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Positive Radial Solutions for Semilinear Biharmonic Equations in Annular Domains

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ABSTRACT. We study the existence of positive radial solutions of $\Delta^2 u = g(|x|)f(u)$ in an annulus with Dirichlet boundary conditions. We establish that the equation has at least one positive radially symmetric solution on any annulus if f and g are nonnegative, $g \not\equiv 0$ and f is superlinear at zero and $+\infty$. We also give a property of positive radial solutions.

1. INTRODUCTION

In this paper we consider the existence of positive radial solutions of the semilinear biharmonic equation

$$\Delta^2 u = g(|x|)f(u) \qquad \text{in } \Omega(a, b) \tag{1.1}$$

$$u = \frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial\Omega(a, b) \tag{1.2}$$

where $0 < a < b < +\infty$, $\Omega(a, b)$ denotes the annulus $\{x \in \mathbb{R}^n;\ a < |x| < b\}$ $(n \geq 2)$, $\frac{\partial}{\partial \nu}$ is the outward normal derivative and f, g satisfy the following hypotheses

(H_1) $f \in C([0, +\infty))$ and $f(u) \geq 0$ for $u > 0$.

(H_2) $g \in C([a, b])$, $g(r) \geq 0$ for $r \in [a, b]$ and $g \not\equiv 0$ in $[a, b]$.

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(H_3) $\lim_{u\to+\infty} f(u)/u = +\infty$.

(H_4) $\lim_{u\to 0} f(u)/u = 0$.

The analogous problem for the Laplace equation has been intensively studied in recent years (see e.g. [1]-[4], [8], [11], [12], [15]) and nearly optimal results have been obtained. In most papers, the shooting method was used to establish the existence of positive radial solutions. In contrast the result of [1] was obtained by a variational method and the use of a priori estimates, while in [15] an expansion fixed point theorem was applied.

Our main result is the following theorem.

Theorem 1.1. *Assume $(H_1)-(H_4)$. Then problem (1.1), (1.2) possesses at least one positive radial solution $u \in C^4(\overline{\Omega(a, b)})$.*

In this paper our method of proof makes use of a priori estimates and well-known properties of compact mappings taking a cone in a Banach space into itself (see [7]).

Since we are interested in positive radial solutions, the problem under consideration reduces to the one-dimensional boundary value problem

$$(1.3) \qquad \Delta^2 u(t) = g(t) f(u(t)), \qquad t \in (a, b)$$

$$(1.4) \qquad u^{(j)}(a) = u^{(j)}(b) = 0, \qquad j = 0,\ 1$$

where Δ denotes the polar form of the Laplacian, i.e. $\Delta = t^{1-n} \dfrac{d}{dt}\left(t^{n-1} \dfrac{d}{dt}\right)$.

Our next result gives a property of nonnegative nontrivial solutions of (1.3), (1.4) when f and g satisfy some monotonicity conditions.

Theorem 1.2. *Suppose that f and g in equation (1.3) satisfy the following assumptions:*

(H_5) $f: [0, +\infty) \to [0, +\infty)$ is nondecreasing.

(H_6) $g: [a, b] \to [0, +\infty)$ is nonincreasing.

Let $u \in C^4([a, b])$ be a nonnegative nontrivial solution of problem (1.3), (1.4). Then $\Delta u(a) > \Delta u(b)$.

Remark 1.1. Note that $\Delta u(a) = u''(a)$ (resp. $\Delta u(b) = u''(b)$) since $u'(a) = 0$ (resp. $u'(b) = 0$).

Remark 1.2. Theorems 1.1 and 1.2 can be easily extended to handle more general nonlinearities of the type $f(|x|, u)$.

Our paper is organized as follows. In Section 2 we give a maximum principle for fourth order equations and we describe the special shape of nontrivial solutions of (1.3), (1.4) when $f \geq 0$ and $g \geq 0$. In Section 3 we prove our a priori bounds for positive solutions of (1.3), (1.4). Theorem 1.1 is proved in Section 4. Finally, Section 5 contains the proof of theorem 1.2.

2. PRELIMINARIES

We have the following theorem.

Theorem 2.1. *Let $u \in C^4([a, b])$ be such that*

$$\begin{cases} \Delta^2 u \geq 0 \text{ in } (a, b) \\ u^{(j)}(a) = u^{(j)}(b) = 0,\ j = 0, 1. \end{cases}$$

Assume that $u \not\equiv 0$. Then:

(i) *There exist $r, s \in (a, b)$ such that $r < s$, $\Delta u > 0$ on $[a, r) \cup (s, b]$ and $\Delta u < 0$ on (r, s).*

(ii) *There exist $d_1, d_2 \in (r, s)$ such that $d_1 \leq d_2$, $(\Delta u)' < 0$ on $[a, d_1)$, $(\Delta u)' > 0$ on $(d_2, b]$ and $(\Delta u)' \equiv 0$ on $[d_1, d_2]$.*

(iii) *$u > 0$ on (a, b). Moreover there exists $c \in (r, s)$ such that $u' > 0$ on (a, c) and $u' < 0$ on (c, b).*

Proof. We first prove (i). Suppose that $\Delta u(a) \leq 0$ and $\Delta u(b) \leq 0$. Then the one-dimensional maximum principle ([14] p. 2) implies that $\Delta u \leq 0$ on $[a, b]$. Since $u(a) = u'(a) = u(b) = u'(b) = 0$, the maximum principle and the Hopf boundary lemma ([14] p. 4) imply that $u\equiv 0$ on $[a, b]$ and we reach a contradiction. Thus $\Delta u(a)>0$ or $\Delta u(b)>0$. Suppose for instance that $\Delta u(a)>0$. If $\Delta u\geq 0$ on $[a, b]$, we get a contradiction as before. Thus there exists $x\in(a, b)$ such that $\Delta u(x)<0$ and we can define $r\in(a, b)$ to be the first zero of Δu in (a, b). Since $u''(a)=\Delta u(a)>0$ we have $u>0$ in $(a, a+\eta]$ for some $\eta>0$. Using the maximum principle and the Hopf lemma we get $u'>0$ on $(a, r]$. Now, if $\Delta u(b)\leq 0$, the maximum principle implies $\Delta u<0$ on (r, b). Since $u(r)>0$ and $u(b)=u'(b)=0$, we again reach a contradiction. Thus we have proved that $\Delta u(b)>0$. Now we can define $s\in(r, b)$ to be the last zero of Δu in (a, b). Since $\Delta u(x)<0$, the maximum principle implies that $\Delta u<0$ on (r, s).

We now prove (ii). Denoting by $m<0$ the minimum value of Δu in $[a, b]$, we define $E=\{t\in(a, b)/\Delta u(t)=m\}$. Suppose first that E contains only one point. Then with the aid of the Hopf lemma we obtain (ii). Now, if E contains at least two points, the maximum principle and the continuity of Δu imply that $E=[d_1, d_2]$ where $r<d_1<d_2<s$. Then, using the Hopf lemma we obtain (ii).

Finally we prove (iii). We have already seen that $u'>0$ on $(a, r]$. In the same way we show that $u'<0$ on $[s, b)$. Now let t_0 (resp. t_1) be the first (resp. the last) zero of u' in (a, b). Clearly $r<t_0\leq t_1<s$. Suppose that $t_0<t_1$. Then the Hopf lemma implies that either $u'(t_0)>0$ or $u'(t_1)<0$, a contradiction. Thus $t_0=t_1$ and (iii) is proved.

3. A PRIORI BOUNDS

Theorem 3.1. *Assume $(H_1)-(H_3)$. Then there exists $M>0$ such that*

$$\|u\|_\infty\leq M$$

for all positive solutions $u\in C^4([a, b])$ of (1.3), (1.4).

Proof. We denote by S the set of all positive solutions of (1.3), (1.4) in $C^4([a, b])$. Let $u\in S$. By theorem 2.1 there exist $c(u)$, $d_1(u)$, $d_2(u)$, $r(u)$ and $s(u)$ in (a, b) such that $u'>0$ on $(a, c(u))$, $u'<0$ on $(c(u), b)$, $(\Delta u)'<0$ on $[a, d_1(u))$, $(\Delta u)'>0$ on $(d_2(u), b]$, $(\Delta u)'\equiv 0$ on $[d_1, d_2]$, $\Delta u>0$ on $[a, r(u))\cup(s(u), b]$ and $\Delta u<0$ on $(r(u), s(u))$. Moreover we have $a<r(u)<c(u)$, $d_1(u)$, $d_2(u)<s(u)<b$ and $d_1(u)\leq d_2(u)$.

We shall divide the proof into several steps. Subsequently C will denote various generic constants which may vary from line to line.

Step 1. We first prove that $\{gf(u);\ u\in S\}$ and $\{gu;\ u\in S\}$ are bounded in $L^1_{loc}(a, b)$. Define

$$\varrho(t)=(t-a)^2(t-b)^2 \qquad \text{for } a \le t\le b.$$

Let $\varphi\in C^4([a,b])$ be the solution of the boundary problem

$$\begin{cases}\Delta^2\varphi=g\varrho & \text{in } (a, b)\\ \varphi^{(j)}(a)=\varphi^{(j)}(b)=0, & j=0, 1.\end{cases}$$

By theorem 2.1 $\varphi>0$ in (a, b) and there exist $c_1>0$ and $c_2>0$ such that

$$c_1\varrho\le\varphi\le c_2\varrho \qquad \text{on } [a, b]. \tag{3.1}$$

By (H_3), there exists $\lambda>c_1^{-1}$ and $u_0\ge 0$ such that

$$f(u)\ge\lambda u \qquad \text{for } u\ge u_0. \tag{3.2}$$

If we multiply equation (1.3) by $t^{n-1}\varphi$ and integrate by parts four times we obtain

$$\int_a^b t^{n-1}\varphi gf(u)dt = \int_a^b t^{n-1}\varrho gudt. \tag{3.3}$$

From (3.2) and (3.3) we deduce

$$\int_a^b t^{n-1}\varrho gudt\ge\lambda\int_a^b t^{n-1}\varphi gudt-C\ge\lambda c_1\int_a^b t^{n-1}\varrho gudt-C$$

that is

$$\int_a^b t^{n-1}\varrho gudt\le \frac{C}{\lambda c_1-1} \tag{3.4}$$

$$\int_a^b t^{n-1}\varrho gf(u)dt\le \frac{C}{c_1(\lambda c_1-1)}. \tag{3.5}$$

Thus $\{gf(u);\ u\in S\}$ and $\{gu;\ u\in S\}$ are bounded in $L^1_{loc}(a, b)$.

Step 2. Now we prove the following lemma.

Lemma 3.1. *Let A be a subset of S. Then:*

(i) *If $\{\Delta u; u\in A\}$ is bounded in $L^1(a, b)$, then there exists a constant $M>0$ such that $\|u\|_\infty \le M$ for all $u\in A$.*

(ii) *If there exist $\gamma>0$, $\eta>0$ and $C>0$ such that $\gamma+\eta\le b-a$ and $u(t)\le C$ for $t\in[a, a+\gamma]\cup[b-\eta, b]$ and $u\in A$, then there exists a constant $M\ge C$ such that $\|u\|_\infty\le M$ for all $u\in A$.*

Proof. (i) follows readily from the fact that u and u' vanish at least once in $[a, b]$. We now prove (ii). Setting $m=\inf_{[a+\gamma, b-\eta]} \varrho(t)$ and using (3.5) we obtain

$$\int_a^b t^{n-1}\Delta^2 u dt = \int_a^b t^{n-1} g f(u) dt = \int_a^{a+\gamma} + \int_{a+\gamma}^{b-\eta} + \int_{b-\eta}^{b}$$

$$\le C + \int_{a+\gamma}^{b-\eta} t^{n-1}\frac{1}{m}\varrho g f(u) dt$$

$$\le C(1+\int_a^b t^{n-1}\varrho g f(u) dt) \le C.$$

Thus $\{\Delta^2 u; u\in A\}$ is bounded in $L^1(a, b)$. Since u, u', Δu and $(\Delta u)'$ vanish at least once in $[a, b]$ the result follows.

Step 3. Finally we prove that S is bounded in $L^\infty(a, b)$. Let $\gamma>0$, $\eta>0$ and $\delta>0$ be such that $\gamma+\eta+2\delta<b-a$ and $g>0$ on $[a+\gamma, b-\eta]$. Define $a'=a+\gamma+\delta$ and $b'=b-\eta-\delta$. Then by (3.4) there exists $K>0$ such that

$$\int_{a'}^{b'} u dt \le K. \tag{3.6}$$

Let $n_0>0$ be such that $1/n_0<(b'-a')/4$. By (3.6) we have

$$meas\ \{t\in[a', b'];\ u(t)\geqslant n_0 K\} \le \frac{1}{n_0} < \frac{b'-a'}{4}. \tag{3.7}$$

Now define

$$S_+=\{u\in S; c(u)>b'\}$$

$$S_-=\{u\in S; c(u)<a'\}$$

and

$$S_0=\{u\in S; a'\leq c(u)\leq b'\}.$$

Thus $S=S_-\cup S_0\cup S_+$. Let $\alpha=a'+(b'-a')/4$ and $\beta=b'-(b'-a')/4$. Clearly by the shape of u, (3.7) implies

(3.8) $$u(t)<n_0K \text{ for } t\in[\alpha, \beta] \text{ and } u\in S_+$$

and

(3.9) $$u(t)<n_0K \text{ for } t\in[\alpha, b] \text{ and } u\in S_-.$$

Lemma 3.2. *S_0 is bounded in $L^\infty(a, b)$.*

Proof. Let $u\in S_0$. From (3.4) we get

$$C\geq\int_{a+\gamma}^{b-\eta} udt\geq\int_{a+\gamma}^{a'} udt+\int_{b'}^{b-\eta} udt$$

which implies

$$u(t)\leq C/\delta \text{ for } t\in[a, a+\gamma]\cup[b-\eta, b]$$

and we get the conclusion by using lemma 3.1 (ii).

Clearly Theorem 3.1 follows from Lemma 3.2 and the next lemma.

Lemma 3.3. *$\underline{S}_+$ and $\underline{S}_-$ are bounded in $L^\infty(a, b)$.*

Proof. We shall show that S_- is bounded in $L^\infty(a, b)$. We first prove that $F=\{u\in S_-;\ s(u)>\beta\}$ is bounded in $L^\infty(a, b)$. We claim that for $u\in F$

$$\|u\|_\infty \leq M = n_0K + T(b-a)$$

where $T = (2/(\beta-\alpha))n_0K(b/a)^{n-1}$. Indeed, suppose this is not the case. Then let $u \in F$ be such that $u(c(u)) > M$. We have $c(u) < \alpha$. Let $t \in [\alpha, \beta]$ be arbitrary. By the mean value theorem there exists $x \in (c(u), t)$ such that, by virtue of (3.9)

$$u'(x) = \frac{u(t) - u(c(u))}{t - c(u)} < -T.$$

Since $r(u) < c(u) < \alpha < \beta < s(u)$, $y^{n-1}u'(y)$ is nonincreasing on $[x, t]$, thus we get

$$u'(t) \leq \left(\frac{x}{t}\right)^{n-1} u'(x) < -\left(\frac{a}{b}\right)^{n-1} T$$

with $t \in [\alpha, \beta]$. This and (3.9) imply

$$u(t) < n_0K - T\left(\frac{a}{b}\right)^{n-1}(t-\alpha) \qquad \text{for } t \in [\alpha, \beta].$$

For $t \in [(\alpha+\beta)/2, \beta]$, we deduce

$$u(t) < n_0K - T\left(\frac{a}{b}\right)^{n-1}\frac{\beta-\alpha}{2} = 0$$

and we reach a contradiction. Thus our assertion is proved

Now we prove that $G = \{u \in S_-;\ s(u) \leq \beta\}$ is bounded in $L^\infty(a, b)$. Let $t \in [\beta, b)$. We first show that $H = \{\Delta u(t);\ u \in G\}$ is bounded. Suppose $n \geq 3$. Since $\Delta u(s) \geq \Delta u(t) \geq 0$ for $s \in [t, b]$, we have

$$u(t) = \frac{1}{n-2}\int_t^b \left(\left(\frac{s}{t}\right)^{n-2} - 1\right) s\Delta u(s)\,ds \geq C(t)\Delta u(t)$$

where $C(t) > 0$ does not depend on u. Thus, if H is not bounded we get a contradiction with (3.9). When $n = 2$ the argument is the same.

Now let $\chi \in C^4([\beta, b])$ be the solution of the boundary problem

$$\begin{cases} \Delta^2\chi = 1 \text{ in } (\beta, b) \\ \chi^{(j)}(\beta) = \chi^{(j)}(b) = 0,\ j = 0, 1. \end{cases}$$

Multiplying equation (1.3) by $t^{n-1}\chi$ and integrating by parts four times we get

$$\int_t^b s^{n-1}\chi g f(u)ds = \int_t^b s^{n-1}\chi \Delta^2 u ds = -t^{n-1}\chi(t)(\Delta u)'(t)$$

$$(3.10) \qquad + t^{n-1}\chi'(t)\Delta u(t) - t^{n-1}\Delta\chi(t)u'(t)$$

$$+ t^{n-1}(\Delta\chi)'(t)u(t) + \int_t^b s^{n-1} u ds$$

for all $t \in [\beta, b)$. Setting $t=\beta$ in (3.10) we obtain

$$(3.11) \quad \int_\beta^b s^{n-1}\chi g f(u)ds = -\beta^{n-1}\Delta\chi(\beta)u'(\beta) + \beta^{n-1}(\Delta\chi)'(\beta)u(\beta) + \int_\beta^b s^{n-1} u ds.$$

Since by Theorem 2.1 $\Delta\chi(\beta)>0$, we deduce from (3.9) and (3.11) that $\{u'(\beta); u \in G\}$ is bounded. Since $\Delta u > 0$ on $(\beta, b]$ when $u \in G$, we have $\beta^{n-1}u'(\beta) < t^{n-1}u'(t) < 0$ for $t \in (\beta, b)$. Hence $\{u'(s); u \in G, s \in [\beta, b]\}$ is bounded. From this, (3.9), (3.10), Theorem 2.1 and the fact that $H = \{\Delta u(t); u \in G\}$ is bounded for each fixed $t \in [\beta, b)$ we deduce that $\{(\Delta u)'(t); u \in G\}$ is bounded for each fixed $t \in (\beta, b)$. Since for $t \in (\beta, b)$ we have

$$b^{n-1}(\Delta u)'(b) - t^{n-1}(\Delta u)'(t) = \int_t^b s^{n-1} g f(u) ds$$

we obtain that $\{(\Delta u)'(b); u \in G\}$ is bounded. Now let d be such that $d_1(u) \le d \le d_2(u)$. Using the fact that $t^{n-1}(\Delta u)'$ is nondecreasing on $[a, b]$ we can write

$$0 = d^{n-1}(\Delta u)'(d) \le t^{n-1}(\Delta u)'(t) \le b^{n-1}(\Delta u)'(b)$$

for $t \in [d, b]$, from which we deduce that $\{(\Delta u)'(t); u \in G, t \in [d, b]\}$ is bounded. Since

$$\Delta u(b) = \int_{s(u)}^b (\Delta u)' ds$$

we obtain that $\{\Delta u(b); u\in G\}$ is bounded. Now we write

$$\Delta u(d)=\Delta u(b)-\int_d^b(\Delta u)'ds$$

and we finally obtain that $\{\Delta u(t); u\in G,\ t\in[d, b]\}$ is bounded. Using the fact that

$$\int_a^b t^{n-1}\Delta u dt=0$$

for all $u\in S$, we easily deduce that $\{\Delta u; u\in G\}$ is bounded in $L^1(a, b)$ and the conclusion follows from lemma 3.1 (i).

It remains to prove that S_+ is bounded in $L^\infty(a, b)$. The proof is similar. Using analogous arguments we show that $\{u\in S_+; r(u)<\alpha]$ and $\{u\in S_+; r(u)\geq\alpha\}$ are bounded in $L^\infty(a, b)$. The proof of lemma 3.3 is complete.

Remark 3.1. Note that the constant M in theorem 3.1 can be chosen independently of the parameter $x\in[0, x_0]$ for each fixed $x_0\in(0, +\infty)$ if we consider positive solutions of (1.3), (1.4) for the family of nonlinearities $f_x(t)=f(t+x)$, $t\geq 0$.

4. PROOF OF THEOREM 1.1

We shall prove that problem (1.3), (1.4) has at least one positive solution $u\in C^4([a, b])$. The proof makes use of the Kranosel'skii type fixed point theorem [7] (proposition 2.1 and remark 2.1).

The homogeneous Dirichlet problem

$$\begin{cases}\Delta^2 v=0 \text{ in } (a, b)\\ v^{(j)}(a)=v^{(j)}(b)=0,\ j=0,\ 1\end{cases}$$

has only the trivial solution. Then it is well-known (see e.g. [13] p. 29) that the operator Δ^2 with Dirichlet boundary conditions has one and only one Green's function $G(t, s)$. Define the closed cone

$$Z = \{u \in C([a, b]); u \geq 0\}.$$

For $(u, x) \in Z \times [0, +\infty)$ we define

$$F(u, x)(t) = \int_a^b G(t, s) g(s) f(u(s) + x) ds$$

and

$$\Phi(u) = F(u, 0).$$

By Theorem 2.1 F maps $Z \times [0, +\infty)$ into Z. Since G is continuous, it is well-known that F is compact. Now the following properties hold:

(i) By Theorem 2.1 and the properties of the Green's function any nontrivial solution of the fixed point equation

$$\Phi(u) = u, \qquad u \in Z,$$

yields a positive solution of (1.3), (1.4) in $C^4([a, b])$.

(ii) $u \neq \theta \Phi(u)$ for all $\theta \in [0, 1]$ and $u \in Z$ such that $\|u\|_\infty = r$ for sufficiently small $r > 0$. Indeed, let $\alpha \in (0, c_2^{-1})$, where c_2 is the constant in (3.1). By (H_4) we can choose $r > 0$ such that $f(s) \leq \alpha s$ for $0 \leq s \leq r$. Now suppose that there exist $\theta \in [0, 1]$ and $u \in Z$ such that $u = \theta \Phi(u)$ with $\|u\|_\infty = r$. Then $\Delta^2 u = \theta g f(u)$. With the notations of step 1 of the proof of Theorem 3.1, we have

$$\int_a^b t^{n-1} \varrho g u dt = \int_a^b t^{n-1} u \Delta^2 \varphi dt = \int_a^b t^{n-1} \varphi \Delta^2 u dt$$

$$= \theta \int_a^b t^{n-1} \varphi g f(u) dt \leq \alpha c_2 \theta \int_a^b t^{n-1} \varrho g u dt$$

and we reach a contradiction.

(iii) By (H_3), there exists $\lambda > c_1^{-1}$ (where c_1 is the constant in (3.1)) and $x_0 > 0$ such that $f(t + x) \geq \lambda t$ for all $t \geq 0$ and $x \geq x_0$. Then using the same arguments as in step 1 of the proof of Theorem 3.1, we can show that the equation $F(u, x) = u$ has no solutions $u \in Z$ for $x \geq x_0$.

(iv) Finally, using Remark 3.1 and (iii) above we can find a constant $R>r$ such that $F(u, x)\neq u$ for all $x\geq 0$ and $u\in Z$ with $\|u\|_{\infty}=R$.

Now we can apply Proposition 2.1 and Remark 2.1 stated in [7] to conclude that Φ has a nontrivial fixed point. The proof of the theorem is complete.

5. PROOF OF THEOREM 1.2

The proof of Theorem 1.2 is based on the maximum principle and the technique of moving parallel planes as in [9], [16] for second order equations and [5], [6] for fourth order equations. Subsequently Δ denotes equally the cartesian form and the polar form of the Laplacian. In the same way we write indifferently $u(x)$ or $u(|x|)$.

Assume that $\Delta u(a)\leq\Delta u(b)$ for some nonnegative nontrivial solution $u\in C^4([a, b])$ of problem (1.3), (1.4). Then by Theorem 2.1 we have $u>0$ in $\Omega(a, b)$ and $\Delta u<\Delta u(b)$ in $\Omega(a, b)$. Let $\lambda\in\left[\frac{a+b}{2}, b\right)$ and define $\Sigma(\lambda)=\Omega(a, b)\cap\{x=(x_1, x')\in\mathbb{R}^n;\ x_1>\lambda\}$. Let $\Sigma'(\lambda)$ denote the reflection of $\Sigma(\lambda)$ in the plane $T_\lambda=\{x=(x_1, x')\in\mathbb{R}^n;\ x_1=\lambda\}$. Define the function

$$u_\lambda(x)=u(2\lambda-x_1, x') \qquad \text{for } x\in\Sigma'(\lambda).$$

We have the following Lemma.

Lemma 5.1. *u (resp. Δu) is strictly increasing (resp. strictly decreasing) as one enters $\Omega(a, b)$ from $\{x\in\mathbb{R}^n;\ |x|=b\}$ along any nontangential direction $\vec{s}$, for some positive distance $d>0$ into $\Omega(a, b)$.*

Proof. We have $u=\frac{\partial u}{\partial\nu}=0$ on $\partial\Omega(a, b)$. Since by Theorem 2.1 Δu is a positive constant on $\{x\in\mathbb{R}^n;\ |x|=b\}$ the proof is immediate.

From Lemma 5.1 we deduce the existence of $\eta\in\left(0, \frac{a+b}{2}\right)$ such that, for $\lambda\in[b-\eta, b)$, we have

$$(5.1) \qquad \begin{cases} u_\lambda - u < 0 \text{ in } \Sigma'(\lambda) \text{ and } \dfrac{\partial u}{\partial x_1} < 0 \text{ in } \Sigma(\lambda) \\ \Delta(u_\lambda - u) > 0 \text{ in } \Sigma'(\lambda) \text{ and } \dfrac{\partial \Delta u}{\partial x_1} > 0 \text{ in } \Sigma(\lambda). \end{cases}$$

Decrease λ until a critical value $\mu \geq (a+b)/2$ is reached, beyond which (5.1) is no longer true. Then (5.1) holds for $\lambda \in (\mu, b)$ while for $\lambda = \mu$ we have by continuity

$$\begin{cases} u_\mu - u \leq 0 \text{ in } \Sigma'(\mu) \text{ and } \dfrac{\partial u}{\partial x_1} < 0 \text{ in } \Sigma(\mu) \\ \Delta(u_\mu - u) \geq 0 \text{ in } \Sigma'(\mu) \text{ and } \dfrac{\partial \Delta u}{\partial x_1} > 0 \text{ in } \Sigma(\mu). \end{cases}$$

Suppose $\mu > (a+b)/2$. We have $u_\mu \not\equiv u$ in $\Sigma'(\mu)$ since $u > 0$ in $\Omega(a, b)$. The maximum principle ([10] p. 15) and the Hopf boundary lemma ([10] p. 33) imply that

$$(5.2) \qquad u_\mu - u < 0 \text{ in } \Sigma'(\mu) \text{ and } \frac{\partial u}{\partial x_1} < 0 \text{ on } T_\mu \cap \Omega(a, b)$$

where the second inequality follows from the fact that $\dfrac{\partial}{\partial x_1}(u_\mu - u) = -2\dfrac{\partial u}{\partial x_1}$ on $T_\mu \cap \Omega(a, b)$. Now (H_5) and (H_6) imply that $\Delta^2(u_\mu - u) \leq 0$ in $\Sigma'(\mu)$. From our assumption we have $\Delta u < \Delta u(b)$ in $\Omega(a, b)$. Thus $\Delta(u_\mu - u) \not\equiv 0$ in $\Sigma'(\mu)$. The maximum principle and the Hopf boundary lemma imply that

$$(5.3) \qquad \Delta(u_\mu - u) > 0 \text{ in } \Sigma'(\mu) \text{ and } \frac{\partial \Delta u}{\partial x_1} > 0 \text{ on } T_\mu \cap \Omega(a, b)$$

where the second inequality follows from the fact that $\dfrac{\partial}{\partial x_1}(\Delta(u_\mu - u)) = -2\dfrac{\partial \Delta u}{\partial x_1}$ on $T_\mu \cap \Omega(a, b)$. (5.2) and (5.3) show that (5.1) holds for $\lambda = \mu$.

Now our definition of μ implies that either there is a strictly increasing sequence (λ_j) with $\lim_{j\to\infty} \lambda_j = \mu (\lambda_j > (a+b)/2 \ \forall j)$ such that for each j there is a point $x_j \in \Sigma'(\lambda_j)$ for which

$$(5.4) \qquad u_{\lambda_j}(x_j) - u(x_j) \geq 0 \qquad \forall j$$

or that there is a strictly increasing sequence (μ_j) with $\lim_{j\to\infty} \mu_j = \mu$ $(\mu_j > (a+b)/2 \ \forall j)$ such that for each j there is a point $z_j \in \Sigma'(\mu_j)$ for which

$$(5.5) \qquad \Delta u_{\mu_j}(z_j) - \Delta u(z_j) \leq 0 \qquad \forall j\,.$$

In the situation (5.4), a subsequence which we still call x_j will converge to some point $x \in \overline{\Sigma'(\mu)}$; then $u_\mu(x) - u(x) \geq 0$. Since (5.1) holds for $\lambda = \mu$ we must have $x \in \partial\Sigma'(\mu)$; If $x \in \partial\Sigma'(\mu) \setminus T_\mu$ then $0 = u_\mu(x) < u(x)$, a contradiction. Therefore $x \in T_\mu$. Using Lemma 5.1, (5.2) and (5.1) with $\lambda = \mu$ we see that for some $\varepsilon > 0$ we have

$$(5.6) \qquad \frac{\partial u}{\partial x_1} < 0 \text{ in } \Omega(a, b) \cap \{x = (x_1, x') \in \mathbb{R}^n\,;\, x_1 > \mu - \varepsilon\}.$$

The straight segment joining x_j to its symmetric with respect to T_{λ_j} belongs to $\Omega(a, b)$ and by the theorem of the mean it contains a point y_j such that

$$\frac{\partial u}{\partial x_1}(y_j) \geq 0.$$

Since $\lim_{j\to\infty} y_j = x$, we obtain a contradiction to (5.6).

In the situation (5.5), a subsequence which we still call z_j will converge to some point $z \in \overline{\Sigma'(\mu)}$; then $\Delta u_\mu(z) - \Delta u(z) \leq 0$. Since (5.1) holds for $\lambda = \mu$ we must have $z \in \partial\Sigma'(\mu)$; If $z \in \partial\Sigma'(\mu) \setminus T_\mu$ then $\Delta u(b) = \Delta u_\mu(z) > \Delta u(z)$, a contradiction. Therefore $z \in T_\mu$. Using Lemma 5.1, (5.3) and (5.1) with $\lambda = \mu$ we see that for some $\varepsilon > 0$ we have

$$(5.7) \qquad \frac{\partial \Delta u}{\partial x_1} > 0 \text{ in } \Omega(a, b) \cap \{x = (x_1, x') \in \mathbb{R}^n\,;\, x_1 > \mu - \varepsilon\}.$$

The straight segment joining z_j to its symmetric with respect to T_{μ_j} belongs to $\Omega(a, b)$ and by the theorem of the mean it contains a point t_j such that

$$\frac{\partial \Delta u}{\partial x_1}(t_j) \leq 0.$$

Since $\lim_{j\to\infty} t_j = z$, we obtain a contradiction to (5.7).

Thus we have proved that $\mu=(a+b)/2$ and that (5.1) holds for $\lambda\in\left(\frac{a+b}{2}, b\right)$. By continuity we have

$$u_\varrho - u \leq 0 \text{ in } \Sigma'(\varrho) \text{ and } \frac{\partial u}{\partial x_1} < 0 \text{ in } \Sigma(\varrho)$$

and

$$\Delta(u_\varrho - u) \geq 0 \text{ in } \Sigma'(\varrho) \text{ and } \frac{\partial \Delta u}{\partial x_1} > 0 \text{ in } \Sigma(\varrho)$$

where $\varrho=\frac{a+b}{2}$. Now let $x=(a, 0)$ then

$$(u_\varrho - u)(x) = \frac{\partial}{\partial \nu}(u_\varrho - u)(x) = 0$$

and the Hopf lemma implies that $u_\varrho - u \equiv 0$ in $\Sigma'(\varrho)$, but this is impossible. The proof of the theorem is complete.

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