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Metric Projections and Best Approximants in Bochner-Orlicz Spaces

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ABSTRACT. In the first section of this paper there are given criteria for strict convexity and smoothness of the Bochner-Orlicz space with the Orlicz norm as well as the Luxemburg norm. In the second one that geometrical properties are applied to the characterization of metric projections and zero mean valued best approximants to Bochner-Orlicz spaces.

INTRODUCTION.

Problems of finding best approximants are important in approximation theory and probability theory. Best approximants in the Hilbert space L^2 are known as conditional expectations; in the space L^p for $p > 1$, as p-preditors [1]; in the space L^1 as conditional medians [17]; in

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an order closed sublattice of the space L^p as p-means [3] and in Orlicz spaces as Φ-approximants [15]. Some existence problems of best approximants in Modular spaces were considered in [7] and [8]. In this paper, we characterize the best approximant in Bochner-Orlicz spaces. Bochner-Orlicz spaces are the natural generalization of classical Orlicz spaces. Our preliminaries, Section 0, give some basic concepts and facts of the theory. Section 1 is devoted to the characterization of the strict convexity and smoothness of the Bochner-Orlicz space with the Orlicz norm as well as the Luxemburg norm. Results from Section 1 are applied to proofs of main theorems included in Section 2. In Section 2 there is described metric projection $\Pi(u|C)$ of any element $u \notin C$ onto convex subset C of the Bochner-Orlicz space. The last theorem of this paper is a theorem on representation of zero mean valued best approximant in Bochner-Orlicz space.

0. PRELIMINARY DEFINITIONS AND LEMMAS.

Let (T, Σ, μ) be a measure space with atomless, finite measure defined on σ-algebra Σ of subsets of T, $\mathbf{R}$ the set of real numbers, $(X, \|\cdot\|_X)$ a reflexive real Banach space and $(X^*, \|\cdot\|_{X^*})$ be the dual space to the space X. Traditionally, symbol $< x, x^* >$ denotes the value of the functional x^* at the point $x \in X$. By $\mathcal{M}(T, X)$ we denote the linear space of all μ-equivalence classes of strongly measurable functions $u(\cdot) : T \to X$.

A convex and even function $\Phi : \mathbf{R} \to [0, \infty)$ is called an $\mathcal{N}$-function if $\Phi(0) = 0$, $\Phi(u) > 0$ for $u \neq 0$,

$$\lim_{u \to 0} \frac{\Phi(u)}{|u|} = 0 \quad \text{and} \quad \lim_{u \to \infty} \frac{\Phi(u)}{|u|} = \infty.$$

For every $\mathcal{N}$-function Φ we define the complementary function $\Psi : \mathbf{R} \to [0, \infty)$ by the formula

$$\Psi(v) = \sup_{u > 0} \{uv - \Phi(u)\}$$

for every $v \in \mathbf{R}$. The function Ψ is also an $\mathcal{N}$-function (see [10]).

We say that the $\mathcal{N}$-function Φ satisfies the Δ_2-condition (write $\Phi \in \Delta_2$) if there exist constants $K > 1$, $u_0 > 0$ such that

$$\Phi(2u) \leq K\Phi(u) \quad \text{for} \quad u \geq u_0.$$

We say that the $\mathcal{N}$-function Φ satisfies the ∇_2-condition (write $\Phi \in \nabla_2$) if the $\mathcal{N}$-function Ψ complementary to Φ satisfies the Δ_2-condition.

Denote by small letters ϕ and ψ the right-hand derivative of $\mathcal{N}$-functions Φ and Ψ, respectively.

The space

$$L_\Phi(X) = \left\{u(\cdot) \in \mathcal{M}(T,X) : \exists_{k>0} I_\Phi(ku) = \int_T \Phi(k\|u(t)\|_X)\,dt < \infty\right\}$$

equipped with so called *Orlicz norm*

$$\|u\|_\Phi = \inf_{k>0} \frac{1}{k}[1 + I_\Phi(ku)]$$

or with equivalent to it *Luxemburg norm*

$$\|u\|_{(\Phi)} = \inf\{k > 0 : I_\Phi(k^{-1}u) \leq 1\}$$

is said to be a *Bochner-Orlicz space.* Elements of the space $L_\Phi(X)$ will be usually denoted by u instead of $u(\cdot)$ if it does not lead to misunderstanding. Further, if a Bochner-Orlicz space is equipped with the Luxemburg norm, then we will denote such space by $L_{(\Phi)}(X)$. In the case $X = \mathbf{R}$, the spaces are reduced to classical Orlicz spaces (see [11] or [16]) and they are denoted shortly by L_Φ.

Relation between spaces $L_\Phi(X)$ and L_Φ as well as $L_{(\Phi)}(X)$ and $L_{(\Phi)}$ are expressed by the following obvious lemma

Lemma 1. *$u \in L_\Phi(X)$ iff $\|u(\cdot)\|_X \in L_\Phi$. Furthermore,*

$$\|u\|_\Phi = \big\|\|u(\cdot)\|_X\big\|_\Phi \quad \textit{and} \quad \|u\|_{(\Phi)} = \big\|\|u(\cdot)\|_X\big\|_{(\Phi)}$$

for every $u \in L_\Phi(X)$.

Bochner-Orlicz spaces $L_\Psi(X^*)$ and $L_{(\Psi)}(X^*)$ are defined analagously. The next lemma shows some connexions between $L_\Phi(X)$ and $L_\Psi(X^*)$.

Lemma 2.

a) The following Hölder inequalities

$$\left| \int_T < u(t), v(t) > dt \right| \le ||u||_\Phi ||v||_{(\psi)},$$

$$\left| \int_T < u(t), v(t) > dt \right| \le ||u||_{(\Phi)} ||v||_\Psi$$

hold for every $u \in L_\Phi(X)$ *and* $v \in L_\Psi(X^*)$;

b) If $\Phi \in \Delta_2$, *then* $(L_\Phi(X))^* = L_{(\Psi)}(X^*)$ *and* $(L_{(\Phi)}(X))^* = L_\Psi(X^*)$;

c) The space $L_\Phi(X)$ *is reflexive iff* $\Phi \in \Delta_2 \cap \nabla_2$, *i.e.* $\Phi \in \Delta_2$ *and* $\Phi \in \nabla_2$.

The proof of Lemma 2 can be found in the monograph [20].

It is well known that the infimum in the definition of the Orlicz norm is realized for some $k > 0$. That fact is very useful in the theory of classical Orlicz spaces. It is also true in the case of Orlicz-Bochner spaces. More precisely, there holds the following

Lemma 3. *If there exists a* $k_0 > 0$ *such that*

$$\int_T \Psi[\phi(k_0||u(t)||_X)]dt = 1,$$

then

$$||u||_\Phi = \int_T ||u(t)||_X \phi(k_0||u(t)||_X)dt.$$

b) For every $u \in L_\Phi(X) \setminus \{0\}$ *there exists a* $k_0 > 0$ *such that*

$$||u||_\Phi = \frac{1}{k_0}\left\{1 + \int_T \Phi(k_0||u(t)||_X)dt\right\}.$$

Proof. The proof of a) is an inmediate consequence of Theorem 1.25 from [20] and Lemma 1. By Lemma 1, b) follows from Theorem 1.27 in [20].

Definition 1. (cf. [5]). *Let* E *be a Banach space,* E^* *its dual space and* $S(E^*)$ *the unit sphere of* E^*. *A multi-valued mapping* $\lambda_E : E \setminus \{0\} \to S(E^*)$ *defined by the formula*

$$\lambda_E(u) = \{u^* \in S(E^*) : \langle u, u^* \rangle = ||u||,\ u \in E \setminus \{0\}\}$$

is called a support mapping of E.

Definition 2. (cf. [2]). *Let* E *and* E^* *be as in Definition 1. A multi-valued mapping* $F_E : E \to E^*$ *defined by the formula*

$$F_E(u) = \{u^* \in E^* : \langle u, u^* \rangle = ||u||^2 = ||u^*||^2,\ u \in E\}$$

is called a duality mapping of E.

Remark 1. A relation between the support mapping of E and the duality mapping of E is expressed by the following formula

$$F_E(u) = ||u||\Lambda_E(u) \text{ for every } u \in E,$$

where

$$\Lambda_E(u) = \begin{cases} \lambda_E(u) & \text{for } u \in E \setminus \{0\} \\ 0^* & \text{for } u = 0 \end{cases}$$

and 0^* is the zero element in E^*.

It turns out that the properties of F_E are closely related to the geometry of the space E. The following results may be found in [2]

Lemma 4. *Let F_E be a duality mapping of the Banach space E. Then*

a) F_E is surjective iff E is reflexive;

b) F_E is single-valued iff E is smooth;

c) F_E is injective iff E is strictly convex.

Definition 3. *Let C be a convex subset of a Banach space E. The multi-valued mapping $\Pi(C|\cdot): E \to C$ defined for each $u \in E$ by the formula*

$$\Pi(C|u) = \{u_0 \in C : ||u - u_0|| = \inf_{v \in C} ||u - v||\}$$

is called a metric projection onto C. If $\Pi(C|\cdot)$ is single-valued, then it is called the best approximate operator and $\Pi(C|u)$ the best approximant of u. In particular, if C is a linear subspace of the space $E = L^2(T, \Sigma, \mu)$, then $\Pi(C|u)$ is said to be a generalized conditional expectation of u.

The set $\Pi(C|u)$ is characterized by the following lemma.

Lemma 5. (cf. [2]) *Let u_0 be an element of a convex subset C of a smooth Banach space E and let $u \in E \setminus C$. TFAE.*

a) $u_0 \in \Pi(C|u)$;

b) $< u_0 - w, \Lambda_E(u - u_0) > \geq 0$ for every $w \in C$.

1. STRICT CONVEXITY AND SMOOTHNESS OF BOCHNER-ORLICZ SPACES.

We will begin the study of geometrical properties with the following theorem

Theorem 1. *A Bochner-Orlicz space $L_{(\Phi)}(X)$ is strictly convex iff the following conditions are satisfied*

a) X is a strictly convex Banach space;

b) $\Phi \in \Delta_2$;

c) Φ is strictly convex.

Proof. That result is a special case of the main theorem in [6].

A criterion for strict convexity of Bochner-Orlicz spaces with Orlicz norm is furnished by the following theorem

Theorem 2. *A Bochner-Orlicz space $L_\Phi(X)$ is strictly convex iff the following conditions are satisfied*

a) X is a strictly convex Banach space;

b) Φ is strictly convex.

Proof of sufficiency. Denote by $S(L_\Phi(X))$ the unit sphere of the space $L_\Phi(X)$. Suppose that a) and b) are satisfied. Let $u, v \in S(L_\Phi(X))$ be such that $||u+v||_\Phi = 2$. We have to proved that

$$u(t) = v(t) \text{ for a.e. } t \in T.$$

To this end observe that $||u(\cdot)||_X$ and $||v(\cdot)||_X$ are elements of the unit sphere of the space L_Φ. Since

$$||u(t)+v(t)||_X \le ||u(t)||_X + ||v(t)||_X \qquad \text{for a.e. } t \in T,$$

by the monotonicity of the Orlicz norm it follows that

$$2 = ||u+v||_\Phi = \Big\| ||u(\cdot)+v(\cdot)||_X \Big\|_\Phi \le$$

$$\le \Big\| ||u(\cdot)||_X + ||v(\cdot)||_X \Big\|_\Phi \le ||u||_\Phi + ||v||_\Phi = 2.$$

Hence

$$\Big\| ||u(\cdot)||_X + ||v(\cdot)||_X \Big\|_\Phi = 2.$$

Since Φ is strictly convex, Theorem 2.4 from [20] implies that L_Φ is strictly convex and hence

$$\|u(t)\|_X = \|v(t)\|_X \qquad \text{for a.e. } t \in T.$$

Denote

$$T_0 = \{t \in T : u(t) \neq v(t)\}.$$

Suppose that $\mu(T_0) > 0$. Then $\|u(t)\|_X = \|v(t)\|_X > 0$ for a.e. $t \in T_0$. Hence, by the strict convexity of the space X, we get

$$\|u(t) + v(t)\|_X < 2\|u(t)\|_X \qquad \text{for a.e. } t \in T_0.$$

Moreover, by Lemma 3 b), constants k_1 and k_2 can be found such that

$$1 = \|u\|_\Phi = \frac{1}{k_1}\left\{1 + \int_T \Phi(k_1\|u(t)\|_X)\, dt\right\}$$

and

$$1 = \|v\|_\Phi = \frac{1}{k_2}\left\{1 + \int_T \Phi(k_2\|v(t)\|_X)\, dt\right\} =$$

$$= \frac{1}{k_2}\left\{1 + \int_T \Phi(k_2\|u(t)\|_X)\, dt\right\}.$$

Consequently, by the convexity of Φ, we obtain

$$2 = \frac{k_1 + k_2}{k_1 k_2}\left\{1 + \frac{k_2}{k_1 + k_2}\int_T \Phi(k_1\|u(t)\|_X)\, dt + \right.$$

$$\left. + \frac{k_1}{k_1 + k_2}\int_T \Phi(k_2\|u(t)\|_X)\, dt\right\} \geq$$

$$\geq \frac{k_1 + k_2}{k_1 k_2}\left\{1 + \int_T \Phi\left[\frac{k_1 k_2}{k_1 + k_2} 2\|u(t)\|_X\right] dt\right\} >$$

$$> \frac{k_1 + k_2}{k_1 k_2}\left\{1 + \int_{T_0} \Phi\left[\frac{k_1 k_2}{k_1 + k_2}\|u(t) + v(t)\|_X\right] dt + \right.$$

$$+ \int_{T\setminus T_0} \Phi\left[\frac{k_1 k_2}{k_1+k_2}\|u(t)+v(t)\|_X\right] dt\Bigg\} \geq$$

$$\geq \inf_{k>0} \frac{1}{k}\left\{1 + \int_T \Phi(k\|u(t)+v(t)\|_X)\, dt\right\} = \|u+v\|_\Phi.$$

This contradiction proves that $\mu(T_0) = 0$, i.e.

$$u(t) = v(t) \qquad \text{for a.e. } t \in T$$

as claimed.

Proof of necessity. Let $x, y \in S(L_\Phi)$ be such that $\|x+y\|_\Phi = 2$. Fix $e \in S(X)$. Define

$$u(t) = x(t)e, \; v(t) = y(t)e, \quad t \in T.$$

Then $u, v \in S(L_\Phi(X))$ and

$$\|u+v\|_\Phi = \|x+y\|_\Phi = 2.$$

By The strict convexity of the space $L_\Phi(X)$, we have

$$u(t) = v(t) \qquad \text{for a.e. } t \in T,$$

and hence

$$x(t) = y(t) \qquad \text{for a.e. } t \in T.$$

Therefore, L_Φ is strictly convex. Using Theorem 2.4 from [20], we conclude that Φ is strictly convex, i.e. b) is satisfied.

Now, suppose that a) is false. Then there exist e_1 and e_2 from the unit sphere $S(X)$ such that

$$\|e_1 + e_2\|_X = 2 \text{ and } e_1 \neq e_2.$$

Let a be a positive real number such that $||a\chi_T||_\Phi = 1$. Define

$$u(t) = ae_1\chi_T(t),\ v(t) = ae_2\chi_T(t) \text{ and } w(t) = \frac{u(t)+v(t)}{2}$$

for every $t \in T$. Then

$$u \neq v \text{ and } ||u||_\Phi = ||v||_\Phi = ||w||_\Phi = 1,$$

which contradicts to the strict convexity of the space $L_\Phi(X)$. Thus a) is satisfied. This completes the proof.

For later use, we present sufficient conditions for smoothness of the space $L_{(\Phi)}(X)$ in the following theorem

Theorem 3. *If $\Phi \in \Delta_2$, ϕ is continuous and X is smooth, then the Bochner-Orlicz space $L_{(\Phi)}(X)$ is smooth.*

Proof. Since X is reflexive and smooth, X^* is strictly convex. Therefore, by the continuity of ϕ and Theorem 2, we conclude that $L_\Psi(X^*)$ is strictly convex. But $(L_{(\Phi)}(X))^* = L_\Psi(X^*)$. Hence $L_{(\Phi)}(X)$ is smooth.

Taking into account the Bochner-Orlicz space with the Orlicz norm, we get the following

Theorem 4. *If $\Phi \in \Delta_2$, ϕ is continuous and X is smooth, then the Bochner-Orlicz space $L_\Phi(X)$ is smooth.*

Proof. Let u be a non-zero element of the space $L_\Phi(X)$. By Lemma 3 b) there is $k > 0$ such that

$$1 = \frac{1}{k}\left\{1 + \int_T \Phi\left[\frac{k||u(t)||_X}{||u||_\Phi}\right] dt\right\}. \tag{1}$$

Moreover Lemma 2 states that $(L_\Phi(X))^* = L_{(\Psi)}(X^*)$ because $\Phi \in \Delta_2$. Suppose that $v \in S(L_{(\Psi)}(X^*))$ is such that

$$||u||_\Phi = \int_T < u(t), v(t) > dt. \tag{2}$$

Note that

$$\|u(t)\|_X = < u(t), \Lambda_X(u(t)) > \qquad \text{for } t \in T.$$

Using (1), (2) and the Young inequality, we have

$$1 + \int_T \Phi\left[\frac{k < u(t), \Lambda_X(u(t)) >}{\|u\|_\Phi}\right] dt = \int_T < \frac{ku(t)}{\|u\|_\Phi}, v(t) > \; dt \leq$$

$$\leq \int_T \frac{k\|u(t)\|_X}{\|u\|_\Phi} \|v(t)\|_{X^*} \; dt \leq$$

$$\leq \int_T \Psi\left[\|v(t)\|_{X^*}\right] dt + \int_T \Phi\left[\frac{k\|u(t)\|_X}{\|u\|_\Phi}\right] dt \leq \tag{3}$$

$$\leq 1 + \int_T \Phi\left[\frac{k < u(t), \Lambda_X(u(t)) >}{\|u\|_\Phi}\right] dt.$$

Taking into account the continuity of the function ϕ and conditions under which both sides of the Young inequality are equal, we get

$$\|v(t)\|_{X^*} = \phi\left[\frac{k\|u(t)\|_X}{\|u\|_\Phi}\right] \qquad \text{for a.e. } t \in T.$$

Hence, by (3), we have

$$\int_T < \frac{ku(t)}{\|u\|_\Phi}, v(t) > \; dt = \int_T \frac{k\|u(t)\|_X}{\|u\|_\Phi} \|v(t)\|_{X^*} \; dt =$$

$$= \int_T < \frac{ku(t)}{\|u\|_\Phi}, \; \phi\left[\frac{k\|u(t)\|_X}{\|u\|_\Phi}\right] \Lambda_X(u(t)) > \; dt.$$

Thus

$$v(t) = \phi\left[\frac{k\|u(t)\|_X}{\|u\|_\Phi}\right] \Lambda_X(u(t)) \qquad \text{for a.e. } t \in T,$$

which implies that the support mapping is single-valued. This means that the space $L_\Phi(X)$ is smooth.

2. MAIN RESULTS.

First theorem of this section gives the complete characterization of values of metric projection on convex subset $C \subset L_\Phi(X)$ at arbitrary point $u \in L_\Phi(X) \setminus C$.

Theorem 5. *Let $\Phi \in \Delta_2$, ϕ be continuous, X be smooth and C be a convex subset of the space $L_\Phi(X)$. If $u_0 \in C$ and $u \in L_\Phi(X) \setminus C$, then the following conditions are equivalent:*

a) $u_0 \in \Pi(C|u)$;

b) $\int_T < u_0(t) - w(t), \Lambda_X(u(t) - u_0(t)) > \phi[k||u(t) - u_0(t)||_X]\ dt \geq 0$ *for any $w \in C$, where*

$$\int_T \Psi[\phi(k||u(t) - u_0(t)||_X)]\ dt \ = \ 1.$$

Proof. By Theorem 4, the space $L_\Phi(X)$ is smooth. To simplify notations we denote $E = L_\Phi(X)$. Then, by Lemma 2 b), we obtain $E^* = L_{(\Psi)}(X^*)$.

Proof of implication. a) $\Rightarrow$ b). Let $u_0 \in \Pi(C|u)$. Using Lemma 5, we get

$$\int_T < u_0(t) - w(t), \Lambda_E(u - u_0)(t) > \ dt \geq 0 \tag{4}$$

for every $w \in C$. Further, by Lemma 3 b), there exists $k > 0$ such that

$$\frac{1}{k}\left\{1 + \int_T \Phi[k||u(t) - u_0(t)||_X]\ dt\right\} \ = \ ||u - u_0||_\Phi \ =$$

$$= \ \int_T < u(t) - u_0(t), \Lambda_E(u - u_0)(t) > \ dt \leq$$

$$\leq \frac{1}{k}\int_T k||u(t) - u_0(t)||_X \ ||\Lambda_E(u - u_0)(t)||_{X^*}\ dt \leq$$

$$\frac{1}{k}\left\{\int_T \Phi[k||u(t) - u_0(t)||_X]\ dt + \int_T \Psi[||\Lambda_E(u - u_0)(t)||_{X^*}]\ dt\right\} \leq$$

$$\leq \frac{1}{k}\left\{1 + \int_T \Phi[k||u(t) - u_0(t)||_X]\, dt\right\},$$

where $||\Lambda_E(u - u_0)||_{(\Psi)} = 1$. Hence

$$\int_T \Psi[||\Lambda_E(u - u_0)(t)||_{X^*}]\, dt = 1 \tag{5}$$

and

$$\int_T \Big\{\Phi[k||u(t) - u_0(t)||_X] + \Psi[||\Lambda_E(u - u_0)(t)||_{X^*}] -$$

$$-k||u(t) - u_0(t)||_X ||\Lambda_E(u - u_0)(t)||_{X^*}\Big\}\, dt = 0.$$

Therefore

$$\Phi[k||u(t) - u_0(t)||_X] + \Psi[||\Lambda_E(u - u_0)(t)||_{X^*}] =$$

$$= k||u(t) - u_0(t)||_X ||\Lambda_E(u - u_0)(t)||_{X^*}$$

for a.e. $t \in T$. Taking into account the continuity of ϕ and conditions under which both sides of the Young inequality are equal, we obtain

$$||\Lambda_E(u - u_0)(t)||_{X^*} = \phi[k||u(t) - u_0(t)||_X] \quad \text{for a.e. } t \in T.$$

Hence ans by previous calculations, we have

$$\int_T < k(u(t) - u_0(t)), \Lambda_E(u - u_0)(t) > \, dt =$$

$$= \int_T k||u(t) - u_0(t)||_X \; ||\Lambda_E(u - u_0)(t)||_{X^*}\, dt =$$

$$= \int_T < k(u(t) - u_0(t)), \phi[k\; ||u(t) - u_0(t)||_X]\Lambda_X(u(t) - u_0(t)) > \, dt.$$

Consequently

$$\Lambda_E(u - u_0)(t) = \Lambda_X(u(t) - u_0(t))\; \phi[k||u(t) - u_0(t)||_X] \tag{6}$$

for a.e. $t \in T$. Moreover, by (5), we have

$$\int_T \Psi[\phi(k||u(t) - u_0(t)||_X)]\, dt \;=\; \int_T \Psi[||\Lambda_E(u - u_0)(t)||_{X^*}]\, dt \;=\; 1.$$

Hence, using (4) and (6), we obtain b).

Proof of implication b) $\Rightarrow$ a). It follows immediately from Lemma 5 and the equality (6).

For the case of Bochner-Orlicz spaces equipped with the Luxemburg norm we can get the following theorem.

Theorem 6. *Let $\Phi \in \Delta_2$, ϕ be continuous, X be smooth and C be a convex subset of the space $L_{(\Phi)}(X)$. If $u_0 \in C$ and $u \in L_{(\Phi)}(X) \setminus C$, then the following conditions are equivalent:*

a) $u_0 \in \Pi(C|u)$;

b) $$\int_T < u_0(t) - w(t),\; \Lambda_X(u(t) - u_0(t)) > \phi\left[\frac{||u(t) - u_0(t)||_X}{||u - u_0||_{(\Phi)}}\right] dt \geq 0$$

for any $w \in C$.

Proof. By Theorem 3, the space $L_{(\Phi)}(X)$ is smooth. Denoting $E = L_{(\Phi)}(X)$, by Lemma 2 b), we obtain $E^* = L_\Psi(X^*)$.

Proof of implication a) $\Rightarrow$ b). Let $u_0 \in \Pi(C|u)$. Lemma 5 implies that

$$\int_T < u_0(t) - w(t),\; \Lambda_E(u - u_0)(t) > \; dt \geq 0 \tag{7}$$

for any $w \in C$. Furthermore

$$||\Lambda_E(u - u_0)||_\Psi \;=\; 1$$

and

$$||u - u_0||_{(\Phi)} = \int_T < u(t) - u_0(t),\ \Lambda_E(u - u_0)(t) > dt.$$

Using Lemma 3 b), a positive number k can be found such that

$$\frac{1}{k}\left\{1 + \int_T \Psi[k||\Lambda_E(u - u_0)(t)||_{X^*}]\, dt\right\} = ||\Lambda_E(u - u_0)||_\Psi = 1 =$$

$$= \int_T < \frac{u(t) - u_0(t)}{||u - u_0||_{(\Phi)}},\ \Lambda_E(u - u_0)(t) > dt \leq$$

$$\leq \frac{1}{k}\int_T \frac{||u(t) - u_0(t)||_X}{||u - u_0||_{(\Phi)}}\, k||\Lambda_E(u - u_0)(t)||_{X^*}\, dt \leq \tag{8}$$

$$\leq \frac{1}{k}\left\{\int_T \Phi\left[\frac{||u(t) - u_0(t)||_X}{||u - u_0||_{(\Phi)}}\right] dt + \int_T \Psi\left[k||\Lambda_E(u - u_0)(t)||_{X^*}\right] dt\right\} \leq$$

$$\leq \frac{1}{k}\left\{1 + \int_T \Psi[k||\Lambda_E(u - u_0)(t)||_{X^*}]\, dt\right\}.$$

It follows from the continuity of ϕ that

$$||\Lambda_E(u - u_0)(t)||_{X^*} = \frac{1}{k}\phi\left[\frac{||u(t) - u_0(t)||_X}{||u - u_0||_{(\Phi)}}\right] \qquad \text{for a.e. } t \in T.$$

Hence, by (8), we have

$$\int_T < \frac{u(t) - u_0(t)}{||u - u_0||_{(\Phi)}},\ \Lambda_E(u - u_0)(t) > dt =$$

$$= \int_T \frac{||u(t) - u_0(t)||_X}{||u - u_0||_{(\Phi)}}\, ||\Lambda_E(u - u_0)(t)||_{X^*}\, dt =$$

$$= \int_T < \frac{u(t) - u_0(t)}{||u - u_0||_{(\Phi)}},\ \frac{1}{k}\phi\left[\frac{||u(t) - u_0(t)||_X}{||u - u_0||_{(\Phi)}}\right]\Lambda_X(u(t) - u_0(t)) > dt.$$

Consequently

$$\Lambda_E(u-u_0)(t) \;=\; \frac{1}{k}\phi\left[\frac{||u(t)-u_0(t)||_X}{||u-u_0||_{(\Phi)}}\right]\Lambda_X(u(t)-u_0(t)) \qquad (9)$$

for a.e. $t \in T$. Combining formula (9) with (7) we complete the proof of the desire implication.

Proof of implication. b) $\Rightarrow$ a). It follows immediately from Lemma 5 and formula (9).

A subspace $L \subset L_\Phi(X)$ is said to be a *zero mean valued subspace* if

$$L \;=\; \left\{w \in L_\Phi(X) : \int_T w(t)dt \;=\; 0\right\},$$

where an integration is in the Bochner sense.

Theorem 7. *If $\Phi \in \Delta_2 \cap \nabla_2$, ϕ and ψ are continuous and X is reflexive, smooth and strictly convex, then for each $u \in L_\Phi(X)$ there exists a unique best approximate element $\Pi(C|u) \in L$. Furthermore*

$$\Pi(L|u) \;=\; u - \frac{\chi_T}{\mu(T)}\int_T u(t)dt \qquad \text{for } u \in L_\Phi(X).$$

Proof. Using Lemma 2, Theorem 2 and Theorem 4, we conclude that $L_\Phi(X)$ is reflexive, strictly convex and smooth. Moreover, L is a linear and closed subspace of $L_\Phi(X)$. Indeed, L is linear in an obvious manner. For the proof of the closure of L, suppose that $\{w_n\}$ is a sequence of elements of the subspace L such that

$$||w_n - w||_\Phi \to 0 \text{ as } n \to \infty.$$

Note that $x^*\chi_T(\cdot) \in L_{(\Psi)}(X^*) \;=\; (L_\Phi(X))^*$ for any $x^* \in X^*$. Hence

$$< \int_T w(t)dt, \; x^* >= \int_T < w(t), x^* > \; dt = \int_T < w(t), x^*\chi_T(t) > \; dt =$$

$$= \lim_{n\to\infty} \int_T < w_n(t), x^*\chi_T(t) > \ dt \ = \ \lim_{n\to\infty} < \int_T w_n(t)dt, x^* > \ = \ 0$$

for each $x \in X$. Therefore

$$\int_T w(t)dt \ = \ 0,$$

i.e. $w \in L$.

Since L is a closed linear subspace of the reflexive strictly convex space $L_\Phi(X)$, there exists a unique element $u_0 \in L$ such that

$$||u - u_0||_\Phi \ = \ \inf_{v\in L} ||u - v||_\Phi,$$

i.e. $u_0 \ = \ \Pi(L|u)$.

By Theorem 5,

$$\int_T < u_0(t) - w(t), \ \Lambda_X(u(t) - u_0(t)) > \phi[k||u(t) - u_0(t)||_X] \ dt \geq 0$$

for any $w \in L$, where

$$\int_T \Psi[\phi(k||u(t) - u_0(t)||_X)] \ dt \ = \ 1.$$

Since L is linear, we have

$$\int_T < w(t), \phi[k||u(t) - u_0(t)||_X]\Lambda_X(u(t) - u_0(t)) > \ dt \ = \ 0$$

for each $w \in L$. Hence, by the definition of the duality mapping $F_E: \ E \to E^*$, we obtain

$$F_E(u - u_0) \ = \ ||u - u_0||_\Phi \Lambda_E(u - u_0) \ =$$

$$||u - u_0||_\Phi \phi[k||u(\cdot) - u_0(\cdot)||_X]\Lambda_X(u(\cdot) - u_0(\cdot)) \in L^\perp \subset L_{(\Psi)}(X^*), \quad (10)$$

where $E = L_\Phi(X)$. Denote

$$H = \{x^*\chi_T(\cdot) : x^* \in X^*\}.$$

Obiously, H is a closed, linear subspace of $L_{(\Psi)}(X^*)$. Moreover,

$$\int_T < w(t), x^*\chi_T(t) > dt = < \int_T w(t)dt, x^* > = 0$$

for any $w \in L$ and $x^* \in X^*$. This inequality implies inmediately that $L = H^\perp$. Therefore,

$$L^\perp = H^{\perp\perp} = H.$$

Hence, by (10), there exists $x_0^* \in X^*$ such that

$$F_E(u - u_0) = x_0^*\chi_T. \tag{11}$$

Let $F_{E^*} : E^* \to E^{**} = E$ be the duality mapping E^*. Since E is reflexive, smooth and strictly convex, F_E and F_{E^*} are bijections (cf. Lemma 4) and $F_E^{-1} = F_{E^*}$. Thus from (11) it follows that

$$u - u_0 = F_E^{-1}(x_0^*\chi_T) = F_{E^*}(x_0^*\chi_T). \tag{12}$$

Since

$$\int_T < x_0^*\chi_T(T), \frac{1}{\mu(T)}\left[\Psi^{-1}\left(\frac{1}{\mu(T)}\right)\right]^{-2} ||x_0^*||_{X^*}\Lambda_{X^*}(x_0^*)\chi_T(t) > dt =$$

$$= \frac{1}{\mu(T)}\int_T ||x_0^*||_{X^*}^2\left[\Psi^{-1}\left(\frac{1}{\mu(T)}\right)\right]^{-2} dt =$$

$$= ||x_0^*||_{X^*}^2\left[\Psi^{-1}\left(\frac{1}{\mu(T)}\right)\right]^{-2} = ||x_0^*\chi_T||_{(\Psi)}^2,$$

we deduce, by the definition of F_{E^*}, that

$$F_{E^*}(x_0^*\chi_T) = x_1\chi_T, \tag{13}$$

where

$$x_1 = \|x_0^*\|_{X^*}\Lambda_{X^*}(x_0^*)\left[\Psi^{-1}\left(\frac{1}{\mu(T)}\right)\right]^{-2}\frac{1}{\mu(T)} \in X^{**} = X.$$

Combining (12) and (13), we obtain

$$u - \Pi(L|u) = x_1\chi_T. \tag{14}$$

Integrating this equality over T, we obtain

$$\int_T u(t)dt = \int_T x_1\chi_T(t)dt = x_1\mu(T)$$

and hence

$$x_1 = \frac{1}{\mu(T)}\int_T u(t)dt.$$

Coming back to the equality (14), we get inmediately

$$\Pi(L|u) = u - \frac{\chi_T}{\mu(T)}\int_T u(t)dt$$

for any $u \in L_\Phi(T)$. This completes the proof.

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