REVISTA MATEMATICA de la
Universidad Complutense de Madrid
Volumen 7, número 1: 1994
http://dx.doi.org/10.5209/rev_REMA.1994.v7.n1.17758

Packing Constant in Musielak-Orlicz Sequence Spaces Equipped with the Luxemburg Norm

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ABSTRACT. A morè precise formula for the Kottman parameter $D(X)$ connected with the packing constant $\Lambda(X)$ in such a way that $\Lambda(X) = D(X)/(2+D(X))$ for a Banach space X, in the case when X is a Musielak-Orlicz sequence spaces l^φ, is given. As a corollary, packing constant of the Nakano space $l^{(p_i)}$, where $1 \leq p_i < +\infty$ for any $i = 1, 2, \dots$, is computed. This generalizes the results of [2] for l^p spaces. It is also proved that $\Lambda(l^\varphi) = \Lambda(h^\varphi)$.

INTRODUCTION

In the sequel $\mathbf{N}, \mathbf{R}, \mathbf{R}_+$ and $\mathbf{R}_+^e$ stand for the set of natural numbers, the set of reals, the set of positive reals and for the interval $[0, +\infty]$, respectively. The space of all real sequences $x = (x_i)_{i=1}^\infty$ is denoted by l^0. A map $\varphi : \mathbf{R} \to \mathbf{R}_+^e$ is said to be an Orlicz function if φ is convex, even, vanishing and continuous at 0, left-hand side continuous on the whole $\mathbf{R}_+$, and not identically equal to zero (see [9], [10] and [11]).

A sequence of $\varphi = (\varphi_i)_{i=1}^\infty$ of Orlicz functions φ_i is called a Musielak-Orlicz function.

1991 Mathematics Subject Classification: 46E30
Editorial Complutense. Madrid, 1994.

Given a Musielak-Orlicz function φ we define on l^0 a convex modular I_φ by

$$I_\varphi(x) = \sum_{i=1}^{\infty} \varphi_i(x_i) \qquad (\forall x = (x_i)_{i=1}^{\infty} \in l^0).$$

A Musielak-Orlicz space generated by φ is defined by

$$l^\varphi = \{x \in l^0 : I_\varphi(\lambda x) < +\infty \text{ for a certain } \lambda > 0\}.$$

The functional

$$||x||_\varphi = inf\{\lambda > 0 : I_\varphi(x/\lambda) \leq 1\}$$

is a norm in l^φ (called the Luxemburg norm) and the couple $(l^\varphi, || \, ||_\varphi)$ is a Banach space (see [9]). We denote by h^φ the subspace of l^φ being the closure in l^φ of all sequences in l^0 with finite number of coordinates different from zero. This subspace will be considered with the norm $|| \, ||_\varphi$ induced from l^φ. In the case when all functions φ_i which define φ are finite-valued (i.e. φ_i are continuous functions) we have simply

$$h^\varphi = \{x \in l^0 : I_\varphi(\lambda x) < +\infty \text{ for any } \lambda > 0\}.$$

We say a Musielak-Orlicz function φ satisfies the δ_2^0-condition if there are positive constants a, K, a natural number m and a sequence $(C_i)_{i=1}^{\infty} \subset \mathbf{R}_+^e$ such that $\sum_{i=m}^{\infty} C_i < +\infty$ and for any $i \in \mathbf{N}$ and $u \in \mathbf{R}$ satisfying $\varphi_i(u) \leq a$ there holds

$$\varphi_i(2u) \leq K\varphi_i(u) + C_i.$$

If a Musielak-Orlicz function φ satisfies the δ_2^0-condition with $m = 1$ we say that φ satisfies the δ_2-condition (see [4], [6] and [11]).

For any Musielak-Orlicz function φ, h^φ coincides with l^φ if and only if φ satisfies the δ_2^0-condition (see [5]).

For any Banach space X denote by $B(X)$ and $S(X)$ the unit ball and the unit sphere of X, respectively. The unit ball in X centered at $x \in X$ and with radius $r > 0$ denote by $B_X(x, r)$.

The packing constant $\Lambda(X)$ of a Banach space X is defined by

$$\Lambda(X) = \sup \{r > 0 : \exists (x^n)_{n=1}^{\infty} \subset B(X),\ \|x^m - x^n\| \geq 2r \text{ for } m \neq n, \text{ and } \bigcup_{n=1}^{\infty} B_X(x^n, r) \subset B(X)\}$$

(see [12]). Kottman [8] proved that for any Banach space X we have $\Lambda(X) = D(X)/(2 + D(X))$, where

$$D(X) = \sup_{(x^n) \subset S(X)} \inf_{m \neq n} \|x^m - x^n\|.$$

It is well known that $D(l^p) = 2^{1/p}$ for $1 \leq p < +\infty$ and $D(l^\infty) = 2$ (see [12]). Moreover, Cleaver [3] obtained some lower and upper estimations for the packing constant of Orlicz sequence (as well as function) spaces and he pointed out that these estimations give an exact formula for the packing constant in a special class of Orlicz spaces, i.e. for Orlicz spaces such that $\|x\|_\varphi = \varphi^{-1}(I_\varphi(x))$ for any $x \in l^\varphi$ (respectively for any $x \in L^\varphi$). However, Zaanen [15] pointed out that this class of Orlicz spaces reduces only to l^p (respectively L^p) spaces (see also Wnuk [13]). Next, Ye [14] obtained a simpler formula for $D(l^\varphi)$, where φ is an Orlicz function (i.e. all functions φ_i in the definition of φ are the same). Namely, he has proved that for any finite-valued Orlicz function φ which satisfies the Δ_2-condition at zero, we have

$$D(l^\varphi) = \sup_{x \in S(l^\varphi)} \left\{c_x > 0 : I_\varphi(x/c_x) = \frac{1}{2}\right\}. \tag{1}$$

In this paper we obtain an analogue for Musielak-Orlicz sequence spaces l^φ generated by finite-valued Musielak-Orlicz function $\varphi = (\varphi_i)_{i=1}^{\infty}$ which satisfies an additional condition (+). In the case when φ is an Orlicz function this formula coincides with formula (1).

We say a Musielak-Orlicz function $\varphi = (\varphi_i)_{i=1}^{\infty}$ satisfies condition $(+)$ if for any $c > 0$ and any $\varepsilon \in (0, c)$ there is $\delta > 0$ such that $\varphi_i((1+\delta)u) \leq c$ whenever $\varphi_i(u) \leq c - \varepsilon$ for $i = 1, 2, \ldots$ and $u \in \mathbf{R}_+$.

Assuming in this definition $c = 1$ we obtain condition (*) defined by *A.* Kaminska in [7].

RESULTS.

To obtain main results we need to give some auxiliary lemmas.

Lemma 1 (see [4] and [6]). *If $\varphi = (\varphi_i)_{i=1}^{\infty}$ is a finite-valued Musielak-Orlicz function i.e. all functions φ_i are finite-valued and φ satisfies the δ_2^0-condition (equivalently, the δ_2-condition), then $\|x\|_\varphi = 1$ if and only if $I_\varphi(x) = 1$.*

Lemma 2. *Let $\varphi = (\varphi_i)_{i=1}^{\infty}$ be a Musielak-Orlicz function satisfying condition (+) and such that all φ_i are finite-valued. For any sequence $(x^k)_{k=1}^{\infty}$ of elements in $S(l^\varphi)$ and $\delta > 0$ there exist a subsequence $(y^k)_{k=1}^{\infty}$ of $(x^k)_{k=1}^{\infty}$ and a strictly increasing sequence $(i_k)_{k=1}^{\infty}$ of natural numbers such that*

$$(i) \qquad \sum_{i=i_{k+1}+1}^{\infty} \varphi_i(y_i^k) < \delta \qquad (k = 1, 2, \ldots),$$

$$(ii) \qquad \sum_{i=1}^{i_k} \varphi_i(y_i^n - y_i^m) < \delta/2 \qquad (k = 1, 2, \ldots;\ m, n \geq k),$$

$$(iii) \qquad \sum_{i=i_k}^{i_{k+1}} \varphi_i(y_i^n) < \delta \qquad (k = 1, 2, \ldots;\ n \geq k),$$

where y_i^k denotes the i-th coordinate of y^k.

The lemma can be proved in the same way as Lemma 4 in [14]. Therefore, we omit the proof.

Lemma 3 (see [7]). *Let $\varphi = (\varphi_i)_{i=1}^{\infty}$ be a finite-valued and satisfying the δ_2-condition and condition (+) Musielak-Orlicz function.*

For a given $\varepsilon > 0$ *and* $c > 0$ *there exist* $\delta = \delta(c, \varepsilon) > 0$ *such that* $I_\varphi(x+y) < I_\varphi(x) + \varepsilon$ *whenever* $x, y \in l^\varphi$, $I_\varphi(x) \le c$ *and* $I_\varphi(y) < \delta$.

Corollary 1. *Under the assumption of Lemma 3 concerning* φ *we have that for any* $\eta > 0$ *there exists* $\alpha > 0$ *such that* $I_\varphi(x-y) \ge \alpha$ *whenever* $x, y \in l^\varphi$, $I_\varphi(x) \ge \eta$, $I_\varphi(y) < \eta/2$.

Proof. Assume that φ satisfies the assumptions and that the assertion from the corollary does not hold, i.e. there exists $\eta > 0$ such that for any $\alpha > 0$ there exist $x, y \in l^\varphi$ such that

$$I_\varphi(x) \ge \eta,\ I_\varphi(y) < \frac{\eta}{2} \text{ and } I_\varphi(x-y) < \alpha.$$

Let $\delta > 0$ be the number corresponding to $c = \varepsilon = \frac{\eta}{2}$ in Lemma 3, and assume $\alpha = \delta$. Then we have

$$I_\varphi(x) = I_\varphi(y + (x-y)) \le I_\varphi(y) + \frac{\eta}{2} < \frac{\eta}{2} + \frac{\eta}{2} = \eta,$$

a contradiction. Therefore, the corollary is proved.

Now, we shall introduce a parameter d_φ for finite-valued Musielak-Orlicz functions $\varphi = (\varphi_i)_{i=1}^\infty$.
Define

$$c(x,m,n) = inf\left\{c > 0 : \sum_{i=n}^{n+m} \varphi_i\left(\frac{x_i}{c}\right) \le \frac{1}{2}\right\} \quad (\forall x \in S(l^\varphi);\ m, n \in \mathbf{N}).$$

The sequence $(c(x,m,n))_{m=1}^\infty$ is nondecreasing for any $x \in S(l^\varphi)$ and $n \in \mathbf{N}$. Therefore, for any $x \in S(l^\varphi)$ and $n \in \mathbf{N}$, the limit

$$d(x,n) = \lim_{m \to +\infty} c(x,m,n)$$

exists. Moreover, we have $d(x,n) \ge c(x,m,n)$ for any $x \in S(l^\varphi)$ and $m, n \in \mathbf{N}$. Let

$$d_n = sup\left\{d(x,n) :\ x \in S(l^\varphi)\right\} \qquad (\forall n \in \mathbf{N}).$$

It is easy to see that $1 \leq d_n \leq 2$ for each $n \in \mathbf{N}$. Since the sequence $(d_n)_{n=1}^{\infty}$ is nonincreasing, we can define

$$d_{\varphi} = \lim_{n \to +\infty} d_n.$$

Obviously, $1 \leq d_{\varphi} \leq 2$.

Remark 1. Note that if we change only finite number of Orlicz functions φ_i in the sequence $\varphi = (\varphi_i)_{i=1}^{\infty}$, the parameter d_{φ} remains the same. Also the δ_2^0-condition holds true for the Musielak-Orlicz function changed in such a way, whenever φ satisfies the δ_2^0-condition.

We are now in a position to give one of the main results of this paper.

Theorem 1. *Let $\varphi = (\varphi_i)_{i=1}^{\infty}$ be a finite-valued Musielak-Orlicz function satisfying the δ_2^0-condition and condition (+). Then $D(l^{\varphi}) = d_{\varphi}$.*

Proof. First, we will prove that $D(l^{\varphi}) \geq d_{\varphi}$. For any $\varepsilon > 0$ and each $n_1 \in \mathbf{N}$ there exist $x^1 \in S(l^{\varphi})$ and $m_1 \in \mathbf{N}$ such that

$$c(x^1, m_1, n_1) > d(x^1, n_1) - \frac{\varepsilon}{4} > d_{n_1} - \frac{\varepsilon}{2} \geq d_{\varphi} - \frac{\varepsilon}{2},$$

i.e.

$$\sum_{i=n_1}^{n_1+m_1} \varphi_i\left(\frac{x_i^1}{d_{\varphi} - \frac{\varepsilon}{2}}\right) \geq \frac{1}{2}.$$

Take $n_2 = m_1 + n_1 + 1$. There exists $x^2 \in S(l^{\varphi})$ and $m_2 > m_1$ such that

$$c(x^2, m_2, n_2) \geq d_{\varphi} - \frac{\varepsilon}{2},$$

i.e.

$$\sum_{i=n_2}^{n_2+m_2} \varphi_i\left(\frac{x_i^2}{d_{\varphi} - \frac{\varepsilon}{2}}\right) \geq \frac{1}{2}.$$

Generally, putting $n_k = n_{k-1} + m_{k-1} + 1$, there exist $x^k \in S(l^\varphi)$ and $m_{i+1} > m_i$ such that

$$c(x^k, m_k, n_k) > d_\varphi - \frac{\varepsilon}{2},$$

i.e.

$$\sum_{i=n_k}^{n_k+m_k} \varphi_i\left(\frac{x_i^k}{d_\varphi - \frac{\varepsilon}{2}}\right) \geq \frac{1}{2}.$$

Let $y^k = (y_i^k)_{i=1}^\infty$, where

$$y_i^k = \begin{cases} x_i^k & \text{when } n_k \leq i \leq n_k + m_k \\ 0 & \text{otherwise} \end{cases}.$$

Define $z = (z^k)_{k=1}^\infty$, where $z^k = y^k/\|y^k\|_\varphi$. Then $z^k \in S(l^\varphi)$ and $\|y^k\|_\varphi \leq \|x^k\|_\varphi = 1$. Therefore,

$$\begin{aligned} I_\varphi\left(\frac{z^k - z^l}{d_\varphi - \frac{\varepsilon}{2}}\right) &= \sum_{i=n_k}^{n_k+m_k} \varphi_i\left(\frac{x_i^k}{\|y^k\|_\varphi(d_\varphi - \frac{\varepsilon}{2})}\right) + \sum_{i=n_l}^{n_l+m_l} \varphi_i\left(\frac{x_i^l}{\|y^l\|_\varphi(d_\varphi - \frac{\varepsilon}{2})}\right) \\ &\geq \sum_{i=n_k}^{n_k+m_k} \varphi_i\left(\frac{x_i^k}{d_\varphi - \frac{\varepsilon}{2}}\right) + \sum_{i=n_l}^{n_l+m_l} \varphi_i\left(\frac{x_i^l}{d_\varphi - \frac{\varepsilon}{2}}\right) \geq \frac{1}{2} + \frac{1}{2} = 1. \end{aligned}$$

This means that $\|z^k - z^l\|_\varphi \geq d_\varphi - \frac{\varepsilon}{2}$, whence it follows immediately that $D(l^\varphi) \geq d_\varphi$.

We shall prove now that $D(l^\varphi) \leq d_\varphi$. Assume that this inequality does not hold. Then there exists a natural number n such that $D(l^\varphi) > d_n$. Denote $\eta = D(l^\varphi) - d_n$. By the definition of $D(l^\varphi)$ there exists a sequence $x = (x^k)_{k=1}^\infty$ such that

$$\|x^k - x^l\|_\varphi > D(l^\varphi) - \frac{\eta}{3} \qquad (\forall k,\ l \in \mathbf{N};\ k \neq l). \tag{13}$$

Put $\varepsilon = (\eta/3)/2(d_n + \eta/3)$. In view of Lemma 3 there exists $\delta \in (0, \varepsilon)$ such that

$$I_\varphi(x+y) < I_\varphi(x) + \frac{\varepsilon}{2} \tag{14}$$

whenever $I_\varphi(x) \leq 1$ and $I_\varphi(y) < \delta$.

In virtue of Lemma 2 we can choose a subsequence $(y^k)_{k=1}^\infty$ of $(x^k)_{k=1}^\infty$ satisfying conditions (i), (ii) and (iii) from this lemma with $\delta/2$ instead of δ.

For any $n_1, n_2 \in \mathbf{N}$, $n \leq n_1 < n_2$, and any $y \in S(l^\varphi)$, we have $d_n \geq c(y, n_2 - n_1, n_1)$. Therefore,

$$\sum_{i=n_1}^{n_2} \varphi_i\left(\frac{y_i}{d_n + \frac{\eta}{3}}\right) \leq \sum_{i=n_1}^{n_2} \frac{d_n}{d_n + \frac{\eta}{3}} \varphi_i\left(\frac{y_i}{d_n}\right) \leq \tag{15}$$

$$\leq \frac{d_n}{d_n + \frac{\eta}{3}} \sum_{i=n_1}^{n_2} \varphi_i\left(\frac{y_i}{c(y, n_1, n_2 - n_1)}\right) \leq \frac{d_n}{2(d_n + \frac{\eta}{3})} = \frac{1}{2} - \varepsilon.$$

For all natural numbers k, l; $k < l$, we have

$$I_\varphi\left(\frac{y^k - y^l}{d_n + \eta/3}\right) = \left(\sum_{i=1}^{i_k} + \sum_{i=i_k+1}^{i_{k+1}} + \sum_{i=i_{k+1}+1}^{\infty}\right) \varphi_i\left(\frac{y_i^k - y_i^l}{d_n + \eta/3}\right). \tag{16}$$

We may assume without loss of generality that $i_k > n$. In view of (ii) in Lemma 2 and the inequality $d_n \geq 1$, we get

$$\sum_{i=1}^{i_k} \varphi_i\left(\frac{y_i^k - y_i^l}{d_n + \eta/3}\right) \leq \sum_{i=1}^{i_k} \varphi_i(y_i^k - y_i^l) < \frac{\delta}{2} \leq \frac{\varepsilon}{2}. \tag{17}$$

In virtue of (iii) in Lemma 2 and the inequality $d_n \geq 1$ we have

$$\sum_{i=i_k+1}^{i_{k+1}} \varphi_i\left(\frac{y_i^l}{d_n + \eta/3}\right) \leq \sum_{i=i_k+1}^{i_{k+1}} \varphi_i(y_i^l) < \delta. \tag{18}$$

Moreover, by (15) we have

$$\sum_{i=i_k+1}^{i_{k+1}} \varphi_i\left(\frac{y_i^k}{d_n+\eta/3}\right) \le \frac{1}{2} - \varepsilon. \tag{19}$$

Applying (14) with $(0,\ldots 0, -y_{i_k+1}^k, \ldots, -y_{i_{k+1}}^k, 0, \ldots)$ in place of x and $(0,\ldots 0, y_{i_k+1}^l, \ldots, y_{i_{k+1}}^l, 0, \ldots)$ in place of y, we get

$$\sum_{i=i_k+1}^{i_{k+1}} \varphi_i\left(\frac{y_i^k - y_i^l}{d_n+\eta/3}\right) < \frac{1}{2} - \frac{\varepsilon}{2}. \tag{20}$$

We have by Lemma 2 (i) and (iii) with $\delta/2$ instead of δ and $l = k+1$ that

$$\sum_{i=i_{k+1}+1}^{\infty} \varphi_i(y_i^j) < \delta \qquad (j = k, k+1),$$

so applying again inequality (14), we get

$$\sum_{i=i_{k+1}+1}^{\infty} \varphi_i\left(\frac{y_i^k - y_i^l}{d_n+\eta/3}\right) \le \sum_{i=i_{k+1}+1}^{\infty} \varphi_i(y_i^k - y_i^l) < \delta + \frac{\varepsilon}{2}. \tag{21}$$

Combining now inequalities (17), (20) and (21), we get

$$I_\varphi\left(\frac{y^k - y^l}{d_n+\eta/3}\right) \le \left(\frac{1}{2} - \frac{\varepsilon}{2}\right) + \left(\delta + \frac{\varepsilon}{2}\right) + \frac{\varepsilon}{2} \le 1,$$

whence it follows that

$$\|y^k - y^l\| \le d_n + \eta/3 = D(l^\varphi) - \frac{2\eta}{3},$$

which contradicts (13), and consequently finishes the proof.

Theorem 2. *If $\varphi = (\varphi_i)_{i=1}^\infty$ is a Musielak-Orlicz function which does not satisfy the δ_2^0-condition, then $\Lambda(l^\varphi) = \frac{1}{2}$.*

Proof. It follows from the assumption that l^φ contains an isomorphically isometric copy of l^∞. In the case when all φ_i are finite-valued this was proved by Kaminska in [6]. The proof can be repeated in the general case. Therefore,

$$2 = D(l^\infty) \leq D(l^\varphi) \leq 2,$$

whence we get $D(l^\infty) = 2$, i.e. $\Lambda(l^\varphi) = \frac{1}{2}$, what completes the proof.

Corollary 1. *Let $l^{(p_i)}$ be a Nakano space, where $l \leq p_i < +\infty$ for any $i \in \mathbf{N}$. Then:*

$$\Lambda(l^{(p_i)}) = \frac{1}{2}, \qquad \text{if } \limsup_{i\to+\infty} p_i = +\infty, \tag{22}$$

$$\Lambda(l^{(p_i)}) = 2^{\frac{1}{p}}/(2+2^{\frac{1}{p}}), \quad \text{if } \limsup_{i\to+\infty} p_i < +\infty\ , \tag{23}$$

where $p = \liminf_{i\to+\infty} p_i$.

Proof. The space $l^{(p_i)}$ is the Musielak-Orlicz space l^φ, where $\varphi = (\varphi_i)_{i=1}^\infty$ is the Musielak-Orlicz function with $\varphi_i(u) = |u|^{p_i}$ for each $i \in \mathbf{N}$ and $u \in \mathbf{R}$. It is obvious that φ satisfies the δ_2^0-condition if and only if $\limsup_{i\to+\infty} p_i < +\infty$ and that the δ_2^0-condition implies in this case condition (+). Therefore, (22) follows inmediately from Theorem 2.

Assume now that $\limsup_{i\to+\infty} p_i < +\infty$ and define $p = \liminf_{i\to+\infty} p_i$. Take an arbitrary $\varepsilon > 0$. There exists $j \in \mathbf{N}$ such that $p_i \geq p - \varepsilon$ for any $i \geq j$.

Given $\alpha > 0$ there exist $m \in \mathbf{N}$ and $x \in S(l^\varphi)$ such that $x_i = 0$ for $i \leq n$ and $c(x,m,n) \geq d_n - \alpha/2$. We have

$$\sum_{i=n}^{n+m} \varphi_i\left(\frac{x_i}{d_n-\alpha}\right) = \sum_{i=n}^{n+m} \left(\frac{x_i}{d_n-\alpha}\right)^{p_i} \geq \sum_{i=1}^{n+m} \left(\frac{x_i}{c(x,m,n)-\alpha/2}\right)^{p_i} \geq \frac{1}{2}$$

whence

$$\sum_{i=n}^{\infty} \varphi_i\left(\frac{x_i}{d_n - \alpha}\right) = \sum_{i=1}^{\infty} \left(\frac{x_i}{d_n - \alpha}\right)^{p_i} \geq \frac{1}{2}.$$

Then we get for any $x \in S(l^\varphi)$:

$$\frac{1}{2} \leq \sum_{i=n}^{\infty} (\frac{x_i}{d_n - \alpha})^{p_i} = \sum_{i=1}^{\infty} \frac{x^{p_i}}{(d_n - \alpha)^{p_i}} \leq \frac{1}{min[(d_n - \alpha)^p, (d_n - \alpha)^{p-\varepsilon}]}.$$

Hence, in view of the arbitrariness of $\varepsilon > 0$, we get $d_n - \alpha \leq 2^{1/p}$, whence $d_\varphi \leq 2^{1/p}$ by the arbitrariness of $\alpha > 0$.

Conversely, for any $\varepsilon > 0$ there is an infinite subset A of $\mathbf{N}$ such that $p_i \leq p + \varepsilon$ for any $i \in A$. Denote the sequence $(\varphi_i)_{i \in A}$ by Ψ. If $\varphi = (\varphi_i)_{i \geq 1}$ is a sequence of Orlicz functions and $\Psi = (\Psi_j)_{j \geq 1}$ is a subsequence of φ, then $d_\Psi \leq d_\varphi$. So, we can assume without loss of generality that $A = \mathbf{N}$ and $\Psi = \varphi$. Take any $m, n \in \mathbf{N}$ and $x \in S(l^\varphi)$ with $supp\ x = \{i \in \mathbf{N} : x_i \neq 0\} \subset \{n, n+1, \ldots, n+m\}$. We have

$$\frac{1}{2} = \sum_{i=n}^{n+m} \varphi_i\left(\frac{x_i}{c(x,m,n)}\right) = \sum_{i=n}^{n+m} \frac{|x_i|^{p_i}}{(c(x,m,n))^{p_i}}$$

$$\geq \frac{1}{[c(x,m,n)]^{p+\varepsilon}} \sum_{i=n}^{n+m} |x_i|^{p_i} = \frac{1}{[c(x,m,n)]^{p+\varepsilon}}.$$

Hence it follows that $d_\varphi \geq 2^{1/p}$ since $\varepsilon > 0$ is arbitrary.

Note. *The packing constant of c_0 is equal to $\frac{1}{2}$.*

Proof. Taking the sequence

$$x^1 = (1, 0, \ldots),\ x^2 = (-1, 1, 0, \ldots), \ldots, \tag{24}$$

$$x^n = (\underbrace{-1, \ldots, -1}_{(n-1)\ times}, 1, 0, \ldots), \ldots,$$

we have $x^n \in S(c_0)$ for any $n \in N$ and $\|x^m - x^n\|_\infty = 2$ for any $m, n \in \mathbf{N}$, $m \neq n$. Therefore, $D(c_0) = 2$, i.e. $\Lambda(c_0) = \frac{1}{2}$.

Theorem 3. *If* $\varphi = (\varphi_i)_{i=1}^\infty$ *is a Musielak-Orlicz function such that all functions* φ_i *are finite-valued then* $\Lambda(h^\varphi) = \Lambda(l^\varphi)$.

Proof. Assume first that φ satisfies the δ_2^0-condition. Then $h^\varphi = l^\varphi$ (see [5]), whence the desired equality follows.

Assume now that φ does not satisfy the δ_2^0-condition and take an arbitrary $\varepsilon > 0$. Then there exists a closed subspace l of h^φ and a linear operator $P: c_0 \xrightarrow{onto} l$ such that

$$\|x\|_\infty \leq \|Px\|_\varphi \leq (1+\varepsilon)\|x\|_\infty \qquad (\forall x \in c_0) \tag{25}$$

(see [1]). Consider the sequence $(x^n)_{n=1}^\infty$ defined in (24) and define a new sequence $(Px^n/\|Px^n\|_\varphi)_{i=1}^\infty$ in l. In view of (25) we have

$$\left\| \frac{x^i}{\|Px^i\|_\varphi} - \frac{x^j}{\|P\underline{x^j}\|_\varphi} \right\|_\infty \leq \left\| P\left(\frac{x^i}{\|Px^i\|_\varphi}\right) - P\left(\frac{x^j}{\|Px^j\|_\varphi}\right) \right\|,$$

and

$$\left| \frac{1}{\|Px^i\|_\varphi} - \frac{1}{\|px^j\|_\varphi} \right| \leq 1 - \frac{1}{1+\varepsilon} = \frac{\varepsilon}{1+\varepsilon} \leq \varepsilon. \tag{27}$$

Applying (27), we get

$$\left\| \frac{x^i}{\|Px^i\|_\varphi} - \frac{x^j}{\|Px^j\|_\varphi} \right\|_\infty \geq \left\| \frac{x^i - x^j}{\|Px^i\|_\varphi} \right\|_\infty - \left| \frac{1}{\|Px^i\|_\varphi} - \frac{1}{\|Px^j\|_\varphi} \right| \tag{28}$$

$$\geq \frac{\|x^i - x^j\|_\varphi}{1+\varepsilon} - \varepsilon = \frac{2}{1+\varepsilon} - \varepsilon.$$

Combining (26) and (28), we have

$$\left\| P\left(\frac{x^i}{\|Px^i\|_\varphi}\right) - P\left(\frac{x^i}{\|Px^j\|_\varphi}\right) \right\|_\varphi \geq \frac{2}{1+\varepsilon} - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary it follows that $D(h^\varphi) \geq D(l) = 2$. In view of the obvious inequality $D(h^\varphi) \leq 2$, we get $D(h^\varphi) = 2$. The proof is finished.

Acknowledgements. The authors thank the referee for valuable comments and remarks.

The results of this paper were obtained in September 1989 when the first named author visited Harbin Institute of Technology in Harbin, China. He want to thank Prof. Wu Congxin for invitation and very good hospitality and the government of Harbin Institute of Technology for financial support.

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Recibido: 9 de octubre de 1992
Revisado: 27 de octubre de 1993