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On the Variational Inequality Approach to Compressible Flows via Hodograph Method

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ABSTRACT. We study the flow of a compressible, stationnary and irrotational fluid with wake, in a channel, around a convex symmetric profile, with assigned velocity q_∞ at infinity and $q_s<q_\infty$ at the wake. In particular, we study the regularity of the free boundary (for a problem which has non-constant coefficients), in the hodograph plane.

Using variational inequalities, we obtain the solutions of some known cases as limit of this one, namely the solutions of the problems of an incompressible fluid in a channel without wake, in the plane with and without wake and the problem of a compressible fluid in the plane.

We also study the convergence of the free boundary of the new problem to the free boundaries of the limit corresponding problems.

1. INTRODUCTION

We study the flow with wake of a stationnary, irrotational, compressible fluid, with non-constant density ϱ, in a channel with semi-height h, around a convex symmetric profile, with given velocity q_∞ at infinity and $q_s<q_\infty$ on the wake.

A change of variable of Baiocchi type in the physical formulation of the problem, after a hodograph transformation, leads us to a variational inequality; we also prove that its solution converges to the limit cases of a fluid

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-in the channel, without wake, when $q_s \to 0$, $[(\varrho, h, 0)]$
-in the plane, with wake, when $h \to +\infty$, $[(\varrho, +\infty, q_s)]$
-in the plane, without wake, when $h \to +\infty$, $q_s \to 0$, $[(\varrho, +\infty, 0)]$
-incompressible, in each of the three situations above, when the density $\varrho \to 1$, $[(1, *, **)]$
being $* = h$, $+\infty$ and $** = q_s$, 0.

The problem $(1, +\infty, 0)$ was studied by Brézis and Stampacchia ([7]), the problem $(1, +\infty, q_s)$ by Brézis and Duvaut ([5]), the problem $(1, h, 0)$ by Tomarelli ([21]) and the problem $(\varrho, +\infty, 0)$ by Brézis ([4]). These problems were also regarded from the numerical point of view, namely by Bourgat and Duvaut ([3]) and by Bruch and Dormiani ([8]). These kind of problems were also studied by Díaz ([9]), Díaz and Dou ([10]), by Hummel ([13], for non-symmetric convex profiles) and extended by Shimborsky ([19], [20]) to plane channels, Venturi tubes and flow around a Joukowski airfoil.

We are going to extend the formulation of these problems to the compressible case (ϱ, h, q_s), in a channel, with wake, establishing that each of the previous cases is a (variational) limit in the hodograph plane, of this more complex case. We shall also prove the convergence of the free boundaries. These results for the incompressible case can be found in [18] and some of the results for the compressible case were announced in [17]. A new result is the study of the convergence of the solution of the problem of a compressible fluid to the solution of the problem of the incompressible one, when the density of the fluid ϱ becomes constant.

In section 2 we formulate the physical problem and introduce the hodograph transformation. In section 3, after a Baiocchi type change of variable ([2]) in the hodograph plane, we present the variational formulation of the problem, for which we know there exists a unique solution. In section 4 we study the regularity of the solution and free boundary of the solution of the variational inequality, concluding that it is possible to turn back to the physical plane and we establish the existence and uniqueness of solution of the physical problem. In section 5 we study, firstly the convergence of the solution of the problem with wake to the solution of the problem without wake, when the velocity at the wake goes to zero and afterwards the convergence of the solution of our problem to the solution of the problem in the plane, when the height of the channel becomes arbitrarily large. In section 6 we extend the convergence results of sections 4 and 5 to the free boundaries. In the last section we prove the convergence of the solution and free boundary of the compressible problem to the solution and free boundary of the incompressible problem, when the density of the fluid goes to one.

2. FORMULATION OF THE PROBLEM

Let $\mathscr{P}$ be a strictly convex profile with height $2H$, symmetric with respect to the OX axe. Consider the flow of a fluid around this profile in a channel of semi-height $h > H > 0$. We suppose the flow is uniform at infinity, with assigned velocity q_∞ parallel to the OX axe. The fluid has a given density ϱ, being

$$\varrho = g(q) \tag{1}$$

where q is the velocity of the fluid and g is a non-increasing C^1 function, bounded below by a strictly positive number m.

Let S be the (unknown) boundary of the wake. We assume S is a line, decreasing when x grows, which intersects the profile in its descending part. Let G be the exterior of the profile and wake in the channel.

By symmetry reasons, it is enough to work in the region $y \geq 0$. We denote by G^+ and $\mathscr{P}^+$ the intersections of G and $\mathscr{P}$ with the region $\{y \geq 0\}$.

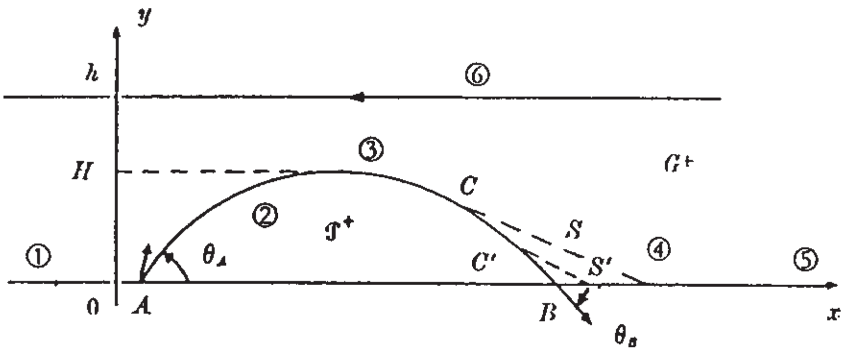

Fig. 1. The physical plane.

Let ψ be the *stream function* defined as follows:

$$\psi_x = -\varrho q_2, \qquad \psi_y = \varrho q_1.$$

Notice that $|\nabla \psi| = qg(q)$ and we can consider ψ as a function of x and y. Our problem can be formulated as follows:

$$\left\{\begin{array}{l} \psi \in C^2(G^+) \cap C^1(\overline{G^+}), \\ \left(1 - \dfrac{q_1^2}{a^2(q)}\right)\psi_{xx} + \left(1 - \dfrac{q_2^2}{a^2(q)}\right)\psi_{yy} - 2\dfrac{q_2 q_2}{a^2(q)}\psi_{xy} = 0 \text{ in } G, \\ |\nabla \psi| = q_s g(q_s) \text{ on } S, \\ \psi(x, h) = hq_\infty g(q_\infty), \ \forall\, x \in \mathbb{R}, \\ \psi = 0 \text{ on } \mathscr{P}^+ \cup S \text{ or if } y = 0, \\ \lim\limits_{|x| \to +\infty} \psi(x, y) = yq_\infty g(q_\infty), \text{ uniformly in } y, \end{array}\right. \tag{2}$$

being $a^2(q) = -q\dfrac{g(q)}{g'(q)}$.

Consider now ψ as function of q and θ, instead of x and y ($q_1 = q\cos\theta$, $q_2 = q\sin\theta$). Then

$$q\left(\frac{1}{qg(q)}\right)_q \psi_{\theta\theta} = \left(\frac{q}{g(q)}\psi_q\right)_q .$$

Suppose that the equation $a(q) = q$ has a positive solution. Denote by q_c the least positive solution of the equation $a(q) = q; q_c$ is called the *velocity of the sound* (later on, we will allow $q_c = +\infty$, $a(q) = q$ has no positive solution).

Notice that the second equation of problem (2) is elliptic in the subsonic domain ($q < q_c$) and hyperbolic in the supersonic domain ($q > q_c$). Our study is restricted to the first case.

Define

$$\sigma = \int_q^{q_c} \frac{g(\tau)}{\tau} dr \tag{3}$$

and

$$\sigma_\infty = \int_{q_\infty}^{q_c} \frac{g(\tau)}{\tau} d\tau, \quad s = \int_{q_s}^{q_c} \frac{g(\tau)}{\tau} d\tau, \quad \sigma_h = \int_{q_h}^{q_c} \frac{g(\tau)}{\tau} d\tau, \tag{4}$$

where $q_h = \max_{x \in \mathbb{R}} |\vec{q}(x, h)|$.

Assuming the fluid is totally subsonic, following Brézis ([4]), the profile $\mathscr{P}^+$ is transformed in a curve $\mathscr{L}$(which is a free boundary) contained in the region $\sigma > 0$ that will be denoted by $\sigma = l(\theta)$.

Let $R(\theta) = -[(X')^2(\theta) + (Y')^2(\theta)]^{\frac{1}{2}}$ the curvature radius of $\partial\mathscr{P}^+$ on the point $(X(\theta), Y(\theta))$, being $(X(\theta), Y(\theta))$ the unique point P of $\mathscr{P}^+$where the tangent at P to $\partial\mathscr{P}^+$ makes an angle $\theta(\theta < \pi/2)$ with the axe OX. We suppose

$$X, Y \in C^{1,\alpha}(\theta_B, \theta_A), \quad 0 < \alpha < 1.$$

We are going to work now in the hodograph plane, i.e., our coordinates are now (θ, σ) instead of (x, y). We still denote our function, now on the variables θ and σ, by ψ. The figure below shows how the hodograph transformation acts.

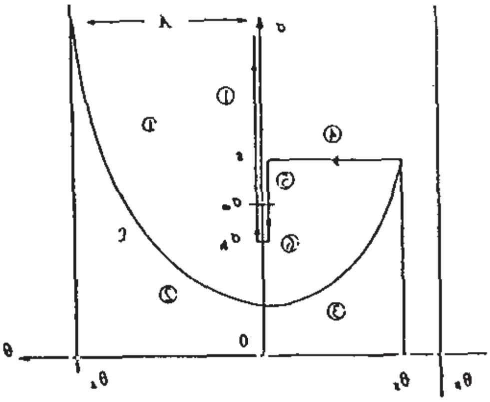

Fig. 2. The hodograph plane.

The angle between the wake and the profile in the point where both intersect is denoted by θ_s. The region G^+ is transformed in a region $\mathscr{D}$, defined as follows:

$$\mathscr{D} = \{(\theta,\sigma) : \theta_s<\theta<0,\ l(\theta)<\sigma<s\} \cup \{(\theta,\sigma) : 0<\theta<\theta_A,\ l(\theta)<\sigma\} \cup \{(0,\sigma) : l(0)<\sigma<\sigma_h\}.$$

Since $\vec{q}\cdot\vec{n}=0$ on S, we conclude that $\psi\equiv 0$ on S.

Define

$$\Omega_s =]\theta_B,0]\times]0,s[\cup]0,\ \theta_A[\times\mathbb{R}^+, \qquad \Gamma_s=\{(\theta,\sigma):\theta_s<\theta<0\},$$

$$\Sigma_h=\{(0,\sigma):\sigma_h<\sigma<\sigma_\infty\}, \qquad (5)$$
$$\Sigma_\infty=\{(0,\sigma):\sigma_\infty<\sigma<+\infty\}.$$

Problem (2) is equivalent to the following one:

To find ψ defined in Ω_s and $\mathscr{D}\subseteq\Omega_s$ such that:

$$\left|\begin{array}{l} k(\sigma)\psi_{\theta\theta}+\psi_{\sigma\sigma}=0 \text{ in } \mathscr{D}, \\ \left\{\begin{array}{l}\psi=0 \\ \psi_\sigma=-\dfrac{R(\theta)q(\sigma)}{1+k(\sigma)l'^2(\theta)}\end{array}\right. \quad \text{on } \mathscr{L}, \\ \psi=0 \text{ on } \Gamma_s, \\ \psi=0 \text{ on } \Sigma_\infty, \\ \psi=hq_\infty\ g(q_\infty) \text{ on } \Sigma_h, \\ \psi\to 0 \text{ when } \sigma\to+\infty,\ \theta>0, \end{array}\right. \qquad (6)$$

being

$$k(\sigma)=\frac{1}{g^2(q(\sigma))}\left(1-\frac{q^2(\sigma)}{a^2(q(\sigma))}\right). \tag{7}$$

3. THE VARIATIONAL INEQUALITY

In this section we assume the existence of a regular solution ψ of the problem in the physical plane, verifying the following physically natural assumption:

«(*) The line in the physical plane, correspondent to $\theta=0$, intersects each line paralell to the axe OX at most in one point».

Remark 3.1. Tomarelli established in [21] that the solution to the problem of the incompressible fluid in a channel verifies this property.

Let us make now a change of variable of Baiocchi type:

$$u(\theta,\sigma)=\begin{cases}\displaystyle\int_{l(\theta)}^{\sigma}\frac{k(\tau)}{q(\tau)}\psi(\theta,\tau)d\tau & \text{if } (\theta,\sigma)\in\mathscr{D}\\ 0 & \text{if } (\theta,\sigma)\in\Omega_s\setminus\mathscr{D}.\end{cases} \tag{8}$$

Proposition 3.2. *The function u verifies the following properties:*

$$u>0 \text{ on } \mathscr{D},\ \frac{1}{q^2}\left(\frac{q^2}{k}u_\sigma\right)_\sigma+u_{\theta\theta}+u=-R(\theta) \text{ on } \mathscr{D},$$

$$u=u_\theta=u_\sigma=0 \text{ on } \mathscr{L},\ u_\sigma=0 \text{ on } \Gamma_s,$$

$$u(0,\sigma)\begin{cases}=H & \text{if } \sigma\geq\sigma_\infty,\\ \displaystyle=H-g(q_\infty)hq_\infty\int_\sigma^{\sigma_\infty}\frac{k(\tau)}{q(\tau)}d\tau & \text{if } \sigma_h\leq\sigma<\sigma_\infty,\\ \displaystyle\geq H-g(q_\infty)hq_\infty\int_\sigma^{\sigma_\infty}\frac{k(\tau)}{q(\tau)}d\tau & \text{if } l(0)<\sigma<\sigma_h.\end{cases}$$

Proof. For the proof of the three first properties, see [4].

Since $u_\sigma=\dfrac{k(\sigma)}{q(\sigma)}\psi$, we have $u_\sigma(\theta,s)=0$, and so $u_\sigma=0$ on Γ_s.

If $\sigma \geq \sigma_\infty$, following [4] we conclude that $u(0,\sigma)=H$.

Notice that a point $(0,\sigma)$ of Σ_h is image, by the hodograph transformation, of a point $(x,\ h)$, and $\psi(x,\ h)=hq_\infty g(q_\infty)$ (see (5)). If σ is such that $l(0)<\sigma<\sigma_h$, $(0,\sigma)$ is image of a point of G^+. As $\psi=0$ on $\partial\mathscr{P}^+\cup S\cup\{y=0\}$, $\psi(x,\ h)=hg(q_\infty)q_\infty$ and $\lim_{|x|\to+\infty}\psi(x,\ y)=yg(q_\infty)q_\infty\leq hg(q_\infty)q_\infty$, by the maximum principle (since ψ satisfies an elliptic equation in G^+), we conclude that $\psi(0,\sigma)<hq_\infty g(q_\infty)$.

Let $\sigma<\sigma_\infty$. Then

$$
\begin{aligned}
u(0,\sigma) &= \int_{l(0)}^{\sigma} \frac{k(\tau)}{q(\tau)}\psi(0,\tau)d\tau \\
&= \int_{l(0)}^{\sigma_\infty} \frac{k(\tau)}{q(\tau)}\psi(0,\ \tau)d\tau - \int_{\sigma}^{\sigma_\infty} \frac{k(\tau)}{q(\tau)}\psi(0,\ \tau)d\tau \\
&= H - \int_{\sigma}^{\sigma_\infty} \frac{k(\tau)}{q(\tau)}\psi(0,\ \tau)d\tau,
\end{aligned}
$$

which concludes the proof. □

Let $\Gamma=]\theta_B,\ 0]\times\{s\}$. Define

$$V_s=\{v: qv\in L^2(\Omega_s),\ qv_\theta\in L^2(\Omega_s),\ \frac{q}{\sqrt{k}}v_\sigma\in L^2(\Omega_s),\ v|_{\partial\Omega\setminus\Gamma_s}\equiv 0\}, \tag{9}$$

$$\|v\|^2_{V_s}=\int_{\Omega_s} q^2\left(\frac{1}{k}\ v_\sigma^2+v_\theta^2+v^2\right)d\theta\, d\sigma. \tag{10}$$

Define on V_s the following bilinear symmetric form

$$a(u,\ v)=\int_{\Omega_s} q^2\left(\frac{1}{k}\ u_\sigma\, v_\sigma+u_\theta v_\theta-uv\right)d\theta\, d\sigma, \tag{11}$$

and the convex set

$$\mathbb{K}=\{v\in V_s: v\geq 0,\ v(0,\sigma)=H \text{ if } \sigma\geq\sigma_\infty,\ \ v(0,\sigma)\geq\eta(\sigma) \text{ if } \sigma<\sigma_\infty\}, \tag{12}$$

where

$$\eta(\sigma)=H-hq_\infty g(q_\infty)\int_{\sigma}^{\sigma_\infty}\frac{k(\tau)}{q(\tau)}d\tau. \tag{13}$$

Lemma 3.3 *a is coercive in* $\mathbb{K}$, *that is,*

$$\exists\, \alpha>0 \ \forall\, u,\ u_0 \in \mathbb{K}\ a(u-u_0,\ u-u_0)\geq \alpha \|u-u_0\|^2_{V_s}. \tag{14}$$

Proof. The proof is analogous to the one found in [4] for the problem in the whole plane, without wake, so we omit it. □

Lemma 3.4 $u \in \mathbb{K}$.

Proof. Results directly from Proposition 3.2.

Lemma 3.5.

$$u_\theta(0^-,\sigma)-u_\theta(0^+,\sigma)\geq 0 \quad \textit{if } \sigma_h<\sigma<\sigma_\infty.$$

Proof. Reasoning as in [4], we conclude that

$$u(\theta,\sigma)=\frac{1}{\varrho q}\ \psi+\sin\theta[x(\theta,\sigma)-X(\theta)]-\cos\theta[y(\theta,\sigma)-Y(\theta)]$$

where $(x(\theta,\sigma),\ y(\theta,\sigma))$ is the point of the physical plane applied by the hodograph transformation in (θ,σ) and $(X(\theta),\ Y(\theta))$ is the parametrization of the profile indicated in the previous section.

Since

$$\sin\theta=-\frac{1}{\varrho q}\psi_x, \qquad \cos\theta=\frac{1}{\varrho q}\psi_y,$$

we conclude that

$$u(\theta,\sigma)=\frac{1}{\varrho q}\ (\psi-x\psi_x-y\psi_y)-X(\theta)\sin\theta+Y(\theta)\cos\theta,$$

and as

$$d(\psi-x\psi_x-y\psi_y)=-x\,d\psi_x-y\,d\psi_y,$$

we have

$$u_\theta(\theta,\sigma) = \frac{1}{\varrho q}[-x(\psi_x)_\theta - y(\psi_y)_\theta] + [-X(\theta)\sin\theta + y(\theta)\cos\theta]_\theta =$$

$$= \frac{1}{\varrho q}[x\varrho q\cos\theta + y\varrho q\sin\theta] + [-X(\theta)\sin\theta + Y(\theta)\cos\theta]_\theta.$$

Then

$$u_\theta(0^-,\sigma) - u_\theta(0^+,\sigma) = x(0^-,\sigma) - x(0^+,\sigma).$$

The property (*) imposed to the solution of the physical problem guarantees us that the line $\theta=0$, $l(0)\leq\sigma\leq\sigma_h$ in the hodograph plane corresponds, in the physical plane, to a one-to-one line that joins the unique point of the profile where $\theta=0$ (the point of the profile of maximum height), to a point of the line $y=h$ where the velocity is maximum. The complement of this curve in G^+ has two connected components, one the set of the points where $\theta<0$ in the hodograph plane and the other the set of the points where $\theta>0$ in the hodograph plane. It is now obvious that $x(0^-,\sigma)-x(0^+,\sigma)\geq 0$. □

Theorem 3.6 *u is the unique solution of the variational inequality*

$$\begin{cases} u\in\mathbb{K}, \\ a(u, v-u)\geq \int_{\Omega_s} q^2(\sigma)R(\theta)(v-u)\,d\theta\,d\sigma, \qquad \forall\, v\in\mathbb{K}. \end{cases} \tag{15}$$

Proof.

$$a(u, v-u) = \int_{\Omega_s} q^2\left[\frac{1}{k}u_\sigma(v-u)_\sigma + u_\theta(v-u)_\theta - u(v-u)\right]d\theta\,d\sigma$$

$$= \int_{\mathcal{S}} q^2\left[\frac{1}{k}u_\sigma(v-u)_\sigma + u_\theta(v-u)_\theta - u(v-u)\right]d\theta\,d\sigma$$

$$= -\int_{\mathcal{S}} \left\{\left(\frac{q^2}{k}u_\sigma\right)_\sigma(v-u) + q^2u_{\theta\theta}(v-u)\right.$$

$$\left. + q^2u(v-u)\right\}d\theta\,d\sigma + \int_{\partial\mathcal{S}}\left(u_\theta, \frac{q^2}{k}u_\sigma\right)\cdot\vec{n}(v-u),$$

where $\partial\mathcal{S} = \mathcal{L}\cup\Sigma_\infty\cup\Sigma_h\cup\Gamma_s$ and $\vec{n}$ is the exterior normal vector to $\partial\mathcal{S}$.

Let $f(u, v)=\left(u_\theta, \frac{q^2}{k}u_\sigma\right)\cdot\vec{n}(u-v)$. On $\mathcal{L}$ we have $u_\theta=u_\sigma=0$, so $\int_{\mathcal{L}} f(u, v)=0$. On Σ_∞, since $v(0,\sigma)=u(0,\sigma)=H$ we have $\int_{\Sigma_\infty} f(u, v)=0$. On Γ_s, $\vec{n}=(0, 1)$ and $u_\sigma=0$, so $\int_{\Gamma_s} f(u, v)=0$.

On $\Sigma_h, f(u, v) = (u_\theta(0^-,\sigma) - u_\theta(0^+,\sigma))(v(0,\sigma) - u(0,\sigma))$. Using the previous lemma we know that $u_\theta(0^-,\sigma) - u_\theta(0^+,\sigma) \geq 0$. On the other hand, $v(0,\sigma) \geq \eta(\sigma) = u(0,\sigma)$ on Σ_h,so $\int_{\Sigma_h} f(u, v) \geq 0$.

Then

$$a(u, v-u) \geq -\int_{\mathscr{D}} \left(\left(\frac{q^2}{k} u_\sigma \right)_\sigma + q^2 u_{\theta\theta} + q^2 u \right) (v-u) d\theta d\sigma =$$

$$= \int_{\mathscr{D}} q^2 R(\theta)(v-u) d\theta d\sigma,$$

by one of the properties of u proved in Proposition 3.2. Since $v \geq 0$ in Ω_s, $u \equiv 0$ in $\Omega_s \setminus \mathscr{D}$, $R<0$, we have

$$\int_{\mathscr{D}} q^2 R(\theta)(v-u) d\theta d\sigma \geq \int_\Omega q^2 R(\theta)(v-u) d\theta d\sigma,$$

concluding then that

$$a(u, v-u) \geq \int_\Omega q^2 R(\theta)(v-u) d\theta d\sigma.$$

The uniqueness of solution is a direct consequence of the fact that a is a symmetric bilinear coercive form. □

Remark 3.7. Related with this problem are the problems of flows of a compressible fluid with prescribed velocity q_∞ at infinity, around a convex symmetric profile $\mathscr{P}$, in the three situations below:

(i) in a channel with semi-height h, without wake, $[(h, 0)]$,
(ii) in the plane, with wake, $[(+\infty, q_s)]$,
(iii) in the plane, without wake, $[(+\infty, 0)]$,

The physical formulation of these problems corresponds to omit in (2) the references to the wake, the channel or both.

Concerning the variational formulation, we have in each problem the variational inequality (15), being the space V_s defined by (9) with the norm (10), being $s = +\infty$ and $\Gamma_s = 0$ in the cases (i), (iii). The convex for each one of the three problems are the following ones:

— in case (i):

$$\mathbb{K}_h = \{v \in V_\infty : v \geq 0,\ v(0,\sigma) = H \text{ if } \sigma \geq \sigma_\infty,\ v(0,\sigma) \geq \eta(\sigma) \text{ if } \sigma < \sigma_\infty\}; \quad (16)$$

— in case (ii):

$$\mathbb{K}_s = \{v \in V_\infty : v \geq 0,\ v(0,\sigma) = H \text{ if } \sigma \geq \sigma_\infty\}; \tag{17}$$

— in case (iii):

$$\mathbb{K}_\infty = \{v \in V_\infty : v \geq 0,\ v(0,\sigma) = H \text{ if } \sigma \geq \sigma_\infty\}. \tag{18}$$

If we need to distinguish among the four diferent problems refered here, we put a subscript ∞, s, h or sh on the solutions, functions or convex sets related with the problems $(\infty, 0)$, (∞, q_s), $(h, 0)$ or (h, q_s), respectively.

The problem treated here and in the next section is the case (h, q_s) but all the results are easily ajustable to the other three cases.

4. REGULARITY OF THE SOLUTION

We are going to look now to (15) as a variational inequality in itself, independently of its origin, being $q(\sigma)$ and $k(\sigma)$ defined by

$$\frac{q'(\sigma)}{q(\sigma)} = -\frac{1}{g(q(\sigma))},\ q(0)=1, \tag{19}$$

$$k(\sigma) = \frac{1}{g^2(q(\sigma))}\left(1 + \frac{q(\sigma)g'(q(\sigma))}{g(q(\sigma))}\right), \tag{20}$$

assuming from now on that $g \in W^{3,\infty}(0, q_c)$, in order to garantee the boundedeness of the first and second derivatives of k and q.

Notice that σ_∞ is given and $q_\infty = q(\sigma_\infty)$; s is equally given, $s > \sigma_\infty$, $q_s = q(s)$.

Since the variational inequality (15) has a unique solution we conclude that the physical problem has at most one solution physically natural, i.e., satisfying the property (*); if we prove that u has enough regularity we can turn back to the physical plane, establishing the existence of a physically natural solution of the initial problem.

We begin with some important properties of the solution of (15), denoted in this section just by u, omiting the subscripts, since there is no risk of confusion.

Proposition 4.1. *Let u be the solution of (15). Then*

$$0 \leq u(\theta,\sigma) \leq H, \quad \forall\ (\theta,\sigma) \in \Omega_s, \tag{21}$$

$$\|u\|_{V_s} \leq C,\ C \text{ constant independent of } s \text{ and } h. \tag{22}$$

Proof. Let $v = u - (u - H\ cos\theta)^+$. It is easy to verify that $v \in \mathbb{K}$ and $a(u-v,\ u-v) \leq 0$, using the fact that u is solution of problem (15) and direct calculations. Then

$$\|(u-H\ cos\theta)^+\|^2 = \|u-v\|^2 \leq \frac{1}{\alpha} a(u-v,\ u-v) \leq 0,$$

and so, $u \leq H cos\theta \leq H$ in Ω_s.

To prove (21) consider $\varphi \in C^2(\Omega_s) \cap V_s$, $\varphi(0,\sigma) = 1$ if $\sigma \geq \sigma_h$, $0 \leq \varphi \leq 1$, and let $w(\theta,\sigma) = H\varphi(\theta,\sigma)$. Obviously $w \in \mathbb{K}$.

Noticing that, for $w \in \mathbb{K}$ we have

$$\|u\|^2 \leq \frac{1}{\alpha} a(u,\ u) = \frac{1}{\alpha} a(u,\ w) - \frac{1}{\alpha} a(u,\ w-u)$$

and

$$-a(u,\ w-u) \leq -\int_{\Omega_s} R(\theta) q^2 (w-u)\, d\theta d\sigma \leq -\int_{\Omega_s} R(\theta) q^2 w\, d\theta d\sigma \leq C_1,$$

C_1 positive, since

$$a(u,\ w) \leq \int_{\Omega_s} q^2 \left(\frac{1}{k} u_\sigma w_\sigma + u_\theta w_\theta + uw \right) \leq \|u\| \|w\|,$$

we have

$$\|u\|^2 \leq \frac{C_1}{\alpha} + \frac{1}{\alpha} \|u\| \|w\|,$$

concluding then that

$\|u\| \leq C$, C depending only on α, C_1 and w. □

Define

$$\mathscr{D}=\{(\theta,\sigma)\in\Omega_s:u(\theta,\sigma)>0\},\qquad (\mathscr{D}\text{ is an open set});$$

$$\theta_*=\inf\,\{\theta\in]\theta_B,\,0[:u(\theta,\,s)>0\},$$

$$\sigma_*=\inf\left\{\sigma:u(0,\sigma)=H-hq_\infty g(q_\infty)\int_\sigma^{\sigma_\infty}\frac{k(\tau)}{q(\tau)}\mathrm{d}\tau\right\},$$

$$\Omega_A=\Omega_s\cap\{\theta>0\},\quad \Omega_B=\Omega_s\cap\{\theta<0\},$$

$$\Sigma_\infty=\{0\}\times[\sigma_\infty,\,+\infty[,\ \Sigma_*=\{0\}\times]\sigma_*,\sigma_\infty[,\ \Gamma_*=]\theta_*,\,0[\times\{s\}.$$

Proposition 4.2. *For every α, $0<\alpha<1$ and some β, $0<\beta<1$, we have:*

a) $u\in C^{2,\alpha}(\mathscr{D}\setminus\Sigma_\infty\cup\Sigma_*\cup\Gamma_*)$,

$u\in C^{1,\alpha}(\Omega_A\cup\Sigma_\infty)$, $u\in C^{1,\alpha}(\Omega_B\cup\Sigma_\infty\cup\Gamma_*)$.

b) $u\in C^{1,\beta}(\Omega_A\cup\Sigma_*)$, $u\in C^{1,\beta}(\Omega_B\cup\Sigma_*)$.

Proof. This proposition is a direct consequence of standard results on partial differential elliptic equations ([11], [14]), since $R\in C^{0,\alpha}(\theta_B,\,\theta_A)$, of a result of Grisvard ([12]) which establishes the $W^{2,p}$ regularity (for all p, $1\leq p<+\infty$) near the corner $\Sigma_\infty\cap\bar{\Gamma}_*=\{(0,\,s)\}$ (for details see Lemma 5.1) and a result of Caffarelli for the Signorini problem ([11]). □

Theorem 4.3. *The free boundary $\mathscr{L}=\Omega_s\cap\partial\{(\theta,\sigma):u\,(\theta,\sigma)=0\}$ is a graph $\sigma=l(\theta)$ uniformly Lipschitz in s and h, on any compact subset of $]\theta_*,\,\theta_A[$. The Lipschitz constant depends on $\|u\|_{C^1(K)}$, (being K any compact subset of Ω_s), $\max_{\theta\in[\theta_B,\,\theta_A]}|R(\theta)|$, $\max_{\theta\in[\theta_B,\,\theta_A]}|R'(\theta)|$, $\|q\|_{W^{2,\infty}}$ and on $\|k\|_{W^{2,\infty}}$.*

Proof. Since $\psi=\dfrac{q}{k}u_\sigma$ on $\mathscr{D}$, using that $\dfrac{1}{q^2}\left(\dfrac{q^2}{k}u_\sigma\right)_\sigma+u_{\theta\theta}+u=-R(\theta)$, differentiating this equality in order to σ and noticing that $\dfrac{k}{q}=-\left(\dfrac{q_\sigma}{q^2}\right)_\sigma$, we conclude that $k(\sigma)\psi_{\theta\theta}+\psi_{\sigma\sigma}=0$ on $\mathscr{D}\setminus(\Sigma_\infty\cup\Sigma_*)$. As $\psi=0$ on $\mathscr{L}\cup\Sigma_\infty\cup\Gamma_*$ and $\psi\geq0$ on Σ_*, since on Σ_* $u(0,\sigma)=H-hq_\infty g(q_\infty)\displaystyle\int_\sigma^{\sigma_\infty}\frac{k(\tau)}{q(\tau)}dr$, we conclude, by the strong maximum principle, that $\psi>0$ in $\mathscr{D}$. Then $u_\sigma>0$ in $\mathscr{D}\setminus(\Sigma_\infty\cup\Sigma_*)$ and

$$\forall\ \theta\in]\ \theta_*,\ \theta_A[\ \exists!\sigma_\theta:\quad u(\theta,\sigma)>0\ \forall\ \sigma\geq\sigma_\theta$$

and we can define an upper semi-continuous parametrization of $\mathscr{L}$,

$$l(\theta)=\inf\ \{\sigma\in\mathbb{R}^+:u(\theta,\sigma)>0\},\ \forall\ \theta\in]\theta_*,\ \theta_A[.$$

Observe that $l(0)<\sigma_*$, since $u_\sigma(0,\sigma_*^+)>0$.

Let us verify now that $\mathscr{L}$ does no contain vertical segments. Reasoning by contradiction, let $\gamma\subset\mathscr{L}$ be such a segment. Then $\psi=0$ and $\psi_\theta=\dfrac{q}{k}(u_\theta)_\sigma=0$ on γ, since $u_\theta=0$ on $\mathscr{L}$; on the other hand, each point of γ is a point of minimum of the function ψ and, by the Hopf maximum principle, $\dfrac{\partial\psi}{\partial n}=\pm\psi_\theta>0$ on γ (n normal unitary vector), which is absurd.

To conclude that l is Lipschitz in a neighborhood of $(\theta_0,\sigma_0)\in\mathscr{L}$, we are going to adapt to this case an argument of Alt ([1], [16]) for elliptic operators with constant coefficients. Let L be the operator

$$Lw=-\left[\frac{1}{q^2}\left(\frac{q^2}{k}w_\sigma\right)_\sigma+w_{\theta\theta}+w\right],$$

$$G=\{u>0\}\cap]\theta_0-\varrho,\theta_0+\varrho[\times]\sigma_0-\varrho,\sigma_0+\varrho[\subset\Omega_s,$$

$$F=\{(\theta,\sigma)\in]\theta_0-\varrho',\ \theta_0+\varrho'[\times]\sigma_0-\varrho',\sigma_0+\varrho'[:\sigma\geq l(\theta)\},\ \varrho'<\varrho,$$

where ϱ will be choosen conveniently, suficiently small (in particular in such a way that $\bar{G}\subset\Omega_s$).

Let $w=u+\varepsilon u_\theta-C\dfrac{1}{\sqrt{k}}u_\sigma$. We will prove that

$$\exists C_0>0\ \forall\ C,\ C_0\leq C\leq 2C_0\ \exists\varepsilon_0>0\ \forall\ \varepsilon,\ |\varepsilon|<\varepsilon_0;\ w\leq 0\text{ in }F. \tag{23}$$

Clearly we have

$$Lw=Lu+\varepsilon(Lu)_\theta-CL\left(\frac{1}{\sqrt{k}}u_\sigma\right).$$

Noticing that

$$L(fu_\sigma)=fLu_\sigma-\frac{1}{q^2}\left(\frac{q^2}{k}f_\sigma u_\sigma\right)_\sigma-\frac{f_\sigma}{k}u_{\theta\theta},$$

where $f=f(\sigma)$, and

$$Lu_\sigma=(Lu)_\sigma+\left[\frac{1}{q^2}\left(\frac{q^2}{k}\right)_\sigma\right]_\sigma u_\sigma+\left(\frac{1}{k}\right)_\sigma u_{\sigma\sigma},$$

we can verify that

$$L\left(\frac{1}{\sqrt{k}}u_\sigma\right)=\left\{\frac{1}{q^2}\left[\frac{q^2}{k}\left(\frac{1}{\sqrt{k}}\right)_\sigma\right]_\sigma-\frac{1}{q^2}\left[\frac{q^2}{k}\left(\frac{1}{\sqrt{k}}\right)_\sigma\right]_\sigma\right\}u_\sigma$$

Let $-m$ be a lower bound ($m\geq 0$) of

$$\left\{\frac{1}{\sqrt{k}}\left[\frac{1}{q^2}\left(\frac{q^2}{k}\right)_\sigma\right]_\sigma-\frac{1}{q^2}\left[\frac{q^2}{k}\left(\frac{1}{\sqrt{k}}\right)_\sigma\right]_\sigma\right\}$$

Then $Lw\leq R(\theta)+\varepsilon R'(\theta)+Cmu_\sigma$.

Let

$$\alpha=\max_{\theta\in[\theta_B,\theta_A]}\{R(\theta)\}<0$$

Since $u_\sigma=0$ on Γ, using the preceding proposition, we conclude that, given $C_0>0$, choosing ϱ suficiently small, we have $2C_0mu_\sigma\leq-\alpha/4$ in G. There exists also an $\varepsilon_1>0$ such that, if $|\varepsilon|\leq\varepsilon_1$, then $\varepsilon R'(\theta)\leq-\alpha/4$. So,

$$\forall\, C,\ C_0\leq C\leq 2C_0\quad \forall\,\varepsilon,\ |\varepsilon|\leq\varepsilon_1\quad Lw\leq\alpha/2.$$

Fix a non-negative function ζ of class $C^2(G)$ such that $\zeta=0$ in F and $\zeta\geq 1$ on $\partial G\setminus(\partial\mathscr{D}\cap\Omega_s)$. Choose $\mu>0$ (that will depend only on L, α, ϱ and ϱ', and consequently on $\|q\|_{W^{2,\infty}}$ and $\|k\|_{W^{2,\infty}}$ through ϱ) such that

$$-\mu L\zeta\leq-\frac{\alpha}{2}.$$

If we prove that

$$w\leq\mu \text{ on } \partial G\setminus(\partial\mathscr{D}\cap\Omega_s), \tag{24}$$

since $w=0$ in $\partial\mathscr{D}\cap\Omega_s$, using the maximum principle, we conclude that $w-\mu\zeta\leq 0$ on G, establishing then (23).

To prove (24), recall that $u_\sigma>0$ in $\mathscr{D}$ and define

$$\gamma(\varepsilon)=\inf\left\{\frac{1}{\sqrt{k(\sigma)}}u_\sigma(\theta,\sigma):(\theta,\sigma)\in G \text{ and } u(\theta,\sigma)>\varepsilon\right\}.$$

Let us see that $\gamma(\varepsilon)>0$. Argue by contradition. Then there exists a sequence $(\xi_n)_n$ of elements of G converging to an element $\xi_0\in G\cup\partial\Omega$ such that $\frac{1}{\sqrt{k(\sigma_n)}}u_\sigma(\xi_n)\to\gamma(\varepsilon)\leq 0$. Then we have $u(\xi_0)\geq\varepsilon$ and $u_\sigma(\xi_0)=\sqrt{k(\xi_0)}\gamma(\varepsilon)\leq 0$, which contradicts the fact that u_σ is strictly positive in $\mathscr{S}$.

Let $M=\|u\|_{C^1(\overline{G})}$ and define

$$\varepsilon_0=\min\left\{\frac{\mu}{1+M},\ \varepsilon_1\right\},\quad C_0=\frac{M}{\gamma(\varepsilon_0)}.$$

Let $(\theta,\sigma)\in\partial G\setminus(\partial\mathscr{S}\cap\Omega_s)$ be such that $u(\theta,\sigma)\leq\varepsilon_0$. Then $w(\theta,\sigma)\leq(1+M)\varepsilon_0\leq\mu$.

Let (θ,σ) be now a point of $\partial G\setminus(\partial\mathscr{S}\cap\Omega_s)$ such that $u(\theta,\sigma)>\varepsilon_0$. Then we have $w\leq M+\varepsilon_0 M-M\leq\mu$. We have then established (24) and from this we easily deduce that F verifies the (uniform) cone property (or equivalently ∂F is Lipschitz) since

$$(-\varepsilon,\ C)\cdot\nabla u\geq 0\quad\forall\ C\geq C_0 a\quad\forall\ \varepsilon:|\varepsilon|\leq\varepsilon_0,$$

where a is a positive lower bound of the function $\sqrt{k}$ in $\overline{G}$, which exists by the continuity and strict positivity of k in $[0,\ b]$, for any b such that $0<b<q_c$. □

Theorem 4.4.

$$l\in C^{1,\alpha}_{loc} \text{ and } u\in C^{1,\alpha}(\mathscr{S}\cup\mathscr{L}).$$

Proof. These results are direct consequence of known theorems on the regularity of the free boundaries (once more since $R\in C^{0,\alpha}$) and of Theorem 4.3 ([11], [14], [16]).

Observe that, established the regularity of the function u it is easy to verify the possibility of returning to the physical plane through the inverse transformation of the hodograph and that $\psi=\frac{q}{k}u_\sigma$ is the unique physically natural solution of the initial problem, with the desired regularity.

5. THE CONVERGENCE OF THE SOLUTION TO THE COMPRESSIBLE LIMIT CASES

In this section we are going to let $s \to +\infty$ firstly (it corresponds to let the velocity of the wake $q_s \to 0$) and prove that the solution of problem $(*, q_s)$ converges to the solution of problem $(*, 0)$, being $* = h, +\infty$. The proofs are presented just for the case $h < +\infty$, being the other case similar. The second step is to let $h \to +\infty$ and prove that the solution of problem $(+\infty, q_s)$ is the limit of the solutions of problems (h, q_s), $q_s \geq 0$. The proof presented here is just for the case $q_s > 0$, since the other case is analogous.

Lemma 5.1.

$$|(u_{sh})_\theta(0^-, s)| \leq C,$$

C contant independent of s and h.

Proof. Denote, to simplify, u_{sh} by u.

Let $\varphi \in C^\infty(\mathbb{R})$ be uniformly Lipschitz, $0 \leq \varphi \leq 1$, and such that $\varphi \equiv 1$ if $s - \sigma < \varepsilon$, $\varphi \equiv 0$ if $\sigma < s - 3\varepsilon$, ε fixed less than $(s - \sigma_\infty)/4$. Let $Q =]\theta_B, 0[\times]\sigma_\infty, s[$ and $w = \varphi u$. Let

$$L = \frac{1}{q^2}\frac{\partial}{\partial\sigma}\left(\frac{q^2}{k}\frac{\partial}{\partial\sigma}\right) + \frac{\partial^2}{\partial\theta^2} - I.$$

As $Lu = -R(\theta)\chi_{\{u>0\}}$, w is the solution of problem

$$\left\{\begin{array}{l} Lw = -\varphi R(\theta)\chi_{\{u>0\}} + \dfrac{1}{q^2}\left(\dfrac{q^2}{k}\varphi_\sigma u\right)_\sigma + \dfrac{1}{k}\varphi_\sigma u_\sigma, \\ w_\sigma = 0 \text{ on } \Gamma_s \cup \Gamma_0, \\ w = \varphi H \text{ on } \Sigma_0, \\ w = 0 \text{ on } \Sigma_{\theta_B}, \end{array}\right. \tag{25}$$

where $\Gamma_0 =]\theta_B, 0[\times \{\sigma_\infty\}$, $\Sigma_0 = \{0\} \times]\sigma_\infty, s[$, $\Sigma_{\theta_B} = \{\theta_B\} \times]\sigma_\infty, s[$.

Let

$$f = -\varphi R(\theta)\chi_{\{u>0\}} + \frac{1}{q^2}\left(\frac{q^2}{k}\varphi_\sigma u\right)_\sigma + \frac{1}{k}\varphi_\sigma u_\sigma.$$

Using the properties of k, q and u, we can easily conclude that $f \in L^2(Q)$ and so w, which is the solution of $Lw=f$, belongs to $H^2(Q)$. But, as

$$H^2(Q) \hookrightarrow W^{1,p}(Q), \ \forall\, p<+\infty,$$

we have $w \in W^{1,p}(Q)$ and $f \in L^p(Q)$, $\forall\, p<+\infty$.

By a result of Grisvard ([12]), since the necessary compatibility conditions are verified on the vertices and interior of the rectangle Q, we can assert that $w \in W^{2,p}(Q)$ and

$$\|w\|_{W^{2,p}(Q)} \leq (\|f\|_{L^p(Q)} + \|\varphi H\|_{W^{2-1/p,p}(\Sigma_0)}). \tag{26}$$

Since $\|u\|_{V_s}$ is independent of s and h, we conclude that $\|f\|_{L^p(Q)}$ is independent of s and h and consequently, $\|w\|_{W^{2,p}(Q)}$ is bounded independently of s and h. Since $w=u$ if $\sigma>s-\varepsilon$, we have that the $C^{0,\alpha}$ norm of u_θ is bounded in a neighborhood of $(0^-, s)$, independently of s and h. □

Lemma 5.2.

$$\int_{\theta_*}^{0} |(\psi_{sh})_\sigma(\theta, s)|\, d\theta \leq C, \tag{27}$$

C constant independent of s and h.

Proof. For simplicity, we omit the subscripts sh. Since $u_\sigma = \dfrac{k}{q}\psi$, and

$$\frac{1}{q^2}\left(\frac{q^2}{k}u_\sigma\right)_\sigma + u_{\theta\theta} + u = -R(\theta) \text{ in } \mathscr{D},$$

we have

$$(q\psi)_\sigma = (-R(\theta) - u_{\theta\theta} - u)q^2 \text{ on } \mathscr{D}.$$

On points of the form (θ, s), $\psi \equiv 0$, so

$$\psi_\sigma(\theta, s) = [-R(\theta) - u_{\theta\theta}(\theta, s) - u(\theta, s)]q(s).$$

Besides that

$$\psi_\sigma(\theta, s) = \lim_{h\to 0^-} \frac{\psi(\theta,\sigma+h)}{h} \le 0,$$

so

$$\begin{aligned}\int_{\theta_*}^{0} |\psi_\sigma(\theta, s)| d\theta &= \int_{\theta_*}^{0} -\psi_\sigma(\theta, s) d\theta \\ &\le \int_{\theta_*}^{0} u(\theta, s) q(s) d\theta + \int_{\theta_*}^{0} u_{\theta\theta}(\theta, s) q(s) d\theta \\ &\leqq \frac{\pi}{2} H q(s) + u_\theta(0^-, s) q(s) \\ &\leqq \frac{\pi}{2} H + u_\theta(0^-, s),\end{aligned}$$

since q is non-increasing and $q(0)=1$. □

Extend now u_{sh} to $\Omega=\Omega_\infty$ as follows:

$$\tilde{u}_{sh}(\theta,\sigma) = \begin{cases} u_{sh}(\theta,\sigma) & \text{if } (\theta,\sigma)\in\Omega_s, \\ u_{sh}(\theta,\sigma) & \text{if } (\theta,\sigma)\in\Omega\setminus\Omega_s \end{cases} \tag{28}$$

Proposition 5.3.

$$\tilde{u}_{sh} \in \mathbb{K}_h \cap L^\infty(\Omega)$$

and $\tilde{u}_{sh}$ *is the unique solution of the variational inequality*

$$\begin{aligned} a(\tilde{u}_{sh}, v-\tilde{u}_{sh}) \ge \int_\Omega [R(\theta) + \\ + \varphi_{sh}(\theta,\sigma)] q^2 (v-\tilde{u}_{sh}) d\theta d\sigma, \quad \forall\, v\in\mathbb{K}\cap L^\infty(\Omega), \end{aligned} \tag{29}$$

being

$$\varphi_{sh}(\theta,\sigma) = \frac{1}{q(s)} (\psi_{sh})_\sigma(\theta, s) \chi_{E_*}, \quad E_* =]\theta_*, 0[\times]s, +\infty[. \tag{30}$$

Proof. Denote $\tilde{u}_{sh}$ by $\tilde{u}$ and u_{sh} by u. We can easily check that

$$\tilde{u}\in C^{1,\alpha}(\mathscr{D}\cup E_*\setminus(\Sigma_\infty\cup\Sigma_*)),\ \tilde{u}\in C^{0,\alpha}(\Sigma_\infty\cup E_*),$$

and also that $\tilde{u}\in\mathbb{K}_h\cap L^\infty(\Omega)$.

Let us evaluate $L\tilde{u}$. Let $(\theta,\sigma)\in E_*$; then

$$\left\{\frac{1}{q^2}\left(\frac{q^2}{k}\tilde{u}_\sigma\right)_\sigma+\tilde{u}_{\theta\theta}+\tilde{u}\right\}(\theta,\sigma) = u_{\theta\theta}(\theta,\ s)+u(\theta,\ s)$$

$$= -R(\theta)-\frac{1}{q(s)}\psi_\sigma(\theta,\ s).$$

Then

$$L\tilde{u}=\begin{cases}Lu & \text{in } \mathscr{D}\\ 0 & \text{in } \Omega\setminus(\Omega_s\cup E_*)\\ -R(\theta)-\dfrac{1}{q(s)}\psi_\sigma(\theta,\ s) & \text{in } E_*.\end{cases}$$

Since $\partial(\mathscr{D}\cup E_*)=\mathscr{L}\cup\Sigma_\infty\cup\Sigma_*\cup\{\theta_*\}\times]s,\ +\infty[$, $\dfrac{\partial\tilde{u}}{\partial n}=0$ on $\{\theta_*\}\times]s,\ +\infty[$ and $\int_{\mathscr{L}\cup\Sigma_\infty\cup\Sigma_*}\left(\tilde{u}_\theta,\ \frac{q^2}{k}\tilde{u}_\sigma\right)\cdot\vec{n}(v-\tilde{u})\geq0$, we have

$$a(\tilde{u},\ v-\tilde{u}) = \int_\Omega q^2\left\{\frac{1}{k}\tilde{u}_\sigma(v-\tilde{u})_\sigma+\tilde{u}_\theta(v-\tilde{u}_\theta)-\tilde{u}(v-\tilde{u})\right\}$$

$$= -\int_{\mathscr{D}\cup E_*}\left\{\left(\frac{q^2}{k}\tilde{u}_\sigma\right)_\sigma(v-\tilde{u})+\tilde{u}_{\theta\theta}(v-\tilde{u})q^2+\tilde{u}(v-\tilde{u})q^2\right\}$$

$$+\int_{\partial(\mathscr{D}\cup E_*)}\left(\tilde{u}_\theta,\ \frac{q^2}{k}\tilde{u}_\sigma\right)\cdot\vec{n}(v-\tilde{u})$$

$$\geq \int_{\mathscr{D}} R(\theta)q^2(\sigma)(v-\tilde{u})+\int_{E_*}\left\{R(\theta)+\frac{1}{q(s)}\psi_\sigma(\theta,\ s)\right\}q^2(\sigma)(v-\tilde{u})$$

$$\geq \int_\Omega\{R(\theta)+\varphi_{sh}(\theta,\sigma)\}\,q^2(\sigma)(v-\tilde{u}),$$

since $R\leq0$, $\varphi_{sh}\leq0$, $\tilde{u}|_{\Omega\setminus(\mathscr{D}\cup E_*)}\equiv0$ and $v\geq0$. □

Theorem 5.4.

$$\|\tilde{u}_{sh}-u_h\|_V \leq Ce^{-s} \quad \forall\ s>\sigma_\infty, \tag{31}$$

C constant independent of s and h.

In particular

$$\tilde{u}_{sh} \underset{s\to+\infty}{\longrightarrow} u \text{ in } V.$$

Proof. Recall that $\tilde{u}_{sh}\in\mathbb{K}_h$ and

$$\exists\alpha>0 \quad \forall\ v,\ w\in\mathbb{K}_h : \|v-w\|_V^2 \leq \frac{1}{\alpha}a(v-w,\ v-w).$$

Then

$$\begin{aligned}
\alpha\|\tilde{u}_{sh}-u_h\|_V^2 &\leq a(\tilde{u}_{sh}-u_h,\ \tilde{u}_{sh}-u_h) \\
&= -a(\tilde{u}_{sh},\ u_h-\tilde{u}_{sh})-a(u_h,\ \tilde{u}_{sh}-u_h) \\
&\leq \int_\Omega \{R(\theta)+\varphi_{sh}(\theta,\sigma)\}q^2(\sigma)(u_h-\tilde{u}_{sh})d\theta d\sigma \\
&\quad -\int_\Omega R(\theta)q^2(\sigma)(\tilde{u}_{sh}-u_h)d\theta d\sigma \\
&= -\int_\Omega \varphi_{sh}(\theta,\sigma)q^2(\sigma)(u_h-\tilde{u}_{sh})d\theta d\sigma \\
&= \int_{E_*} |(\psi_{sh})_\sigma(\theta,\sigma)|\frac{q^2(\sigma)}{q(s)}(u_h-\tilde{u}_{sh})d\theta d\sigma \\
&\leq H\int_{\theta_*}^{0} |(\psi_{sh})_\sigma(\theta,\ s)|d\theta\int_s^{+\infty}\frac{q^2(\sigma)}{q(s)}d\sigma,
\end{aligned}$$

since $u_h\leq H$ and $\tilde{u}_{sh}\geq 0$.

As the function q is decreasing since $q'(\sigma)=-\dfrac{q(\sigma)}{g(q(\sigma))}<0$, we have $\dfrac{q(\sigma)}{q(s)}<1$ if $\sigma>s$. Using this fact and Lemma 5.2 we conclude that

$$\|\tilde{u}_{sh}-u_h\|_V^2 \leq \frac{C_1}{\alpha}\int_s^{+\infty} q(\sigma)\,d\sigma, \tag{32}$$

C_1 constant independent of s and h.

Since $q(\sigma)\leq e^{-\sigma}$, the estimation results directly from (32). □

Extending now u_s to Ω as we did with u_{sh}, it is possible to prove equally that $(u_s)_\theta(0^-, s)$ is bounded independently of s, $\tilde{u}_s \in \mathbb{K}_\infty \cap L^\infty(\Omega)$ and it is solution of a variational inequality analogous to (29). We have then

Theorem 5.5.

$$\|\tilde{u}_s-u\|_V^2 \leq Ce^{-s}, \tag{33}$$

C constant independent of s, and so

$$\tilde{u}_s \underset{s\to+\infty}{\longrightarrow} u \text{ in } V.$$

□

Let us study now what happens when $h\to+\infty$, firstly with s fixed. We have

Lemma 5.6. *Let $u_{s\hat{h}}$, u_{sh} and u_s be the solutions of* (15) *with convex sets $\mathbb{K}_{s\hat{h}}$, $\mathbb{K}_{sh}$ and $\mathbb{K}_s$ respectively. Then*

$$u_{s\hat{h}} \geq u_{sh} \geq u_s \text{ in } \Omega_s, \text{ if } h<\hat{h}. \tag{34}$$

Proof. Since the obstacle $\eta_h(\sigma)=H-hq_\infty g(q_\infty)\int_\sigma^{\sigma_\infty}\frac{k(\tau)}{q(\tau)}d\tau$ decreases on the segment $\{0\}\times\mathbb{R}^+$ when h increases, we have

$$\mathbb{K}_{s\hat{h}} \subseteq \mathbb{K}_{sh} \subseteq \mathbb{K}_s \text{ when } h<\hat{h}.$$

and, using comparision arguments, we conclude (34). □

Theorem 5.7.

$$\|u_{sh}-u_s\|^2 \leq Ct_h^{1/2},\ C \text{ constant independent of } s \text{ and } h, \tag{35}$$

where t_h is the unique solution $\eta_h(\sigma_\infty - \sigma)=0$, η defined in (13).

Proof. Let

$$W_\infty(\Omega_s)=\{v\in V_s: qv\in L^\infty,\ qv_\theta\in L^\infty,\ \frac{q}{\sqrt{k}}v_\sigma\in L^\infty\},$$

and suppose that

$$\forall\ v\in\mathbb{K}\cap W_\infty(\Omega_s)\quad \exists\ v_h\in\mathbb{K}_h\cap W_\infty(\Omega_s): \|v_h-v\|^2_{V_s}\leq C_1 t_h, \tag{36}$$

C_1 constant independent of s and h.

Since $\mathbb{K}\cap W_\infty(\Omega_s)$ is dense in $\mathbb{K}$, and $u_s\in\mathbb{K}$, there exists $v\in\mathbb{K}\cap W_\infty(\Omega_s)$ such that $\|v-u_s\|^2_{V_s}\leq C_1 t_h$ and so

$$\exists\ v_h\in\mathbb{K}_h\cap W_\infty(\Omega_s):\ \|u_s-v_h\|^2_{V_s}\leq 2C_1 t_h.$$

Then, since

$$\begin{aligned}\alpha\|u_{sh}-u_s\|^2_{V_s} &= a(u_{sh}-u_s,\ u_{sh}-u_s)\\ &= -a(u_s,\ u_{sh}-u_s)-a(u_{sh},\ v_h-u_{sh})-a(u_{sh},\ u_s-v_h),\end{aligned}$$

and $a(u,\ v)\leq\|u\|\|v\|$ (as it was established in the proof of Proposition 21), we have

$$\begin{aligned}\alpha\|u_{sh}-u_s\|^2_{V_s} \leq & \int_{\Omega_s} R(\theta)q^2(\sigma)(u_{sh}-u_s)d\theta d\sigma\\ &+\int_{\Omega_s} R(\theta)q^2(\sigma)(v_h-u_{sh})d\theta d\sigma+\|u_{sh}\|_{V_s}\|u_s-v_h\|_{V_s}\\ &\leq \|R(\theta)q(\sigma)\|_{L^2(\Omega_s)}\|q(\sigma)(v_h-u_s)\|_{L^2(\Omega_s)}+\|u_{sh}\|_{V_s}\|v_h-u_s\|_{V_s}\\ &\leq D\|v_h-u_s\|_{V_s},\end{aligned}$$

D constant independent of s and h.

Then

$$\|u_{sh}-u_s\|\leq Ct_h^{1/2},\ C\text{ constant independent of } s \text{ and } h.$$

To prove (36), given $v\in\mathbb{K}\cap W_\infty(\Omega_s)$, define $v_h(\theta,\sigma)$ as follows:

$$\begin{cases} v(\theta,\sigma)\psi(\theta) & \text{if } \sigma\notin[\sigma_\infty-2t_h,\sigma_\infty],\\ v(\theta,\sigma_\infty)\psi(\theta) & \text{if } \sigma\in[\sigma_\infty-t_h,\sigma_\infty],\\ \{v(\theta,\sigma_\infty)+[(\sigma+t_h-\sigma_\infty)/t_h][v(\theta,\sigma_\infty)-v(\theta,\sigma_\infty-2t_h)]\}\psi(\theta) & \text{otherwise,}\end{cases}$$

where $\psi\in C^1([\theta_B,\theta_A])$, $\psi\geq 0$, $\psi(0)=1$, $\psi(\theta_B)=\psi(\theta_A)=0$.

Observe that $v_h(0,\sigma)=H$ if $\sigma\geq\sigma_\infty$.

If $\sigma\in[\sigma_\infty-t_h,\sigma_\infty]$, $v_h(0,\sigma)=v(0,\sigma)=H\geq\eta_h(\sigma)$, η_h defined in (13).

If $\sigma\in[\sigma_\infty-2t_h,\sigma_\infty-t_h]$, $v_h(0,\sigma)\geq 0\geq\eta_h(\sigma)$, since t_h is the only solution of $\eta_h(\sigma_\infty-\sigma)=0$.

Since $v_h\in V_s$, $v_h\in\mathbb{K}_h$. Besides, $v_h\in W_\infty(\Omega_s)$.

Direct calculations show that

$$\|v-v_h\|^2_{V_s}\leq C_1 t_h,\ C_1 \text{ constant independent of } s \text{ and } h. \qquad \square$$

Remark 5.8. Since $k(\sigma)\neq 0$ in $\left[\frac{\sigma_\infty}{2},\sigma_\infty\right]$ and k is continuous,

$$\exists\xi: k(\sigma)\geq\xi \quad \forall\, \sigma\in\left[\frac{\sigma_\infty}{2},\sigma_\infty\right].$$

Then, for t_h small enough,

$$\int_{\sigma_\infty-t_h}^{\sigma_\infty}\frac{k(\tau)}{q(\tau)}d\tau\geq\xi e^{\sigma_\infty}(1-e^{-t_h}),$$

using the fact that $q(\sigma)\leq e^{-\sigma}$. Then

$$1-e^{-t_h}\leq\frac{H}{hq_\infty g(q_\infty)\xi e^{\sigma_\infty}}$$

and, for h big, we conclude that

$$t_h\leq-\log\left(1-\frac{H}{q_\infty g(q_\infty)\xi e^{\sigma_\infty}}\frac{1}{h}\right)\sim\frac{C}{h}.$$

Corollary 5.9. *We have the following order of convergence*

$$\|u_{sh}-u_s\|^2_{V_s}\leq Ch^{1/2},\ C \text{ constant independent of } s \text{ and } h$$

and consequently

$$u_{sh}\underset{h\to+\infty}{\longrightarrow}u_s \text{ in } V_s. \qquad \square$$

Theorem 5.10.

$$\|u_h - u\|_V^2 \le C t_h^{1/2},\ C \text{ constant independent of } h, \tag{37}$$

being t_h the unique solution of $\eta_k(\sigma_\infty - \sigma) = 0$, η_h defined in (13).
Furthermore

$$\|u_h - u\|_V^2 \le C_1 h^{-1/2},\ C_1 \text{ constant independent of } h, \tag{38}$$

and so,

$$u_h \underset{h\to+\infty}{\longrightarrow} u \text{ in } V. \qquad \square$$

6. STABILITY OF THE FREE BOUNDARIES

Let

$$\mathscr{L}_{sh} = \partial\{u_{sh}=0\} \cap \Omega_s, \qquad \mathscr{L}_s = \partial\{u_s = 0\} \cap \Omega_s,$$

$$\mathscr{L}_h = \{u_h = 0\} \cap \Omega \quad \text{and} \quad \mathscr{L} = \{u = 0\} \cap \Omega,$$

be the free boundaries of problems (s, h), $(s, +\infty)$, $(+\infty, s)$ and $(+\infty, +\infty)$. It is known, by Theorem 4.3, that the free boundaries are graphs of functions l_{sh}, l_s, l_h and l respectively. Our aim in this section is to establish convergence results for these functions when $s \to +\infty$ or $h \to +\infty$.

We begin with the following lemma, for the problems with wake:

Lemma 6.1. *$\theta_* \to \theta_B$ when $s \to +\infty$, being θ_* the angle beetwen the profile and wake on the point of intersection of both.*

Proof. Since $\tilde{u}_{sh}$ are uniformly bounded in $W^{2,p}_{\text{loc}}(\Omega)$, $\tilde{u}_{sh} \underset{s\to+\infty}{\longrightarrow} u_h$ in $W^{2,p}_{\text{loc}}$ – weak and, since $W^{2,p} \hookrightarrow C^{1,\alpha}$, being the inclusion compact, we conclude that $\tilde{u}_{sh} \underset{s\to+\infty}{\longrightarrow} u_h$ uniformly in the compact subsets of $\overline{\Omega}$. Since $l_h(\theta) \to +\infty$ when $\theta \to \theta_B$, we must have $\theta_* \to \theta_B$ when $s \to +\infty$. $\square$

Fix $\delta > 0$ arbitrarily small. By Lemma 6.1

$$\exists\, s_\delta \ \forall\, s \ge s_\delta \quad \theta_* < \theta_B + \delta/2,$$

(observe that $\theta_* = \theta_*(s)$). Let $I_\delta = [\theta_B + \delta, \theta_A - \delta]$. Then

Theorem 6.2.

(1) $\|l_{sh}-l_h\|_{C^{0,\alpha}(I_\delta)}\leq Ce^{-s(1-\alpha)/4}, \quad \forall s\geq s_\delta,$

(2) $\|l_h-l\|_{C^{0,\alpha}(I_\delta)}\leq Ch^{-(1-\alpha)/8},$

(3) $\|l_{sh}-l_s\|_{C^{0,\alpha}(I_\delta)}\leq Ch^{-(1-\alpha)/8}, \quad \forall s\geq s_\delta,$

(4) $\|l_s-l\|_{C^{0,\alpha}(I_\delta)}\leq Ce^{-s(1-\alpha)/4}, \quad \forall s\geq s_\delta,$

being C constants, independent of s and h and α any number belonging to the interval $]0, 1[$.

Proof. The proof will be presented just for case (1), since the others are similar.

Let $\gamma\in C_0^\infty(\Omega\setminus(\Sigma_*\cup\Sigma_\infty))$ be a *cut off* function such that $0\leq\gamma\leq 1$, with $\gamma\equiv 0$ in a neighbourhood of $(\mathscr{L}_{sh}\cup\mathscr{L}_h)\cap]\theta_B+\delta/2, \theta_A-\delta/2[\times\mathbb{R}^+$ and *supp* $\gamma\subseteq\Omega_s$.

Then $\gamma\tilde{u}_{sh}$ is solution of the variational inequality

$$a(\gamma\tilde{u}_{sh}, v-\gamma\tilde{u}_{sh})\geq\int_\Omega\varphi_1(\theta,\sigma)(v-\gamma\tilde{u}_{sh}), \quad \forall v\in\mathbb{K}, \tag{39}$$

and γu_h is solution of the variational inequality

$$a(\gamma u_h, v-\gamma u_h)\geq\int_\Omega\varphi_2(\theta,\sigma)(v-\gamma u_h), \quad \forall v\in\mathbb{K}, \tag{40}$$

being

$$\mathbb{K}=\{v\in V: v\geq 0\}, \tag{41}$$

$$\varphi_1=\gamma q^2[R\theta)+\varphi_{sh}(\theta,\sigma)]-\left(\frac{q^2}{k}\gamma_\sigma u_{sh}\right)_\sigma-\gamma_\sigma\frac{q^2}{k}(u_{sh})_\sigma$$
$$-q^2\gamma_{\theta\theta}u_{sh}-2q^2\gamma_\theta(u_{sh})_\theta,$$

and

$$\varphi_2=\gamma q^2R(\theta)-\left(\frac{q^2}{k}\gamma_\sigma u_h\right)_\sigma-\gamma_\sigma\frac{q^2}{k}(u_h)_\sigma-q^2\gamma_{\theta\theta}u_h-2q^2\gamma_\theta(u_h)_\theta,$$

being φ_{sh} defined in (30).

Let $K=\gamma^{-1}(1)$. Since *supp*γ is a compact set, K is a compact subset of Ω_s. Let

$$\beta = \min_K |R(\theta)q^2(\sigma)| > 0.$$

Observe that

$$\varphi_1 = \varphi_2 = q^2(\sigma)R(\theta) \leq -\beta < 0, \text{ on } K,$$

so, using a result of [16], we have

$$\|l_{sh}(\theta) - l_h(\theta)\|_{L^1(\theta_B+\delta,\theta_A-\delta)} \leq \|\chi_1 - \chi_2\|_{L^1(K)} \tag{42}$$

$$\leq \frac{1}{\beta}\int_\Omega |\varphi_1 - \varphi_2| d\theta d\sigma,$$

where $\chi_1 = \chi_{\{u_{sh}=0\}}$ and $\chi_2 = \chi_{\{u_h=0\}}$. But, since q, q_σ and k are bounded on $\dot{K}$ (notice that $(\theta, 0) \notin K$), we conclude that

$$\int_\Omega |\varphi_1 - \varphi_2| \leq \int_{\Omega_s} q^2 |\gamma_\sigma| \left[\frac{q^2}{k}(u_{sh} - u_h)\right]_\sigma + 2\gamma_\theta (u_{sh} - u_h)_\theta| + |\gamma_{\theta\theta}(u_{sh} - u_h)|$$

$$\leq C\|u_{sh} - u_h\|_V^2.$$

We are going to use now the following interpolation inequality, due to Gagliardo-Nirenberg ([15]): for all α such that $0<\alpha<1$, for all $\varepsilon>0$,

$$\|f\|_{C^{0,\alpha}(a,b)} \leq C_1 \varepsilon^{2/(1+\alpha)} \|f'\|_{L^\infty(a,b)} + (C_2 + C_1 \varepsilon^{2/(\alpha-1)})\|f\|_{L^1(a,b)},$$

Since $l_{sh} - l_h$ is a Lipschitz function, with Lipschitz constant independent of s and h, we have

$$\|l_{sh} - l_h\|_{C^{0,\alpha}(\theta_B+\delta,\theta_A-\delta)} \leq C_1 \varepsilon^{2/(1+\alpha)} \|l'_{sh} - l'_h\|_{L^\infty(\theta_B+\delta,\theta_A-\delta)}$$

$$+ (C_2 + C_1 \varepsilon^{2/(\alpha-1)})\|l_{sh} - l_h\|_{L^1(\theta_B+\delta,\theta_A-\delta)}$$

$$\leq Ce^{-s(1-\alpha)/4},$$

choosing $\varepsilon = e^{-s(1-\alpha)/4}$, and using Theorem 4.3 and property 42. □

7. THE INCOMPRESSIBLE CASE AS A LIMIT CASE

We are going to show in this section that, if g_n is a sequence of density functions (of the fluid) such that

$$g_n \in W^{3,\infty}(\mathbb{R}_0^+), \tag{44}$$

$$\exists\, m, M>0 \quad \forall\, x \in \mathbb{R}^+ \ \forall\, n \in \mathbb{N} \quad m \leq g_n(x) \leq M, \tag{45}$$

$$g_n \xrightarrow[n]{} 1,\ g_n' \xrightarrow[n]{} 0 \text{ uniformly on the compact subsets of } \mathbb{R}_0^+, \tag{46}$$

then, if v_n is the solution of the problem (after a convenient translation) in one of the four situations refered before (in the channel with wake, in the channel without wake, in the plane with wake and in the plane without wake), with density function g_n, then $v_n \underset{n}{\to} v$ in a convenient space, being v the solution of the limit problem with density function $g \equiv 1$; besides $e^{-\sigma}v$ is the solution of the problem in the incompressible case, formulated as it appears in the literature.

Given g_n, define

$$\xi_n = \int_1^{q_c^n} \frac{g_n(\tau)}{\tau} d\tau, \tag{47}$$

where q_c^n denotes the velocity of the sound, which means the least solution of the equation $\frac{g_n(x)}{g_n'(x)} = -x$. For n sufficiently large, ξ_n is positive, since $q_c^n \underset{n}{\to} +\infty$. Let

$$s_n = \int_{q_s}^1 \frac{g_n(\tau)}{\tau} d\tau, \quad \sigma_n = \int_{q_\infty}^1 \frac{g_n(\tau)}{\tau} d\tau.$$

Let q_n and k_n be defined in $]0, +\infty[$ by the following relations:

$$\frac{q_n'(\sigma)}{q_n(\sigma)} = -\frac{1}{g_n(q_n(\sigma))}, \quad q_n(\xi_n) = 1, \tag{48}$$

$$k_n(\sigma) = \frac{1}{g_n^2(q_n(\sigma)))} \left(1 + \frac{q_n(\sigma) g_n'(q_n(\sigma))}{g_n(q_n(\sigma))}\right). \tag{49}$$

Recall that the variable σ (which depends on n) is related with function q_n (taken as a variable) by the following relation: $\sigma(q_n) = \int_{q_n}^{q_c^n} \frac{g_n(\tau)}{\tau} d\tau$. Redefining the problem with a different initial condition for q_n corresponds to make a translation in the variable σ, more specificly, $\sigma \rightsquigarrow \sigma + \xi_n$, and, in the new variables, $\sigma(q_n) = \int_{q_n}^1 \frac{g_n(\tau)}{\tau} d\tau$, the function q_n evaluates 1 in zero and is defined on $]-\xi_n, +\infty[$.

Let

$$\begin{cases} \Gamma_{s_n} =]\theta_B, 0[\times \{s_n\}, \\ \Omega_{s_n} =]\theta_B, 0[\times]-\xi_n, s_n[\cup]0, \theta_A[\times]-\xi_n, +\infty[. \end{cases}$$

Observe that, from now on, everything done for $n\in\mathbb{N}$, has also a meaning for $n=\infty$, being, for simplicity, omited the subscript ∞ whenever convenient.

Let $n\in\mathbb{N}\cup\{+\infty\}$. The space V_n in which we are going to work is

$$V_n=\{v: q_n v\in L^2(\Omega_{s_n}),\ q_n v_\theta\in L^2(\Omega_{s_n}),\quad \frac{q_n}{\sqrt{k_n}}v_\sigma\in L^2(\Omega_{s_n}),\ v_{|\Omega_{s_n}\setminus\Gamma_{s_n}}\equiv 0\},\tag{50}$$

with the canonical norm

$$\|v\|_n^2=\int_{\Omega_s} q_n^2\left(\frac{1}{k_n}v_\sigma^2+v_\theta^2+v^2\right)d\theta\, d\sigma.\tag{51}$$

Let

$$\zeta_n(\sigma)=H-h+\frac{hq_\infty g_n(q_\infty)}{q_n(\sigma)g_n(q_n(\sigma))}\tag{52}$$

and

$$\eta_n=(H\wedge\zeta_n)\vee 0.\tag{53}$$

The convex that appears in the variational formulation of the problem can be defined as follows:

$$\mathbb{K}_n=\{v\in V_n: v\geq 0,\ v(0,\sigma)=\eta_n(\sigma) \text{ if } \sigma\geq\sigma_n,\quad v(0,\sigma)\geq\eta_n(\sigma) \text{ if } \sigma<\sigma_n\}.\tag{54}$$

Consider the bilinear form

$$a_n(u,v)=\int_{\Omega_{s_n}} q_n^2\left(\frac{1}{k_n}u_\sigma v_\sigma+u_\theta v_\theta-uv\right)d\theta\, d\sigma.\tag{55}$$

The solution of the problem in the compressible case with density function g_n is the unique solution v_n of the problem

$$\begin{cases} v_n\in\mathbb{K}_n, & n\in\mathbb{N}\cup\{+\infty\},\\ a_n(v_n, v-v_n)\geq\int_{\Omega_{s_n}} R(\theta)q_n^2(\sigma)(v-v_n)d\theta\, d\sigma, & \forall\, v\in\mathbb{K}_n.\end{cases}\tag{56}$$

We are going to verify now that all the problems (for any n) can be defined in the same open subset of $]\theta_B,\theta_A[\times\mathbb{R}$ and afterwards relate the limit problem ($n=+\infty$) with the problem of the incompressible fluid.

The following theorem (see [4]), gives us an *a priori* estimation of the maximum velocity of the fluid, or equivalently, of a lower bound of $\{\sigma{:}\exists\theta\ v(\theta,\sigma)>0\}$, where u_n is the solution of problem (56) with density function g_n. In the next theorem, since we are working in a fixed situation, we omit the n.

Theorem 7.1. *Consider the problem* (56) *with density function* g. *Let* $\sigma_*=\inf\ \{\sigma:\zeta(\sigma)>0\}$, *being* ζ *defined in* (52), and $q(\sigma_*)=q_*$. *Let* $\alpha=\min_{\theta\in[\theta_B,\theta_A]}|R(\theta)|>0$ *and* q_μ *the least positive solution of the equation*

$$\frac{q_\mu}{q_*}\left[-1+\frac{1}{g(q_*)}\int_{q_*}^{q_\mu}\frac{g(\tau)}{\tau}\mathrm{d}\tau\right]=\frac{H}{\alpha}-1. \tag{57}$$

If $q_\mu\leq q_c$ *then the maximum velocity of the fluid is less or equal to* q_μ.

Proof. Let $\mu=\displaystyle\int_{q_\mu}^{1}\frac{g(\tau)}{\tau}d\tau$ and define, for $\sigma\geq\mu$

$$\varphi(\sigma)=\alpha q_\mu\int_\mu^\sigma\frac{k(\tau)}{q(\tau)}(\tau-\mu)d\tau.$$

It is easily verified that $\varphi\geq0$, $\varphi(\mu)=\varphi_\sigma(\mu)=0$ and $q_*\leq q_\mu$. Besides, since $\dfrac{k}{q}=-\left(\dfrac{q_\sigma}{q^2}\right)_\sigma$, we conclude that

$$\varphi(\sigma)=\alpha q_\mu\left[\frac{1}{qg(q)}\int_q^{q_\mu}\frac{g(\tau)}{\tau}d\tau-\frac{1}{q}+\frac{1}{q_\mu}\right]. \tag{58}$$

Using (58) we conclude that $\varphi(\sigma_*)=H$. As $\varphi_\sigma\geq0$, $\varphi\leq H$ if $\sigma\leq\sigma_*$. Extend φ by zero to $\sigma<\mu$ and define $\Phi=\varphi\wedge H$. Simple calculations show that

$$\frac{1}{q^2}\left(\frac{q^2}{k}\varphi_\sigma\right)_\sigma+\varphi=\alpha.$$

Let $w=u-(u-\Phi)^+$, being $f^+=\max\{f,0\}$. Observe that $w\in\mathbb{K}$ ($\mathbb{K}$ defined in (54)). Then

$$a(u,(u-\Phi)^+)\leq\int_{\Omega_s}R(\theta)q^2(\sigma)(u-\Phi)^+. \tag{59}$$

Let us evaluate now $a(\Phi, (u-\Phi)^+)$. Since Φ depends only on the variable σ and $u(\theta, -\xi)=0$, integrating by parts we obtain

$$
\begin{aligned}
a(\Phi, (u-\Phi)^+) &= -\int_{\Omega_s}\left\{\left(\frac{q^2}{k}\Phi_\sigma\right)_\sigma + \Phi\right\}(u-\Phi)^+ \\
&= \int_{\{\mu<\sigma<\sigma_*\}\cap\{u>\Phi\}} -\alpha q^2(u-\Phi)^+ \qquad (60)\\
&\geq \int_{\Omega_s} -\alpha q^2(u-\Phi)^+.
\end{aligned}
$$

As it was refered before, there exists a positive constant C such that $\|(u-\Phi)^+\|^2 \leq Ca(u-\Phi, (u-\Phi)^+)$ and, since

$$
a(u-\Phi, (u-\Phi)^+) \leq \int_{\Omega_s}\{R(\theta)+\alpha\}\, q^2(u-\Phi)^+ \leq 0,
$$

we conclude that $u\leq\Phi$ and, since $u\geq 0$, $u=0$ *on* $\{\sigma<\mu\}$. □

Remark 7.2. The preceding theorem is valid whenever $q_\mu \leq q_c$. When $g_n \xrightarrow[n]{} 1$ and $g_n' \xrightarrow[n]{} 0$ (uniformly on the compact sets), then $q_c^n \xrightarrow[n]{} +\infty$. On the other hand, observing equation (57) we easily verify that, if q_μ^n is the solution of (57) with density function g_n, $q_\mu^n \xrightarrow[n]{} q_\mu < +\infty$, q_μ solution of the same equation with density function $g\equiv 1$. Then, at least for n big, the preceding theorem is true in our case.

We verify then that there exists $\mu\in\mathbb{R}$ such that, for n big, $\mu > -\xi_n$ and $v_n(\theta,\sigma)=0$ for $\sigma\leq\mu$.

Consider the open set

$$
\tilde{\Omega}_{s_n} =]\theta_B, 0[\times]\mu, s_n[\cup]0, \theta_A[\times]\mu, +\infty[. \tag{61}
$$

Let $\tilde{v}_n$ be the restriction of v to $\tilde{\Omega}_{s_n}$. Define $\tilde{V}_n$, $\tilde{\mathbb{K}}$ and $\tilde{a}_n$ in the natural way, that is considering the intervenient functions with domain of definition $\tilde{\Omega}_{s_n}$. Then

Proposition 7.3. *$\tilde{v}_n$ is the unique solution of the variational inequality*

$$
\begin{cases}
u_n \in \tilde{\mathbb{K}}_n, \qquad n\in\mathbb{N}\cup\{+\infty\}, \\
\tilde{a}_n(u_n, v-u_n) \geq \int_{\tilde{\Omega}_{s_n}} R(\theta) q_n^2(\sigma)(v-u_n)\,d\theta\,d\sigma, \quad \forall\, v\in\tilde{\mathbb{K}}_n.
\end{cases} \tag{62}
$$

Proof. The proof is immediate.

Remark 7.4. The formulation (62) of the problem will be the one used from now on; since there is no risk of confusion, we will omit the tilde on the functions, for simplicity of writing.

Lemma 7.5.

(a) $q_n(\sigma)=e^{-\int_0^\sigma \frac{d\tau}{g_n(q_n(\tau))}}$.

(b) $q_n \in C^3([\mu, +\infty[)$ *and* $k_n \in C^1([\mu, +\infty[)$.

(c) $q_n \underset{n}{\rightarrow} e^{-\sigma}$, $q_n' \underset{n}{\rightarrow} e^{-\sigma}$, *uniformly on the compact subsets of* $[\mu, +\infty[$ *and* $q_n \underset{n}{\rightarrow} e^{-\sigma}$ *in* $H^1(\mu, +\infty)$.

(d) $k_n \underset{n}{\rightarrow} 1$, $k_n' \underset{n}{\rightarrow} 0$ *uniformly on the compact subsets of* $[\mu, +\infty[$ *and* $k_n \underset{n}{\rightarrow} 1$ *in* $H^1(\mu, +\infty)$.

Proof. These results are a direct consequence of the definitions of q_n and k_n. □

The limit problem is defined letting $n=\infty$ in problem (62). Notice that $q(\sigma)=e^{-\sigma}$, $k(\sigma)=1$, $\eta_\infty=(H\wedge\zeta_\infty)\vee 0$, where $\zeta_\infty(\sigma)=H-h+hq_\infty e^\sigma$.

The problem for an incompressible fluid (see [7]) has the following formulation: the open subset of $\mathbb{R}^2$ considered is

$$\Omega_s'=]\theta_B, 0]\times]m, s[\cup]0, \theta_A[\times]m, +\infty[,$$

(see [16]) being $s=-\log(q_s)$ and m any lower bound of $\tau_h+\sigma_\infty+\log(1-H/h)$, where τ_h is the unique negative solution of the equation

$$(1+H/\varrho)(1-H/h)=e^\tau(\tau+1+\log(1-H/h));$$

$$\varrho=\max_{\theta\in[\theta_B,\theta_A]} R(\theta)<0.$$

Recall that $\Gamma_s=]\theta_B, 0[\times\{s\}$. The space considered in this case is

$$V'=\{v\in H^1(\Omega_s'): v_{|\partial\Omega_s'\setminus\Gamma_s}\equiv 0\},$$

the convex set is

$$\mathbb{K}'=\{v\in V':v\geq 0,\ v(0,\sigma)=He^{-\sigma} \text{ if } \sigma\geq\sigma_\infty,\\ v(0,\sigma)\geq\Phi(\sigma) \text{ if } \sigma<\sigma_\infty\},$$

where $\Phi(\sigma)=hq_\infty+(H-h)e^{-\sigma}$, and the bilinear form is

$$b(u,\ v)=\int_{\Omega_s'}\nabla u\cdot\nabla v d\theta\ d\sigma+\int_{\Gamma_s}uvd\theta.$$

Then, the solution u of the problem for an incompressible fluid, formulated in the hodograph plane, using variational inequalities, is the unique solution of

$$b(u,\ v-u)\geq\int_{\Omega_s}R(\theta)e^{-\sigma}(v-u),\quad\forall\ v\in\mathbb{K}'. \tag{63}$$

Obviously we can suppose $m=\mu$, extending u or v by zero, whether we are in the situation $m>\mu$ or in the other one. We need to extend the domain of definition of all functions and to make the appropriate change on the definition of the bilinear form. Since to $\mathbb{K}$ and $\mathbb{K}'$ only belong non-negative functions and $R(\theta)<0$, the extension of u by zero is the solution of the extended problem.

Since, given $v,\ w\in V'$,

$$a(e^\sigma u,\ e^\sigma v)=\int_{\Omega_s}\{e^{2\sigma}[(uv+(uv)_\sigma+u_\sigma v_\sigma)+u_\theta v_\theta-uv]\}e^{-2\sigma}d\theta d\sigma$$

and

$$\int_{\Omega_s}(uv)_\sigma=\int_{\theta_B}^{0}uv(\sigma)]_\mu^s+\int_0^{\theta_A}uv(\sigma)]_\mu^{+\infty}=\int_{\Gamma_s}uvd\theta,$$

because, as $0\leq u\leq He^{-\sigma}$, $\lim_{\sigma\to+\infty}u(\theta,\sigma)=0$, we conclude that

$$a(e^\sigma u,\ e^\sigma v)=b(u,\ v)\quad\forall\ u,\ v\in V'.$$

Since $e^\sigma v\in\mathbb{K}$ if and only if $v\in\mathbb{K}'$, we conclude that u is solution of (56) for $n=+\infty$ with the definition of Ω_s given in (61) if and only if $e^{-\sigma}u$ is the solution of (63).

Define, for $n\in\mathbb{N}\cup\{+\infty\}$,

$$\hat{\mathbb{K}}_n=\{v\in V_n(\Omega_s):v\geq 0,\ v(0,\sigma)=\eta_n(\sigma) \text{ if } \sigma\geq\sigma_n,$$

$$v(0,\sigma)\geq\eta_n(\sigma) \text{ if } \sigma<\sigma_n\},\tag{64}$$

where $V_n(\Omega_s)$ has a definition analogous to V_n, being the domains of definition of the functions just restricted to Ω_s. Being v_n the solution of problem (56) we are going to define (by extension or restriction) $\hat{v}_n\in\hat{\mathbb{K}}_n$.

Suppose that $s_n>s$. Then define $\hat{v}_n=v_{n|\Omega_s}$. If $s\geq s_n$ extend v_n to Ω_{s_n} as it was done in section 5, that is

$$\hat{v}_n(\theta,\sigma)=\begin{cases} v_n(\theta,\sigma) & \text{if } (\theta,\sigma)\in\Omega_{s_n}, \\ v_n(\theta,s_n) & \text{if } (\theta,\sigma)\in\Omega_s\setminus\Omega_{s_n}. \end{cases}\tag{65}$$

Let $\alpha=1/m, m$ defined in (45) and $W=W(\Omega_s)=\{v\in H^1(\Omega_s): v_{|\partial\Omega_s\setminus\Gamma_s}\equiv 0\}$. Let $Z=e^{-\alpha\sigma}W$ with the canonical norm, that is, if $w\in Z$, $\|w\|_Z=\|e^{\alpha\sigma}w\|_{H^1}$.

Theorem 7.6. *If v_n is the solution of problem (56) with density function g_n and v is the solution of the same problem with density function 1 and if the assumptions (44), (45) and (46) are verified, then*

$$\|\hat{v}_n-v\|_Z^2\underset{n}{\to}0\tag{66}$$

Proof. By Lemma 3.3, the bilinear form a_n is coercive in

$$K_n=\{v\in V_n(\Omega_s): v(0,\sigma)=H \text{ if } \sigma>\sigma_n\}.$$

Notice that K_n contains $\hat{\mathbb{K}}_n$. Since $(q_n)_n$ and $(k_n)_n$ are sequences uniformly convergent in the compact subsets, the constant of coerciveness can be choosen independently of n, as we can verify in the proof of the refered lemma.

Since

$$\exists a_0,b_0>0 \quad \forall\sigma\in[\mu,+\infty] \quad a_0\leq k_n(\sigma)\leq a_1,$$

and (45) is verified, we have that $V_n(\Omega_s)=q_n^{-1}W(\Omega_s)$ and

$$\exists\, C_1,C_2>0 \quad \forall\, v\in V_n \quad C_1\|q_nv\|_{H^1}\leq\|v\|_n\leq C_2\|q_nv\|_{H^1},$$

C_1 e C_2 constants independent of n.

Since $g_n\underset{n}{\to}1$ uniformly on the compact sets, there exists $c_n\geq 1$, $c_n\underset{n}{\to}1$. So, $g_n(0)\leq c_n$ and, since g_n is decreasing, $g_n\leq c_n$. Notice that $r_n=e^{-\sigma/c_n}$ is the solution of equation (48) and $j_n=1/c_n^2$ the solution of equation (49) (after translation of the variable σ) with g_n substituted by c_n. Let $Z_n = r_n^{-1}W$ the canonical norm (that is, $\|w\|_Z = \|r_n w\|_{H^1}$). Then $Z_n \hookrightarrow V_n$ and $\|v\|_{V_n} \leq C\|v\|_{Z_n}$, C constant independent of n. Let b_n be the following bilinear form, defined for v, $w \in Z_n$:

$$b_n(v,w) = \int_{\Omega_s} e^{-2\sigma/c_n}(c_n^2 v_\sigma w_\sigma + v_\theta w_\theta - vw), \tag{67}$$

and w_n the solution of (56) with convex set $\hat{\mathbb{M}}_n$ defined as in (64), but for the functions τ_n and j_n.

To prove the result it is sufficient to establish that

$$\|w_n - \hat{v}_n\|_Z \underset{n}{\to} 0 \text{ and } \|w_n - v\|_Z \underset{n}{\to} 0.$$

Recall that $Z = r_n^{-1}W \subset V_n = q_n^{-1}W$ and $w_n(0,\sigma) = H$ for $\sigma \geq \sigma_n$, since $w_n \in \hat{\mathbb{M}}_n$.

It was proved in section 2 that $\|v_n\|_{V_n}$ are bounded independently of n (and the same is verified by the extensions $\hat{v}_n$). Letting α be the constant of coerciveness of all the bilinear forms a_n and b_n (for n big enough), we have

$$\begin{aligned} \|w_n - \hat{v}_n\|_Z^2 &\leq \frac{1}{\alpha} a_n(w_n - \hat{v}_n, w_n - \hat{v}_n) \\ &= \frac{1}{\alpha}[-a_n(\hat{v}_n, w_n - \hat{v}_n) - a_n(w_n, \hat{v}_n - w_n)]. \end{aligned} \tag{68}$$

Let $\psi \in C^1([\theta_B, \theta_A])$ verifying

$$\psi \geq 0, \quad \psi(\theta_B) = \psi(\theta_A) = 0, \quad \psi(0) = 1 \tag{69}$$

and $\bar{w}_n \in q_n^{-1}W^{1,\infty} \cap V_n$ verifying $\|w_n - \bar{w}_n\|_{V_n} \leq \frac{1}{n}$, $\bar{w}_n(0,\sigma) = H$ if $\sigma \geq \sigma_n$ ($\bar{w}_n$ exists because $W^{1,\infty} \cap W$ is dense in W). Observe that $\|w_n - \bar{w}_n\|_{V_n} \underset{n}{\to} 0$. Define

$$\tilde{w}_n(\theta,\sigma) = \bar{w}_n(\theta,\sigma) + [\bar{w}_n(0,\sigma) \vee \eta_n(\sigma) - \bar{w}_n(0,\sigma)]\psi(\theta).$$

Easy verifications allow us to conclude that $\tilde{w}_n \in \mathbb{K}_n \cap q_n^{-1}W^{1,\infty}$. Notice that $\|\tilde{w}_n - w_n\|_{V_n} \underset{n}{\to} 0$ and

$$-a_n(\hat{v}_n, w_n-\hat{v}_n)=-a_n(\hat{v}_n, \tilde{w}_n-\hat{v}_n)-a_n(\hat{v}_n, w_n-\tilde{w}_n)$$

$$\leq \int_\Omega q_n^2[R(\theta)+\varphi_n](\hat{v}_n-\tilde{w}_n)+\|\hat{v}_n\|_{V_n}\|w_n-\tilde{w}_n\|_{V_n},$$

being φ_n defined in (30).

Analogously we construct $z_n \in \hat{\mathbb{M}}_n \cap r_n^{-1}W^{1,\infty}, \|z_n-\hat{v}_n\|_{V_n}\underset{n}{\to}0$. Then

$$-a_n(w_n, \hat{v}_n-w_n)\leq$$

$$\|w_n\|_{V_n}\|\hat{v}_n-z_n\|_{V_n}+\int_\Omega\{r_n^2[R+\psi_n](z_n-w_n)+\xi_n(w_n, z_n-w_n)\},$$

being ψ_n defined as φ_n, but for the problem with density of the fluid equal to c_n, and being $\xi_n=a_n-b_n$.

Then

$$\begin{aligned}\|w_n-\hat{v}_n\|_Z^2 &\leq O(1)+\int_{\Omega_s}q_n^2[R+\varphi_n](\tilde{w}_n-\hat{v}_n)\\ &\quad+\int_{\Omega_s}r_n^2[R+\psi_n](z_n-w_n)\\ &=\int_{\Omega_s}r_n^2R[(\tilde{w}_n-w_n)+(z_n-\hat{v}_n)]\\ &\quad+\int_{\Omega_s}(q_n^2-r_n^2)R(\tilde{w}_n-w_n)+\int_{\Omega_s\setminus\Omega_{s_n}}\Phi_n,\end{aligned} \tag{70}$$

where $\Phi_n=H(|(v_n)_\theta(0^-,s_n)|+|(w_n)_\theta(0^-,s_n)|)$ (for details, see calculations on section 3). Since $\|\tilde{w}_n-w_n\|_{V_n}\underset{n}{\to}0$, $q_n^2-r_n^2\underset{n}{\to}0$ uniformly on the compact subsets of $[\mu,+\infty]$ and $\int_{\Omega_s\setminus\Omega_{s_n}}\Phi_n\leq C|s-s_n|$, C constant independent of n (due to the uniform boundedeness of v_n and w_n in $C^{1,\alpha}(\Omega_{s_n})$), we conclude that $\|w_n-\hat{v}_n\|_Z\underset{n}{\to}0$.

Analogously, we prove that $\|w_n-v\|_Z\underset{n}{\to}0$, since $Z_n=r_n^{-1}W\subset V=e^\sigma W$. Seeming this case simpler than the case treated before, it has an additional difficulty, since σ_n changes, converging to σ_∞ when $n\to+\infty$. Nevertheless, this difficulty can be easily solved, choosing the functions $\tilde{w}_n$ in $q_n^{-1}W^{1,\infty}\cap V_n$ verifying $\tilde{w}_n(0,\sigma)=H$ for $\sigma\geq\sigma_n\wedge\sigma_\infty$. □

We are going to study now the free boundaries. Recall that $g_n\in W^{3,\infty}$, and so the free boundaries are graphs of Lipschitz functions (with Lipschitz constants independent of n).

The limit problem was not included in what was done in the preceding sections. However, we have seen in this section, after the translation of the coordinate σ done above, that the limit problem is similar to the others and so, all the results previously obtained are also true for the limit problem.

Let l_n and l be parametrizations of $\partial\{u_n=0\}\cap\Omega$ and of $\partial\{u=0\}\cap\Omega$ respectively. We are going to prove the convergence of the free boundaries when $n\to+\infty$, which means that the free boundary of the problem of the compressible flow converges to the free boundary of the problem of the incompressible flow, when $g_n \underset{n}{\to} 1$.

Let us begin with a result of convergence of the angles of the wakes with the profile.

Lemma 7.7.

$$\theta_*^n \underset{n}{\to} \theta_*,$$

being θ_*^n the angle between the wake and the profile on the point of intersection of both, in the problem of the flow with density function g_n, and θ_* defined in the same way, but for the incompressible case.

Proof. Notice that

$$\theta_*^n=\inf\{\theta:v_n(\theta,s_n)>0\}=\sup\{\theta:v_n(\theta,\ s_n)=0\}$$

Let us see that θ_*^n converges on the right for θ_*. Suppose, by contradiction, that this does not happen, that is,

$$\exists\ \varepsilon>0\quad \forall\ p\in\mathbb{N}\quad \exists\ n_p\geq p\quad \theta_*^{n_p}>\theta_*+\varepsilon.$$

Observe that $v_n(\theta,s_n)\underset{n}{\to}v(\theta,s)$ uniformly, since $\hat{v}_n$ is uniformly bounded in $W^{2,p}_{loc}(\Omega_\infty)$. Since $v_{n_p}(\theta,s_{n_p})=0\ \ \forall\ \theta<\theta_*^{n_p}$, we have $v_{n_p}(\theta,s_{n_p})=0$, $\forall\,\theta<\theta_*+\varepsilon$ and so, passing to the limite, v will be zero for all (θ,s) such that $\theta_*<\theta<\theta+\varepsilon$, which is against the definition of θ_*.

Let us see now that θ_*^n converges on the left to θ_*. Suppose not; then

$$\exists\ \varepsilon>0\quad \forall\ p\in\mathbb{N}\quad \exists n_p\geq p\quad \theta_*^{n_p}<\theta_*-\varepsilon.$$

Fix $r, 0<r<\varepsilon$. Let $x_0=(\theta_*^{n_p},s_{n_p})$ and notice that $x_0\in\bar{N}_{n_p}$, where $N_{n_p}=\{(\theta,\sigma)\in\Omega_{s_{n_p}}:v_{n_p}(\theta,\sigma)>0\}$. Then, by a result of Caffarelli, ([14])

$$\sup_{B_r(x_0)}\{v_{n_p}(\theta,\sigma)-v_{n_p}(\theta_*^{n_p},s_{n_p})\}\geq\frac{\alpha}{4}r^2,$$

where $B_r(x_0)$ is the ball of center x_0 and radius r and $\alpha=\min_\theta|R(\theta)|>0$.

Observe that $v_{n_p}(\theta_*^{n_p},s_{n_p})=0$ and, by the definition of least upper bound, there exists $(\theta_{n_p},\sigma_{n_p})\in B_r(x_0)$ such that $v_{n_p}(\theta_{n_p},\sigma_{n_p})\geq\frac{\alpha}{8}r^2$. Since $v_{n_p}(\theta,\sigma)=v_{n_p}(\theta,s_{n_p})$ if $\sigma>s_{n_p}$, and $(v_{n_p})_\sigma(\theta,\sigma)\geq 0$, we conclude that $v_{n_p}(\theta_{n_p},s_{n_p})\geq v_{n_p}(\theta_{n_p},\sigma_{n_p})$. Since $(\theta_{n_p},\sigma_{n_p})\in B_r(x_0)$, we also have $(\theta_{n_p},s_{n_p})\in B_r(x_0)$ and $|\theta_{n_p}-\theta_*^{n_p}|<r<\varepsilon$. Notice that $\theta_{n_p}<\theta_*^{n_p}+r<\theta_*+(r-\varepsilon)<\theta_*$. Since $(\theta_{n_p})_{p\in\mathbb{N}}$ is a bounded sequence, it has a convergent subsequence to a number θ_0 and $\theta_0\leq\theta_*+(r-\varepsilon)<\theta_*$. Since $v_{n_p}(\theta_{n_p},s_{n_p})\geq\frac{\alpha}{8}r^2$, passing to the limit we conclude that $v(\theta_0,s)\geq\frac{\alpha}{8}r^2$, which is in contradiction with the definition of θ_*, since $\theta_0<\theta_*$. □

Fix $\delta>0$ and let $p\in\mathbb{N}$ be such that

$$\forall\ n\geq p \qquad \theta_*^n<\theta_*+\delta. \tag{71}$$

Proposition 7.8.

$$\|l_n-l\|_{C^{0,\alpha}(\theta_*+\delta,\theta_A-\delta)}\underset{n}{\to}0. \tag{72}$$

Proof. Let γ be a *cut off* function with compact support such that $0\leq\gamma\leq 1$, and $\gamma\equiv 1$ in a neighbourhood of $(\mathscr{L}_n\cup\mathscr{L})\cap]\theta_B+\delta/2, \theta_A-\delta/2[\times\mathbb{R}^+$ and such that $supp\gamma\subseteq\Omega_s\setminus\Sigma$, where $\Sigma=\{0\}\times[\inf_n\{\sigma_n\}, +\infty[$, n large enough.

Then, if v_n is the solution of problem (56) and v is the solution of the same problem for $n=+\infty$, γv_n is the solution of the variational inequality

$$a_n(\gamma v_n,v-\gamma v_n)\geq\int_{\Omega_s}\xi_n(\theta,\sigma)(v-\gamma v_n),\quad\forall\ v\in\mathbb{K}, \tag{73}$$

and γv is the solution of the variational inequality

$$a(\gamma v,v-\gamma v)\geq\int_{\Omega_s}\xi(\theta,\sigma)(v-\gamma v),\quad\forall\ v\in\mathbb{K}, \tag{74}$$

where $\mathbb{K}=\{v\in V: v\geq 0\}$,

$$\xi_n = \gamma q_n^2[R(\theta)+\varphi_n(\theta,\sigma)] - \left(\gamma_\sigma \frac{q_n^2}{k_n} v_n\right)_\sigma - \gamma_\sigma \frac{q_n^2}{k_n}(v_n)_\sigma$$

$$-q_n^2\gamma_{\theta\theta}v_n - 2q_n^2\gamma_\theta(v_n)_\theta,$$

being $\varphi_n\equiv 0$ if $\Omega_s\subset\Omega_{s_n}$ and being φ_n defined in (30) otherwise, and

$$\xi=\gamma e^{-2\sigma}R(\theta)-(\gamma_\sigma e^{-2\sigma}v)_\sigma-\gamma_\sigma e^{-2\sigma}v_\sigma-e^{-2\sigma}\gamma_{\theta\theta}v-2e^{-2\sigma}\gamma_\theta v_\theta.$$

Observe that

$$a(\gamma v_n, v-\gamma v_n) = a_n(\gamma v_n, v-\gamma v_n) + (a-a_n)(\gamma v_n, v-\gamma v_n)$$

$$\geq \int_{\Omega_s}\xi_n(v-\gamma v_n) - \int_{\Omega_s}\zeta_n(v-\gamma v_n), \quad \forall\, v\in\mathbb{K},$$

where

$$\zeta_n = \left[\left(\frac{q_n^2}{k_n}-e^{-2\sigma}\right)(\gamma v_n)_\sigma\right]_\sigma + [(q_n^2-e^{-2\sigma})\,(\gamma v_n)_{\theta\theta}]$$

$$+\ [(q_n^2-e^{-2\sigma})\gamma v_n].$$

Let $K=\gamma^{-1}(1)$. Since K is a compact subset of Ω_s,

$$a=d(K,\mathbb{R}^2\setminus\Omega_s)>0 \text{ and } K\subset\Omega_{s-a/2}$$

and, since $s_n \underset{n}{\rightarrow} s$,

$$\exists p\in\mathbb{N} \quad \forall\, n\geq p \qquad s_n>s-a/2,$$

and so, $K\subset\Omega_{s_n}$, for $n\geq p$.

Defining

$$\beta=\min_{K,n}\{q_n^2|R(\theta)|, \quad e^{-2\sigma}|R(\theta)|\},$$

we have, in K,

$$(\xi_n-\zeta_n)(\theta,\sigma)=q_n^2R(\theta)-(q_n^2-e^{-2\sigma})\leq-\frac{\beta}{2}<0$$

and

$$\xi(\theta,\sigma)=e^{-2\sigma}R(\theta)\leq-\beta<0,$$

for n suficiently large, since $q_n \underset{n}{\rightarrow} e^{-\sigma}$ uniformly in K (compact).

Using (just as for the stability of the free boundaries in s and h) a result of [16], we conclude that

$$\|l_n-l\|_{L^1(\theta_B+\delta,\theta_A-\delta)} = \|\chi_n-\chi\|_{L^1(K)} \leq \frac{2}{\beta}\int_\Omega|(\xi_n+\zeta_n)-\xi|,$$

where $\chi_n=\chi_{\{v_n=0\}}$ and $\chi=\chi_{\{v=0\}}$.

But

$$\int_\Omega|(\xi_n-\zeta_n)-\xi|\leq\int_\Omega|\xi_n-\xi|+\int_\Omega|\zeta_n|\ .$$

Observe that

$$\int_\Omega|\xi_n-\xi| \leq \int_\Omega\gamma|q_n^2-e^{-2\sigma}||R(\theta)|+\int_\Omega\left|\left(\gamma_\sigma\frac{q_n^2}{k_n}v_n\right)_\sigma-(e^{-2\sigma}\gamma_\sigma v)_\sigma\right|$$

$$+\int_\Omega\left|\gamma_\sigma\frac{q_n^2}{k_n}(v_n)_\sigma-\gamma_\sigma e^{-2\sigma}v_\sigma\right|+\int_\Omega|q_n^2\gamma_{\theta\theta}v_n-e^{-2\sigma}\gamma_{\theta\theta}v|$$

$$+\int_\Omega|2q_n^2\gamma_\theta(v_n)_\theta-2e^{-2\sigma}\gamma_\theta v_\theta|\underset{n}{\rightarrow}0,$$

and that

$$\int_\Omega|\zeta_n|\underset{n}{\rightarrow}0,$$

since $q_n\underset{n}{\rightarrow}e^{-\sigma}$, $q_n'\underset{n}{\rightarrow}-e^{-\sigma}$, $k_n\underset{n}{\rightarrow}1$, and $k_n'\underset{n}{\rightarrow}0$ uniformly in the compact subsets of $\mathbb{R}$, v_n and v are bounded in $W^2_{p,loc}$ (independently of n).

Since l_n are Lipschitz function, by Theorem 4.3, and the Lipschitz constants are independent of n, using the Gagliardo-Nirenberg inequality ([15]), we conclude,as in the preceding section that

$$\|l_n - l\|_{C^{0,\alpha}(\theta_* + \delta, \theta_\Lambda - \delta)} \underset{n}{\longrightarrow} 0. \tag{75}$$

□

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