)) - M(k_0 x_0(t)) d\mu + \varepsilon \leq 2\varepsilon. \quad (11)$$

When $t \in E_n$, then $|\eta k_n x_n(t)| \leq |k_0 x_0(t)| \leq |k_n x_n(t)|$,
and $M(k_n x_n(t)) \geq \eta k_n |x_n(t)| p(k_n |x_n(t)|)$, and

$$M\left(\frac{k_n x_n(t) + k_0 x_0(t)}{2}\right) > (1 - \delta) \frac{M(k_n x_n(t)) + M(k_0 x_0(t))}{2}.$$

Hence

$$p((1 - \eta)|k_n x_n(t)|) \leq \alpha p((1 - \delta)|k_0 x_0(t)|).$$

Moreover, from Lemma 4 and condition (5) we get for n large enough that

$$\begin{aligned}
 & \int_{E_n} (k_n x_n - k_0 x_0) y_n \, d\mu \\
 &= \eta \int_{E_n} k_n x_n y_n \, d\mu + \int_{E_n} (1 - \eta) k_n x_n y_n \, d\mu - \int_{E_n} k_0 x_0 y_n \, d\mu \\
 &\leq \eta \bar{k} + \int_{E_n} M((1 - \eta) k_n x_n(t)) \, d\mu + \int_{E_n} N(y_n(t)) \, d\mu - \int_{E_n} M(k_0 x_0(t)) \, d\mu \\
 &\quad - \int_{E_n} N(y_n(t)) \, d\mu + \varepsilon \\
 &\leq \eta \bar{k} + \varepsilon + \int_{E_n} (1 - \eta) k_n |x_n(t)| p((1 - \eta) k_n |x_n(t)|) \, d\mu \\
 &\leq \eta \bar{k} + \varepsilon + (1 - \eta) \frac{\alpha}{\eta} \int_{E_n} k_0 |x_0(t)| p((1 - \delta) |k_0 x_0(t)|) \, d\mu \\
 &\leq \eta \bar{k} + \varepsilon + \frac{\alpha \eta \varepsilon}{\eta \alpha} = \eta \bar{k} + 2\varepsilon \leq (2 + \bar{k})\varepsilon.
 \end{aligned}$$

Combining (7) - (12), and (1), we deduce that

$$0 \leq k - k_0 < 0(\varepsilon),$$

where $0(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Hence $k = k_0$, which completes the proof of the sufficiency.

Necessity.

$LW^*UR \Rightarrow M \in SC$. Since $LW^*UR \Rightarrow R$, it follows (i), by [5].

$LW^*UR \Rightarrow M \in \nabla_2$. Indeed, if we suppose that it is not true, then there exist $u_n \nearrow \infty$, satisfying $\frac{u_n p(u_n)}{N(p(u_n))} > 2^n$ ($n = 1, 2, \dots$) (cf [6,3]). Choose $c > 0$, $G_0 \subset G$, with $\mu(G \setminus G_0) > 0$ and $N(p(c))\mu G_0 = 1$. Moreover, choose $G_n \subset G \setminus G_0$, with $u_n p(u_n)\mu G_n = 1$. Hence $N(p(u_n))\mu G_n < \frac{1}{2^n}$. Then take $T_n \subset G_0$, such that $N(p(c))\mu T_n + N(p(u_n))\mu G_n = 1$. Hence $\mu T_n \rightarrow \mu G_0$. Now set

$$k_0 = cp(c)\mu G_0; \quad k_n = cp(c)\mu T_n + u_n p(u_n)\mu G_n. \quad (n = 1, 2, \dots)$$

Obviously, $k_n \rightarrow k_0 + 1$. Define

$$x_0(t) = \frac{c}{k_0} \chi_{G_0}(t); \quad x_n(t) = \frac{1}{k_n} (c \chi_{T_n}(t) + u_n \chi_{G_n}(t)). \quad (n = 1, 2, \dots)$$

Since $R_N(p(k_0x_0)) = R_N(p(k_nx_n)) = 1$, by Theorem 10.5 of [1], it follows that

$$\|x_0\| = \frac{1}{k_0} cp(c)\mu G_0 = 1;$$

$$\|x_n\| = \frac{1}{k_n} (cp(c)\mu T_n + u_n p(u_n)\mu G_n) = 1 \quad (n = 1, 2, \dots)$$

and

$$\|x_n + x_0\| \geq \left(\frac{1}{k_n} + \frac{1}{k_0}\right) cp(c)\mu T_n + \frac{1}{k_n} (u_n p(u_n)\mu G_n) \rightarrow 2.$$

But on the other hand, we have

$$x_0 - x_n = \left(\frac{1}{k_0} - \frac{1}{k_n}\right) c\chi_{T_n}(t) + \frac{c}{k_0} \chi_{G_0 \setminus T_n}(t) - \frac{u_n}{k_n} \chi_{G_n}(t)$$

Since $\mu(G_0 \setminus T_n) \rightarrow 0$, $\mu G_n \rightarrow 0$, $T_n \nearrow G_0$, by Theorem 14.6 in [1], we derive that

$$x_0 - x_n \xrightarrow{w^*} \left(\frac{1}{k_0} - \frac{1}{1+k_0}\right) c\chi_{G_0} \neq 0.$$

This contradicts to the fact that L_M is LW^*UR , which show that $M \in \nabla_2$.

$LW^*UR \Rightarrow$ (iii). Otherwise, suppose there exist $\varepsilon > 0$, $u_n, v_n \nearrow \infty$ such that $\varepsilon^2 \leq \varepsilon u_n \leq v_n < u_n$, $M(u_n) \geq \varepsilon u_n p(u_n)$,

$$M\left(\frac{u_n + v_n}{2}\right) > \left(1 - \frac{1}{n}\right) \frac{M(u_n) + M(v_n)}{2}, \text{ and}$$

$$p((1 - \varepsilon)u_n) > 2^n p\left(\left(1 - \frac{1}{n}\right)v_n\right).$$

In view of the continuity of $M(u)$, we select Θ_n , $0 < \Theta_n < 1$ with $\Theta_n \nearrow 1$ and

$$M\left(\frac{u_n + \Theta_n v_n}{2}\right) \geq \left(1 - \frac{1}{n}\right) \frac{M(u_n) + M(\Theta_n v_n)}{2}. \quad (13)$$

We first construct two sequences $\{w_n\}_{n=1}^\infty$ and $\{\tau_n\}_{n=1}^\infty$ satisfying

$$\tau_n \searrow 1, \Theta_n v_n \leq w_n \leq (1 - \frac{\varepsilon}{3})u_n, \text{ and } p(\tau_n w_n) > 2^n p(w_n). \quad (14)$$

Since $1 \leq \frac{u_n}{v_n} \leq \frac{1}{\varepsilon}$, without loss of generality, if necessary we can pass to a subsequence, we assume that $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = b \geq 1$. Denote

$$\xi = \sup\{\xi' > 0 : \overline{\lim}_{n \rightarrow \infty} \frac{p((1 - \frac{\varepsilon}{2})bv_n)}{p(\xi'v_n)} = \infty\}.$$

Obviously, $1 \leq \xi \leq (1 - \frac{\varepsilon}{2})b$. In the following we discuss two cases.

$$(I) \text{ Let } \overline{\lim}_{n \rightarrow \infty} \frac{p((1 - \frac{\varepsilon}{2})bv_n)}{p(\xi v_n)} = \infty.$$

For any $\lambda > 1$,

$$\infty \leftarrow \frac{p((1 - \frac{\varepsilon}{2})bv_n)}{p(\xi v_n)} = \frac{p((1 - \frac{\varepsilon}{2})bv_n)}{p(\lambda \xi v_n)} \cdot \frac{p(\lambda \xi v_n)}{p(\xi v_n)}$$

Since on the right side of the identity the first quotient formula is bounded, $\overline{\lim}_{n \rightarrow \infty} \frac{p(\lambda \xi v_n)}{p(\xi v_n)} = \infty$. Passing to a subsequence if necessary, we assume that $p((1 + \frac{1}{n})\xi v_n) > 2^n p(\xi v_n)$. Easily we know that for n large enough, $v_n \leq \xi v_n \leq (1 - \frac{\varepsilon}{2})bv_n \leq (1 - \frac{\varepsilon}{3})u_n$. For $w_n = \xi v_n$, $\tau_n = 1 + \frac{1}{n}$, condition (14) is satisfied.

$$(II) \text{ Let } \overline{\lim}_{n \rightarrow \infty} \frac{p((1 - \frac{\varepsilon}{2})bv_n)}{p(\xi v_n)} < \infty.$$

For any $\Theta_n < 1$,

$$\infty \leftarrow \frac{p((1 - \frac{\varepsilon}{2})bv_i)}{p(\Theta_n \xi v_i)} = \frac{p((1 - \frac{\varepsilon}{2})bv_i)}{p(\xi v_i)} \cdot \frac{p(\xi v_i)}{p(\Theta_n \xi v_i)} \text{ as } i \rightarrow \infty.$$

Hence $\overline{\lim}_{i \rightarrow \infty} \frac{p(\xi v_i)}{p(\Theta_n v_i \xi)} = \infty$. Passing to a subsequence if necessary, we get $p(\xi v_n) > 2^n p(\Theta_n \xi v_n)$. Obviously $\Theta_n v_n \leq \Theta_n \xi v_n \leq \xi v_n \leq (1 - \frac{\varepsilon}{2}) b v_n < (1 - \frac{\varepsilon}{3}) u_n$. If we take $w_n = \Theta_n \xi v_n$, and $\tau_n = \frac{1}{\Theta_n}$, then (14) is satisfied.

By (14), we can choose disjoint subsets $G_n \subset G$, $G_n \cap G_m = \emptyset$ ($n \neq m$) such that

$$N(p(w_n))\mu G_n = \frac{1}{2^{n+1}} \quad (n = 1, 2, \dots)$$

For n large enough,

$$N(p(u_n))\mu G_n > N(p(\tau_n w_n))\mu G_n > 2^n N(p(w_n))\mu G_n = \frac{1}{2}.$$

Pick out $\bar{G}_n \subset G_n$ satisfying

$$N(p(u_n))\mu \bar{G}_n = \frac{1}{2} \quad (n = 1, 2, \dots)$$

Now set

$$k_0 = 1 + \sum_{i=1}^{\infty} M(w_i)\mu G_i,$$

$$k_n = 1 + \sum_{i \neq n}^{\infty} M(w_i)\mu G_i + M(u_n)\mu \bar{G}_n \quad (n = 1, 2, \dots)$$

By $M \in \nabla_2$, it yields that $\frac{u p(u)}{N(p(u))} \leq d$ ($u \geq u_0$) (cf [6, 3]). Then

$$\sum_{i=1}^{\infty} M(w_i)\mu G_i \leq \sum_{i=1}^{\infty} w_i p(w_i)\mu G_i \leq d \sum_{i=1}^{\infty} N(p(w_i))\mu G_i = \frac{d}{2},$$

$$M(u_n)\mu \bar{G}_n \leq u_n p(u_n)\mu \bar{G}_n \leq d N(p(u_n))\mu \bar{G}_n = \frac{d}{2}.$$

So $\{k_n\}_{n=1}^{\infty}$ is bounded. Passing to a subsequence if necessary, we assume that $k_n \rightarrow k$. From

$$M(u_n)\mu\bar{G}_n \geq \varepsilon u_n p(u_n)\mu\bar{G}_n \geq \varepsilon N(p(u_n))\mu\bar{G}_n = \frac{\varepsilon}{2}$$

we get that $k - k_0 \geq \frac{\varepsilon}{2}$. Define

$$x_0(t) = \frac{1}{k_0} \sum_{i=1}^{\infty} w_i \chi_{G_i}(t);$$

$$x_n(t) = \frac{1}{k_n} \left(\sum_{i \neq n} w_i \chi_{G_i}(t) + u_n \chi_{\bar{G}_n}(t) \right) \quad (n = 1, 2, \dots)$$

We have

$$\int_G N(p(k_n x_n(t))) d\mu = \sum_{i \neq n} N(p(w_i))\mu G_i + N(p(u_n))\mu\bar{G}_n < 1.$$

In addition, for any $\lambda > 1$, take $i_0 > n$ such that $\lambda > \tau_{i_0}$. Then

$$\begin{aligned} \int_G N(p(\lambda k_n x_n(t))) d\mu &= \sum_{i \neq n} N(p(\lambda w_i))\mu G_i + N(p(\lambda u_n))\mu\bar{G}_n \\ &> \sum_{i=i_0}^{\infty} N(p(\tau_i w_i))\mu G_i > \sum_{i=i_0}^{\infty} 2^i N(p(w_i))\mu G_i = \infty. \end{aligned}$$

By Lemma 2, it follows that

$$\begin{aligned} \|x_n\| &= \frac{1}{k_n} (1 + R_M(k_n x_n)) = \frac{1}{k_n} \left(1 + \sum_{i \neq n} M(w_i)\mu G_i + M(u_n)\mu\bar{G}_n \right) = \\ &= 1. \quad (n = 1, 2, \dots) \end{aligned}$$

Similarly,

$$\|x_0\| = \frac{1}{k_0} (1 + R_M(k_0 x_0)) = \frac{1}{k_0} (1 + \sum_{i=1}^{\infty} M(w_i) \mu G_i) = 1.$$

Since

$$\frac{k_n k_0}{k_n + k_0} (x_n(t) + x_0(t)) = \begin{cases} w_i & t \in G \quad (i \neq n) \\ \frac{k_n}{k_n + k_0} w_n & t \in G_n \setminus \bar{G}_n \\ \dots & \dots \\ \frac{k_n}{k_n + k_0} w_n + \frac{k_0}{k_n + k_0} u_n & t \in \bar{G}_n \\ 0 & \text{otherwise} \end{cases}$$

we derive that

$$\begin{aligned} \int_G N(p \frac{k_n k_0}{k_n + k_0} (x_n + x_0)) d\mu &< \sum_{i \neq n} N(p(w_i)) \mu G_i + \\ &+ N(p(w_n)) \mu (G_n \setminus \bar{G}_n) + N(p(u_n)) \mu \bar{G}_n \leq 1 \end{aligned}$$

But for any $\lambda > 1$,

$$\int_G N(p(\lambda \frac{k_n k_0}{k_n + k_0} (x_n + x_0))) d\mu > \sum_{i \neq n} N(p(\lambda w_i)) \mu G_i = \infty.$$

Hence, by Lemma 2, it yields that

$$\|x_n + x_0\| = \frac{k_n + k_0}{k_n k_0} (1 + R_M(\frac{k_n k_0}{k_n + k_0} (x_n + x_0))).$$

From (13), we get

$$M(\frac{u_n + w_n}{2}) \geq (1 - \frac{1}{n}) \frac{M(u_n) + M(w_n)}{2}$$

By the remark after Lemma 1, we deduce that there exist $\delta_n \searrow 0$ with

$$M\left(\frac{k_n w_n + k_0 u_n}{k_n + k_0}\right) \geq (1 - \delta_n) \left(\frac{k_n}{k_n + k_0} M(w_n) + \frac{k_0}{k_n + k_0} M(u_n)\right).$$

Therefore,

$$\begin{aligned} \|x_n + x_0\| &= \frac{k_n + k_0}{k_n k_0} \left[1 + \sum_{i \neq n} M(w_i) \mu G_i + M\left(\frac{k_n}{k_n + k_0} w_n\right) \mu(G_n \setminus \bar{G}_n) + \right. \\ &\quad \left. + M\left(\frac{k_n w_n + k_0 u_n}{k_n + k_0}\right) \mu \bar{G}_n \right] \\ &> \frac{k_n + k_0}{k_n k_0} \left[1 + \sum_{i \neq n} M(w_i) \mu G_i + (1 - \delta_n) \left(\frac{k_n}{k_n + k_0} M(w_n) + \right. \right. \\ &\quad \left. \left. + \frac{k_0}{k_n + k_0} M(u_n)\right) \mu \bar{G}_n \right] \\ &> (1 - \delta_n) \left[\frac{1}{k_0} \left(1 + \sum_{i \neq n} M(w_i) \mu G_i\right) + \right. \\ &\quad \left. + \frac{1}{k_n} \left(1 + \sum_{i \neq n} M(w_i) \mu G_i + M(u_n) \mu \bar{G}_n\right) \right] \\ &\rightarrow 2 \end{aligned}$$

Since $\mu G_n \rightarrow 0$, we have that

$$x_0(t) - x_n(t) \xrightarrow{w^*} \left(\frac{1}{k_0} - \frac{1}{k}\right) \sum_{i=1}^{\infty} w_i \chi_{G_i}(t)$$

which contradicts with the fact L_M is LW^*UR . ■

Finally we give an example of an N -function M that satisfies (i) and (ii), but not (iii). So L_N is separable and L_M is rotund, but not LW^*UR .

Let

$$p(t) = \begin{cases} t & 0 \leq t < 1 \\ (k+1)^{k+1} + \frac{t-2^k}{2^{2k}} & 2^k \leq t < 2^{k+1} \end{cases} \quad (k = 0, 1, 2, \dots)$$

and

$$M(u) = \int_0^{|u|} p(t) dt.$$

Then $M \in SC$, since $p(t)$ is strictly increasing on the whole axis. And $M \in \nabla_2$. Indeed, from $q(s) = \sup_{p(t) \leq s} t$ (cf § 2 of [1]), it yields that

$$g(s) = \begin{cases} s & 0 \leq s < 1 \\ 2^k & k^k + \frac{1}{2^{k-1}} \leq s \leq (k+1)^{k+1} \\ \text{linear} & (k+1)^{k+1} \leq s \leq (k+1)^{k+1} + \frac{1}{2^k} \\ 2^{k+1} & (k+1)^{k+1} + \frac{1}{2^k} \leq s \leq (k+2)^{k+2} \end{cases} \quad (k = 1, 2, \dots)$$

For any s , there is k such that $k^k + \frac{1}{2^{k-1}} < s \leq (k+1)^{k+1} + \frac{1}{2^k}$, so $2s \leq (k+2)^{k+2}$. Hence $\frac{q(2s)}{q(s)} \leq \frac{2^{k+1}}{2^k} = 2$. By the Young inequality, it yields that $N(2s) \leq 2sq(2s) \leq 4sq(s) \leq 8sq(\frac{s}{2}) = 16\frac{s}{2}q(\frac{s}{2}) \leq 16N(s)$, i.e., $M \in \nabla_2$.

But M does not satisfy (iii). In fact for $v_k = 2^k$, $u_k = 2^{k+1}$ ($k = 1, 2, \dots$), we have

$$1) u_k = 2v_k > v_k = \frac{u_k}{2} \geq 2 > \left(\frac{1}{2}\right)^2.$$

$$2) \frac{M(u_k)}{u_k p(u_k)} \geq \frac{p(v_k)(u_k - v_k)}{u_k p(u_k)} = \frac{p(2^k)(2^{k+1} - 2^k)}{p(2_-^{k+1})2^{k+1}} = \frac{(k+1)^{k+1} \frac{1}{2}}{(k+1)^{k+1} + \frac{2^{k+1} - 2^k}{2^{2k}}} \rightarrow \frac{1}{2}$$

(where $2_-^{k+1} = 2^{k+1} - 0$).

3) $\frac{M(\frac{u_k+v_k}{2})}{M(\frac{u_k+v_k}{2})} \rightarrow 1$. Indeed,

$$\begin{aligned} M(u_k) + M(v_k) - 2M\left(\frac{u_k+v_k}{2}\right) &= \int_{\frac{u_k+v_k}{2}}^{u_k} p(t) dt - \int_{v_k}^{\frac{u_k+v_k}{2}} p(t) dt \\ &= \int_{\frac{u_k+v_k}{2}}^{u_k} p(t) - p\left(t - \frac{u_k-v_k}{2}\right) dt \leq \frac{u_k-v_k}{2} (p(u_k) - p(v_k)) \\ &= \frac{2^{k+1} - 2^k}{2} \left[(k+1)^{k+1} + \frac{2^{k+1} - 2^k}{2^{2k}} - (k+1)^{k+1} \right] = \frac{2^k}{2} \frac{2^k}{2^{2k}} = \frac{1}{2}. \end{aligned}$$

Since $M\left(\frac{u_k+v_k}{2}\right) \rightarrow \infty$. It follows that 3) holds.

$$\begin{aligned} 4) p\left(\left(1-\frac{1}{2}\right)u_k\right) &= p\left(\frac{u_k}{2}\right) = p(v_k) = (k+1)^{k+1} > (k+1)\left(k^k + \frac{2^k - 2^{k-1}}{2^{2(k-1)}}\right) \\ &> (k+1)p\left(\left(1-\frac{1}{k}\right)v_k\right) \quad k = 1, 2, \dots \end{aligned}$$

Combining 1) - 4), we see that M does not satisfy (iii).

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