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# *Dunford-Pettis-like Properties of Projective and Natural Tensor Product Spaces*

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**ABSTRACT.** Several properties of weakly $p$-summable sequences and of the scale of *p-converging* operators (i.e., operators transforming weakly $p$-summable sequences into convergent sequences) in projective and natural tensor products with an $l_p$ space are considered. The last section studies the *Dunford-Pettis property of order p* (i.e., every weakly compact operator is $p$-convergent) in those spaces.

## 0. INTRODUCTION

In this paper several properties of the scale of p-converging operators in projective and natural tensor products with an $l_p$ space are considered. This scale, introduced in [2] and [3], is intermediate between the ideals of *unconditionally converging* operators and the ideal of *completely continuous* or *Dunford-Pettis* operators. Since $p$-converging operators are characterized by the property of sending *weakly-p-summable* sequences into convergent ones, a part of the study is devoted to a special class of subsets of vector sequence spaces, termed *almost compact* sets, nontrivial examples of which are, in certain spaces, precisely the weakly-p-summable sequences, $1 \leq p < +\infty$. Section 2 characterizes the compact sets of $l_p \hat{\otimes}_\pi X$ and $l_p \hat{\otimes}_{\Delta_p} X$, extending results of Leonard [8] and Bombal [1]. Section 3 considers the *Dunford-Pettis properties of order p* in projective and natural tensor product spaces of $l_p$ and a Banach space $X$. For $l_p$-sums of sequences of Banach spaces, generalizations of results of Bombal [1] are obtained. Those properties were introduced in [3].

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## 1. BACKGROUND

Throughout the paper $p^*$ denotes the conjugate number of $p$. We base our approach to the properties of natural and projective tensor products on the use of the representations of those spaces as sequence spaces. A sequence $(x_n)$ in a Banach space $X$ is said to be *weakly-p-summable* $(p\geq 1)$ if for every $x^*\in X^*$ the sequence $(x^*(x_n))$ is in $l_p$; equivalently (see [7] 19.4), if there is a constant $C>0$ such that, for each $(\xi_n)$ in $l_{p^*}$, $w_p((x_n))=\sup_k \{\|\Sigma_{n=1}^k \xi_n x_n\| : \|(\xi_n)\|_{l_{p^*}}\leq 1\}\leq C<+\infty$. (Here, if $p=1$, $c_0$ plays the role of $l_\infty$.) It is said to be *absolutely-p-summable*, when $p\geq 1$, if $s_p((x_n))=(\Sigma_{n=1}^{+\infty} \|x_n\|^p)^{\frac{1}{p}}<+\infty$. (If $p=+\infty$, the $l_p$ norm has to be replaced by the *sup* norm.) It is said to be *strongly-p-summable* for $p\geq 1$ if $\sigma_p((x_n))=\sup\{|\Sigma_{n=1}^{+\infty} f_n(x_n)| : w_{p^*}(\{f_n\})\leq 1,\ (f_n)\in X^*\}<+\infty$. Following [7], we shall denote by $l_p[X]$, $l_p\{X\}$ and $l_p<X>$ respectively the spaces of weakly-$p$-summable, absolutely-$p$-summable and strongly-$p$-summable sequences of $X$, endowed with their natural topologies: those induced by the norms $w_p$, $s_p$ and $\sigma_p$, respectively. The following isometries are well-known (see [7] 19.4.3): $l_p[X]=L(l_{p^*}, X)$, for $1<p<+\infty$, and $l_1[X]=L(c_0, X)$. The symbols $\pi$ and $\varepsilon$ shall denote the projective and injective norms on the space $l_p \otimes X$: they are, respectively, the strongest and coarsest *crossnorms* (i.e. norms satisfying $\|x\otimes y\|=\|x\|\,\|y\|$) which is possible to define on that space. The symbol $\Delta_p$ denotes the norm induced by $s_p$ over $l_p\otimes X$; the topology induced by $s_p$ is termed the *natural* topology. We shall denote by $l_p\hat{\otimes}_\varepsilon X$, $l_p\hat{\otimes}_\pi X$ *and* $l_p\hat{\otimes}_{\Delta_p} X=l_p\{X\}$ the completion of $l_p\otimes X$ with respect to $\varepsilon$, $\pi$, and $\Delta_p$, respectively. The space $l_p\hat{\otimes}_\pi X$ also admits a representation as a vector sequence space: it is the closed subspace of the space $l_p<X>$ formed by those sequences which are the limit of their finite sections; this can be deduced without difficulty from [5], where it is proved that the norm $\sigma_p$ induces $\pi$ over $l_p\otimes X$.

Let $E$ be any of the spaces $l_p\hat{\otimes}_\pi X$ or $l_p\hat{\otimes}_{\Delta_p} X$, $p_k$ be the continuous projection onto the $k^{th}$ coordinate, and $i_k$ the canonical inclusion of $X$ into the $k^{th}$ coordinate. If $T:E\rightarrow Y$ is a continuous operator, then a sequence of operators $T_k\in L(X, Y)$ exists such that $T=\Sigma_k T_k p_k$: explicitely, $T_k=Ti_k$. We shall say that $(T_k)$ is the representing sequence of $T$. If $(X_n)$ is a sequence of Banach spaces, and $T$ is an operator from the Banach space $(\Sigma_n\oplus X_n)_p=\{x=(x_n)\in\Pi_n X_n : \|x\|_p=(\Sigma_n\|x_n\|^p)^{1/p}<+\infty\}$ into $Y$, then the sequence $(T_k)$ defined by $T_k=Ti_k$ is again called the representing sequence of $T$ (cf. [1]).

We shall consider the following operator ideals: The ideal $L$ of all continuous operators; the ideal $W$ of *weakly compact* operators; the ideal $U$ of *unconditionally converging* operators, i.e., those sending weakly-1-summable sequences into unconditionally summable sequences; the ideal $K$ of compact operators; and the ideal $DP$ of *completely continuous or Dunford-Pettis operators*, i.e., those sending weakly convergent sequences into convergent ones.

**Definition.** *We say that an operator $T \in L(X, Y)$ is p-converging, $1 \leq p < +\infty$, if it transforms weakly-p-summable sequences of X into norm null sequences of Y. We shall use $C_p$ to denote the ideal of p-converging operators.*

The classes $C_p$ form injective, non-surjective closed operator ideals. It is not difficult to see that $C_1 = U$ and, with the convention that the weakly-$\infty$-summable are the weakly null sequences, that $C_\infty = DP$. A characterization of $p$-converging operators is contained in the following proposition (see [3]):

**Proposition 0.** *Let X be a Banach space, and $1 \leq p < +\infty$. If $p > 1$ the operator $Id(X)$ belongs to $C_p$ if and only if all operators from $l_{p^*}$ into X are compact. If $p = 1$, $Id(X)$ belongs to $C_1$ if and only if all operators from $c_0$ into X are compact.*

## 2. COMPACT SETS

We shall study in Section 3 the relation between the membership of an operator $T$ in a class $C_p$, and the membership of the operators forming its representing sequence in that same class. To this end, we shall introduce a class of subsets which have something of the flavour of compact sets.

**Lemma 1.** *Let $1 \leq p < +\infty$. Let X and Y be Banach spaces. Consider a set $A \subset l_p \hat{\otimes}_\pi X$ (resp. $A \subset l_p \hat{\otimes}_{\Delta_p} X$). The following are equivalent:*

1. *For each continuous operator $T \in L(l_p \hat{\otimes}_\pi X, Y)$ (resp. $T \in L(l_p \hat{\otimes}_{\Delta_p} X, Y)$), the representing sequence of T converges to T uniformly over A.*

2. $\lim_{N\to+\infty} \sup_{x\in A} \pi\, [(x_k)_{k\geq N}]=0$, (resp., $\lim_{N\to+\infty} \sup_{x\in A} \Delta_p\, [(x_k)_{k\geq N}]=0$).

**Proof.** That $1\Rightarrow 2$ is obvious. Let us show that $2\Rightarrow 1$ for the case of the projective tensor product. Let $(x_n)$ be any sequence in $A$. Then

$$\|T((x_n))-(T_1(x_1),\, T_2(x_2),\, \ldots,\, T_N(x_N),\, 0,\, 0,\ldots)\|_Y =$$
$$=\|T(0,\, 0,\ldots,\, 0,\, x_{N+1},\, x_{N+2},\ldots)\|_Y \leq \|T\|\pi[(0,\, 0,\ldots,\, 0,\, x_{N+1},\, x_{N+2},\ldots)]$$

and this converges uniformly on $A$ by Condition 2.

The computations for the natural product are very similar. ■

**Definition.** *Let $p<+\infty$. A set $A\subseteq l_p \hat{\otimes}_\pi X$, (resp. $A\subset l_p \hat{\otimes}_{\Delta_p} X$), is said to be almost-compact if it satisfies either of the equivalent conditions of Lemma 1.*

**Proposition 2.** *Let $p<+\infty$. A subset $A\subseteq l_p \hat{\otimes}_\pi X$, (resp. $A\subset l_p \hat{\otimes}_{\Delta_p} X$), is relatively compact if and only if it is bounded, almost-compact, and its continuous projections $p_k(A)$ are relatively compact in $X$ for all $k\in\mathbb{N}$.*

**Proof.** It is easy to see that all the conditions are necessary. They are also sufficient: Let $(x^n)$ be a sequence contained in $A\subseteq l_p\hat{\otimes}_\pi X$. Condition 1 of Lemma 1 and a diagonal argument show that a certain sub-sequence, again denoted $(x^n)$, exists having pointwise convergence to an element $x$. To verify that the convergence occurs in the projective norm, it is only necessary to take, in the following expression, the supremum over all elements $x^*$ in the unit ball of $l_{p^*}[X^*]$:

$$\sum_{k=1}^{+\infty} |<x_k^n - x_k, x_k^*>| \leq \sum_{k=1}^{k=N} |<x_k^n - x_k\,,\, x_k^*>| +$$

$$+ \sum_{k=N+1}^{+\infty} |<x_k^n, x_k^*>| + \sum_{k=N+1}^{+\infty} |<x_k\,,\, x_k^*>|,$$

and observe that the first summand can be made, for large $N$, less than $\varepsilon$; since $A$ is almost compact, the second and third summands tend to zero when $N$ tends to infinity.

The proof for the natural product is analogous. ■

**Remark.** Lemma 1 and Proposition 2 have been proved in [1] and [8] for $l_p\{X\}$. The referee has informed to us that this proposition is a particular case of an old theorem due to Mazur, who stated it for the case of a Banach space having a basis, and that a more general result has been established by Goes and Welland as follows:

**Theorem ([6] Thm. 1.)** *Let X be a complete locally convex topological vector space. Let A be a bounded subset of X and $\{P_\beta\}_{\beta\in I}$ a net in L(X, X). Then A is relatively compact if $\{P_\beta\}_{\beta\in I}$ converges uniformly to the identity on A and $P_\beta(A)$ is relatively compact for each $\beta\in I$.*

Proposition 2 follows taking $P_N((x_n)_n)=(x_1, x_2,\dots, x_N, 0, 0,\dots)$ for $N\in\mathbb{N}$. We have left the proof of Proposition 2 for the sake of completeness.

Nontrivial examples of almost compact sets in natural and projective tensor products are provided by the next proposition.

**Proposition 3.** *Assume that X is a Banach space and that $1\le p,\ r<+\infty$. If $r<p^*$, then a weakly-r-summable sequence of $l_p\widehat{\otimes}_\pi X$ or $l_p\widehat{\otimes}_{\Delta_p} X$ is an almost compact set. For $p=1$, a weakly null sequence of $l_1\{X\}$ is an almost compact set.*

**Proof.** We first show the proof for the projective product. Let $(a^n)$ be a weakly-$r$-summable sequence in $l_p\widehat{\otimes}_\pi X$. Assume that $A=\{a^n:n\in\mathbb{N}\}$ is not an almost compact set. In that case, an $\varepsilon>0$ and two sequences $(n_i)$ and $(N_i)$ of naturals exist such that if $I_i$ denotes the set $\{N_i+1, \dots, N_{i+1}\}$ and $P_i:l_p\widehat{\otimes}_\pi X \to l_p\widehat{\otimes}_\pi X$ denotes the projection over the indices of $I_i$ then

$$\pi_p(P_i(a^{n_i}))>\varepsilon.$$

Elements $z_i\in(l_p\widehat{\otimes}_\pi X)^*=L(l_p, X^*)$ with $\|z_i\|\le 1$ can be chosen such that $|<P_i(a^{n_i}), z_i>|>\varepsilon$. The proof of [4, Thm. 1] shows that if $Q_i:l_p\to l_p$ denotes the projection over the indices of $I_i$, then $|<P_i(a^{n_i}), z_iQ_i>|>\varepsilon$. Once more, the proof of [4, Thm. 1] shows that the operator $B:l_p\widehat{\otimes}_\pi X\to l_p$

defined by $B(Y)=(<P_i y, z_i Q_i>)$ is continuous. By [3, Prop. 1.6.], it transforms $(a^n)$ into a norm-null sequence of $l_p$, which is a contradiction.

The proof for the natural product is essentially the same. We shall give it for the sake of completeness: If $A=\{a^n : n\in\mathbb{N}\}$ is not almost compact, then an $\varepsilon>0$ and two sequences $(n_i)$ and $(N_i)$ of naturals exist such that

$$\sum_{k=N_i}^{k=N_{i+1}} \|a_k^{n_i}\|^p > \varepsilon .$$

Normalized elements $x_i^*(k)\in X^*$ can be chosen such that: $<x_i^*(k), a_k^{n_i}> = \|a_k^{n_i}\|$. If $y_k^* = x_i^*(k)$, for $N_i \leq k < N_{i+1}$, then $(y_k^*)$ is a bounded sequence of $X^*$ which defines an element of $L(l_p\{X\}, l_p)$. This operator transforms $(a^n)$ into a weakly-$r$-summable sequence of $l_p$, which must be norm-null (see [3, Prop. 1.6.]). Thus one has

$$\lim_{N\to+\infty} \sup_{n\in\mathbb{N}} \left[\sum_{k=N}^{k=+\infty} |<y_k^*, a_k^n>|^p\right]^{\frac{1}{p}} = 0,$$

which is a contradiction.

The proof for the case $p=1$ follows closely that of the natural product, and it is only necessary to recall that $l_1$ has the Schur property, i.e.: weakly null sequences are norm null. That yields the proof for the projective tensor product since $l_1 \widehat{\otimes}_\pi X = l_1\{X\}$. In other words: the statement holds for $p=1$ and $r=\infty$. ∎

**Remark.** Let $X_n$ be a sequence of Banach spaces, and $1\leq p<+\infty$. A set $A\subseteq(\Sigma_n \oplus X_n)_p$ is said to be *almost compact* if Conditions 1 or 2 of Lemma 1, with suitable modifications, are satisfied. In this form, Propositions 2 and 3 can be translated to $l_p$-sums of sequences of Banach spaces.

## 3. DUNFORD-PETTIS PROPERTIES

A Banach space $X$ is said to have the *Dunford-Pettis property* (*DPP*) if weakly compact operators defined on $X$ are completely continuous, that is, if for any Banach space $Y: W(X, Y)\subseteq DP(X, Y)$. Typical examples of Banach spaces having *DPP* are $L_\infty$ and $L_1$ spaces. No reflexive Banach space can have *DPP*. Weakened versions of the Dunford-Pettis property were introduced in [3]. A Banach space $X$ is said to have the Dunford-Pettis property of order $p \geq 1$, if $W(X, Y) \subseteq C_p(X, Y)$ for all Banach spaces $Y$. We shall call this property $DPP_p$. Notice that $DPP_\infty = DPP$. Every Banach space has $DPP_1$. Other examples are (see [3] for details): $l_p$ has $DPP_r$ for all $r < p^*$; $L_p[0, 1]$ has $DPP_r$ for $r < min\{p^*, 2\}$; Tsirelson's space has $DPP_r$ for all $r < +\infty$, but not $DPP$ since it is reflexive; if $id(X) \in C_p$ then $C(K, X)$ has $DPP_p$.

**Lemma 4.** *Let $1 \leq p < +\infty$. Let $(X_n)$ be a sequence of Banach spaces. Assume that $E$ represents any of the spaces $(\Sigma \oplus X_n)_p$ or $l_p \hat{\otimes}_\pi X$, and that $T$ is a continuous operator from $E$ into a Banach space $Y$, having $(T_k)$ as a representing sequence. If $r < p^*$ (or $p = 1$ and $r = \infty$), then $T$ is $r$-converging if and only if each $T_k$ is $r$-converging.*

**Proof.** Let $(a^n)$ be a weakly-$r$-summable sequence of $E$. Since $(a^n)$ is an almost compact set, the convergence of $(T_k)$ to $T$ is uniform over the set $\{a^n\}$. Furthermore, $T_k(\{a^n\})$ is relatively compact in $Y$ since $T_k$ is $r$-converging. The relationship

$$T(\{a^n\}) \subseteq \sum_{k=1}^{k=N(\varepsilon)} T_k(\{a^n\}) + \varepsilon B_Y$$

implies that $T(\{a^n\})$ is relatively compact, and therefore $(Ta^n)$ must be norm-null. ∎

**Remark.** When $p^* \leq r < \infty$ the result is clearly false: simply consider the example $l_p\{l_1\}$ and $T = id$.

**Proposition 5.** *Let $A$ denote an operator ideal and $r < p^*$ (or $p = 1$ and $r = \infty$). With the same notation as in Lemma 4, $A((\Sigma \oplus X_n)_p, Y) \subseteq C_r((\Sigma \oplus X_n)_p, Y)$ if and only if, for all $n$, $A(X_n, Y) \subseteq C_r(X_n, Y)$. Moreover $A(l_p \hat{\otimes}_\pi X) \subseteq C_r(l_p \hat{\otimes}_\pi X)$ if and only if $A(X, Y) \subseteq C_r(X, Y)$.*

**Remark.** Recalling that $C_1 = U$ and that $C_\infty = DP$, one sees that these results include and generalize the following results of Bombal [1]: Theorem 1.5, part a) for the unconditionally converging operators ($p = 1$,

$r=1$ in Lemma 4) and Dunford-Pettis operators ($p=1$, $r=\infty$ in Lemma 4; Corollary 1.6. part a) for the unconditionally converging operators ($p=1$, $r=1$ in Proposition 5) and Dunford-Pettis operators ($p=1$, $r=\infty$ in Proposition 5) and Proposition 2.6. a) ($p=1$, $r=\infty$, $A=L$), and c) ($p=1$, $r=1$, $A=L$); this last case also appears in [4].

**Theorem 6.** *Let $1\leq p<+\infty$. Assume that $r<p^*$ (or $p=1$ and $r=\infty$): these are the cases when $l_p$ has $DPP_r$. Assume that $X$ also has $DPP_r$. Then $l_p\hat{\otimes}_\pi X$ and $l_p\hat{\otimes}_{\Delta_p}X$ also have $DPP_r$.*

**Proof.** Let $E$ denote any of those spaces, and let $T:E\to Y$ be a weakly compact operator. Since $X$ has $DPP_r$ the operators $(T_k)$ in the representing sequence of $T$, which necessarily are weakly compact, are $p$-converging. By Lemma 4, $T$ must also be $p$-converging. ■

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