Existence and Nonexistence of Nontrivial Solutions for Some Nonlinear Elliptic Systems

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ABSTRACT. In this paper we give some existence and nonexistence results of non trivial solutions of nonlinear elliptic systems involving the p-Laplacian.

0. INTRODUCTION

In this paper, we give some existence and nonexistence results concerning nonlinear elliptic systems. The case of one equation has been studied by many authors.

Let $\Omega$ be a bounded regular open set in $\mathbb{R}^n$ and consider the problem
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\[
\begin{align*}
(P_\lambda) \\
\begin{cases}
-\Delta u = \lambda f(u) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\end{align*}
\]

where \( f(u) \in C^0(\mathbb{R}) \), \( 0 < \alpha < 1 \), is such that: \( f(0) = 0 \) and \( |f(u)| \leq A \cdot B \cdot |u|^\alpha \).

Any solution \( u^* \) of \( (P_\lambda) \) satisfies the Pohožaev’s identity [21]:

\[
\int_\Omega \left[ \frac{n-2}{2n} u \cdot f(u^*) - \int_0^{u^*} f(s)ds \right] dx = -\frac{1}{2} \int_{\partial \Omega} |\nabla u^*|^2 (x \cdot \nu) d\sigma,
\]

whence \( u^* = 0 \) if \( \Omega \) is starshaped and

\[
\lambda \left[ \frac{n-2}{2n} u \cdot f(u^*) - \int_0^{u^*} f(s)ds \right] > 0.
\]

On the other hand, if

\[
0 < m+1 < \frac{2n}{n-2},
\]

Pohožaev [21] has shown that \( (P_\lambda) \) admits an eigenfunction \( u^* = 0 \) corresponding to \( \lambda \).

Always in the scalar case, Ōtani [19], [20] and de Thélín [25] generalize these results for the \( p \)-Laplacian \( \Delta_p u = div(|\nabla u|^{p-2} \nabla u) \).

For example, they give the following results concerning the equation

\[
(E_\lambda) \quad -\Delta_p u = \lambda \cdot |u|^{m-1}u
\]

- If \( \Omega \) is a strictly starshaped open set and \( (m+1)(n-p) \geq np \) the only solution \( u^* \in W^{1,p}_0(\Omega) \) of \( (E_\lambda) \) is \( u^* \equiv 0 \).

- If \( (m+1)(n-p) < np \) and \( m+1 \neq p \), then for any \( \lambda > 0 \), \( (E_\lambda) \) admits a positive solution \( u^* \in W^{1,p}_0(\Omega) \).
- If \( m+1 = p \), we have an eigenvalue problem [3].

More recently, in [32], we have given some results concerning the existence and nonexistence of a nontrivial solution \((u^*,v^*) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)\) of the following system

\[
\begin{cases}
-\Delta u = \frac{u^{\alpha-1}}{\alpha} + \frac{v^{\beta+1}}{\beta+1} & \text{in } \Omega \\
-\Delta v = \frac{u^{\alpha+1}v}{\alpha+1} & \text{in } \Omega
\end{cases}
\]

We prove

1) nonexistence results when

\[
(\alpha+1) \frac{n-p}{np} + (\beta+1) \frac{n-q}{nq} \geq 1
\]

when \( \Omega \) is a strictly starshaped open set;

2) existence results when

\[
(\alpha+1) \frac{n-p}{np} + (\beta+1) \frac{n-q}{nq} < 1
\]

and when

\[
\frac{\alpha+1}{p} + \frac{\beta+1}{q} \neq 1.
\]

Now, in this paper, we extend the study of existence and nonexistence of positive solutions of the nonlinear elliptic problem

\[
(P) \quad \begin{cases}
-\Delta u = f(x;u,v) & \text{in } \Omega \\
-\Delta v = g(x;u,v) & \text{in } \Omega \\
u = 0, \ v = 0 & \text{on } \partial \Omega.
\end{cases}
\]
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We say that $(P)$ is a potential system if there is a $C^1$ function $H$ such that

$$f(x,s,t) = \frac{\partial H}{\partial s}(x,s,t), \quad g(x,s,t) = \frac{\partial H}{\partial t}(x,s,t).$$

In a first part, following Egnell [10] and Pucci-Serrin [22], we obtain a Pohožaev type identity for potential systems. In the case when $\Omega$ is a starshaped bounded open set, this identity gives nonexistence results.

In a second part, we give some existence results for non potential systems. Following Deuel and Hess [7], we construct appropriate sub-supersolutions for $(P)$ and use a suitable comparison principle.

In a third part, we give some existence results for potential systems. Following Nirenberg [18], we apply Mountain-Pass Lemma to find a nontrivial solution; after that, we extend an iterative method previously used by Ōtani [20] for the equation $(E_0)$ to prove that the solution is bounded.

Concerning the systems, we can notice the existence results obtained in [4], [6], [11], [12], [28]. Independently, [13], [22] give nonexistence results.

1. NONEXISTENCE RESULT

In this first section, we propose to extend the non-existence study, made by de Thélın [26] and Egnell [10] in the scalar case, to the following problem $(P)$

$$\begin{align*}
\text{Find } (u,v) \in X \cap [L^\infty(\Omega)]^2 \text{ such that } \\
(1) \quad & -\Delta u = \frac{\partial H}{\partial u}(x,u,v) \quad \text{in } \Omega \\
(2) \quad & -\Delta v = \frac{\partial H}{\partial v}(x,u,v) \quad \text{in } \Omega \\
& u > 0 \quad \text{in } \Omega \\
& v > 0 \quad \text{in } \Omega
\end{align*}$$

$$\tag{P}$$
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Hereafter, $X$ denotes the space $W^{0,q}_0(\Omega) \times W^{0,q}_0(\Omega)$.

1.1. Properties and Results.

**Theorem 1.1.** Assume the following hypotheses

i) $H(x;0,0) = 0$ and $\frac{\partial H}{\partial s}(x;0,0) = \frac{\partial H}{\partial t}(x;0,0) = 0$

ii) $\frac{\partial H}{\partial s}(x;s,t), \frac{\partial H}{\partial t}(x;s,t)$ are in $C(\Omega \times \mathbb{R} \times \mathbb{R})$ and $\frac{\partial H}{\partial s}(x;s,t) \geq 0$

\[
\frac{\partial H}{\partial t}(x;s,t) \geq 0 \text{ for any } s,t \geq 0 \text{ and } x \in \Omega
\]

iii) $\forall (s,t) \in \mathbb{R}^2$

\[
H(x;s,t) \leq \frac{n-p}{np} \left( \frac{\partial H}{\partial s}(x;s,t) \right)^+ + \frac{n-q}{nq} \left( \frac{\partial H}{\partial t}(x;s,t) \right)^- - \frac{x}{n} \nabla H(x;s,t)
\]

iv) $\Omega$ is a bounded strictly starshaped domain in $\mathbb{R}^n$ containing $0$.

Then, $(u^*,v^*) = 0$ is the only solution of $(P)$ in $X \cap [L^\infty(\Omega)]^2$.

**Corollary 1.1.** Let $\Omega$ be a bounded strictly starshaped domain in $\mathbb{R}^n$ and $H(x;s,t) = |s|^{\alpha+1} |t|^{\beta+1}$.

If

\[
(\alpha+1) \frac{n-p}{np} + (\beta+1) \frac{n-q}{nq} \geq 1,
\]

$(P)$ has only the trivial solution $(0,0)$ in $X \cap [L^\infty(\Omega)]^2$.

**Proof of the Corollary 1.1.** Since

\[
(\alpha+1) \frac{n-p}{np} + (\beta+1) \frac{n-q}{nq} \geq 1,
\]

we have
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\begin{equation}
H(x,s,t) \leq \left[ (\alpha+1)\frac{n-p}{np} + (\beta+1)\frac{n-q}{nq} \right] H(x,s,t)
\end{equation}

(1.1)

\begin{equation}
\leq \frac{n-p}{np} \left\{ \frac{\partial H}{\partial s}(x,s,t) \right\} + \frac{n-q}{nq} \left\{ \frac{\partial H}{\partial t}(x,s,t) \right\}
\end{equation}

and all the hypotheses of Theorem 1.1 are satisfied.

The proof of Theorem 1.1 needs the following lemma which extends Egnell’s one [10].

**Lemma 1.1.1.** Let \((u^*, v^*)\) be a solution of the problem \((P)\); then for all \(x\) on the boundary of \(\Omega\), we have: \(|\nabla u^*(x)| \neq 0\) and \(|\nabla v^*(x)| \neq 0\).

**Proof.** Let \(x_0 \in \partial \Omega\); there is a ball \(B_r \subset \Omega\).

By translation we assume that \(B_{r_0} = \{x \in \Omega; |x| < r_0\}\) and, proceeding as in [10], we introduce the function

\[ g(x) = k(e^{-\alpha|x|^p} - e^{-\alpha|x|^q}) \]

For \(p > 1\), a suitable choose of \(\alpha\) gives \(g_p\) such that

\begin{equation}
-\text{div}(\nabla g_p)^{p-2}\nabla g_p \leq a g_p^{-1} \quad \text{in} \quad B_r \setminus B_{r/2}
\end{equation}

(1.2)

Multiplying (1) and (1.2), [resp. (2) and (1.2).] by the test function \(\varphi_p = (g_p - u^*)\), [resp \(\varphi_q = (g_q - v^*)\)] and integrating on the set \(B^*_p = \{x \in B_{r} \setminus B_{r/2}; \varphi_p > 0\}\) [resp. \(B^*_q\)] where \(u^*\) and \(v^*\) are regular, we obtain

\[ 0 \leq \int_{B^*_p} (|\nabla g_p|^{p-2}\nabla g_p - |\nabla u^*|^{p-2}\nabla u^*) \nabla \varphi_p \, dx \leq -\int_{B^*_p} \frac{\partial H}{\partial u}(x; u^*, v^*) \varphi_p \, dx \]

whence, \(g_p \leq u^*\) in \(B_r \setminus B_{r/2}\).
By construction $g_p(x_0) = u^*(x_0) = 0$, therefore

\begin{equation}
|\nabla u^*(x_0)| > 2k_p \alpha_p e^{-3} > 0
\end{equation}

**Proof of Theorem 1.1.** Let \((u^*, v^*)\) be a nontrivial solution of \((P)\). For \(i = 1, \ldots, n; \ell = 1, \ldots, n\) let

\[ P_i = \sum_{i=1}^{n} |\nabla u^*|^{p-2} \frac{\partial u^*}{\partial x_i} \frac{\partial u^*}{\partial x_i} \quad \text{and} \quad Q_{i\ell} = \sum_{i=1}^{n} |\nabla v^*|^{p-2} \frac{\partial v^*}{\partial x_i} \frac{\partial v^*}{\partial x_{i\ell}} \]

Let \(K_p = \{x \in \Omega; |\nabla u^*(x)| = 0\}, K_q = \{x \in \Omega; |\nabla v^*(x)| = 0\}.

Lemma 1.1. allows us to consider as in [10], the sets \(\Omega_k \supset \Omega_k \subset \Omega, K_p \subset \Omega_k \subset \Omega, \) with \(\text{dist}(K_p, \partial \Omega_k) \to 0, \text{dist}(K_p, \partial \Omega_k) \to 0\), as \(k \to +\infty\) and we define \(\Omega_k = \Omega \cap \Omega_k, \Omega_k = \Omega \cap \Omega_k\).

\begin{equation}
\sum_{i=1}^{n} \int_{\Omega} \frac{\partial P_i}{\partial x_i} dx = \sum_{i=1}^{n} \int_{\Omega_k} \left( \sum_{i=1}^{n} x_{i1} \frac{\partial u^*}{\partial x_i} \frac{\partial u^*}{\partial x_i} \right) dx + \int_{\Omega_k} |\nabla u^*|^p dx
\end{equation}

\begin{equation}
= -\int_{\Omega_k} \left( \sum_{i=1}^{n} x_{i1} \frac{\partial u^*}{\partial x_i} \frac{\partial H(x;u^*,v^*)}{\partial u} \right) dx + \int_{\Omega_k} |\nabla u^*|^p dx
\end{equation}

\begin{equation}
= -\int_{\Omega_k} \frac{1}{p} \frac{\partial}{\partial x_i} \left( x_{i1} |\nabla u^*|^p \right) dx - \int_{\Omega_k} \frac{n}{p} |\nabla u^*|^p dx
\end{equation}
\( \nabla u^* \) do not vanish in \( \Omega_k \) and therefore \( u^* \) is of class \( C^2 \) in \( \Omega_k \), so we can use the Gauss's formula to obtain

\[
(1.5) \quad \sum_{j=1}^{n} \frac{\partial P}{\partial x_j} dx = \int_{\partial \Omega_k} \sum_{i=1}^{n} P_i \nu_i d\sigma = \int_{\partial \Omega_k} |\nabla u^*|^p (x \cdot \nabla u^*) (v \cdot \nabla u^*) d\sigma
\]

and

\[
(1.6) \quad \left[ \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left( x_j \frac{1}{p} |\nabla u^*| \right) \right] = \int_{\partial \Omega_k} \frac{1}{p} |\nabla u^*|^p (x \cdot v) d\sigma
\]

Whence, by (1.4), (1.5) and (1.6)

\[
\int_{\partial \Omega_k} |\nabla u^*|^p (x \cdot \nabla u^*) (v \cdot \nabla u^*) d\sigma = \int_{\partial \Omega_k} |\nabla u^*|^p (x \cdot v) d\sigma
\]

In the same way, an analogous relation is also obtained relatively to \( v^* \). Summing up these relations, we have

\[
\int_{\partial \Omega_k} |\nabla u^*|^p (x \cdot \nabla u^*) (v \cdot \nabla u^*) d\sigma + \int_{\partial \Omega_k} |\nabla v^*|^p (x \cdot \nabla v^*) (v \cdot \nabla v^*) d\sigma
\]

\[
- \frac{1}{p} \int_{\partial \Omega_k} |\nabla u^*|^p (x \cdot v) d\sigma - \frac{1}{q} \int_{\partial \Omega_k} |\nabla v^*|^q (x \cdot v) d\sigma
\]
(1.8)

\[
\frac{p-n}{p} \int_{\Omega} u^* \frac{\partial H}{\partial u}(x;u^*,v^*)dx + \frac{q-n}{q} \int_{\Omega} v^* \frac{\partial H}{\partial v}(x;u^*,v^*)dx
- \left( \sum_{i=1}^{n} x_i \frac{\partial u^*}{\partial x_i} \frac{\partial H}{\partial u}(x;u^*,v^*) \right) dx - \left( \sum_{i=1}^{n} x_i \frac{\partial v^*}{\partial x_i} \frac{\partial H}{\partial v}(x;u^*,v^*) \right) dx.
\]

Passing to the limit on \(k\) in this equality, as \(u^*\) and \(v^* \equiv 0\) on \(\partial \Omega\) and using the results of Egnell (2.1 [10, p. 64]).

\[
\frac{p-1}{p} \int_{\Omega} |\nabla u^*|^p(x,v) d\sigma + \frac{q-1}{q} |\nabla v^*|^q(x,v) d\sigma
= - \frac{n-p}{p} \int_{\Omega} u^* \frac{\partial H}{\partial u}(x;u^*,v^*)dx - \frac{n-q}{q} \int_{\Omega} v^* \frac{\partial H}{\partial v}(x;u^*,v^*)dx
- \left( \sum_{i=1}^{n} x_i \frac{\partial u^*}{\partial x_i} \frac{\partial H}{\partial u}(x;u^*,v^*) + \frac{\partial v^*}{\partial x_i} \frac{\partial H}{\partial v}(x;u^*,v^*) \right) dx.
\]

(1.9)

We have the following relation

\[
\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( x_i H(x,s,t) \right) = 2H(x,s,t) + x_i \nabla_i H(x,s,t)
\]
Moreover, since the application \( x \to H(x,u^*(x),v^*(x)) \) is of class \( C^1(\Omega) \), using again the Gauss’s formula then we have from hypothesis \( i) \)
\[
\int_{\partial \Omega} H(x,u^*(x),v^*(x)) (x \cdot \nu) d\sigma = 0.
\]
Hence, we obtain
\[
(1.11) \quad \int_{\partial \Omega} \nabla u^* \cdot \nu d\sigma (x \cdot \nu) d\sigma
\]
by integration by parts. According to the hypothesis \( iii) \) the integral on \( \Omega \) is nonnegative, whence a contradiction.

2. EXISTENCE RESULTS VIA COMPARISON ARGUMENTS

\( \Omega \) denotes a bounded regular open set in \( \mathbb{R}^n \) and \( X = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega) \).

Throughout this second section, we shall prove some existence results for the following problem.

\[
(P) \quad \begin{cases}
\text{Find } (u,v) \in X \text{ such that } \\
-\Delta u = f(x;u,v) & \text{on } \Omega \\
-\Delta v = g(x;u,v) & \text{on } \Omega.
\end{cases}
\]
We make the following assumptions

\((H1)\) 
\[ f \text{ and } g \text{ belong to } C(\Omega \times \mathbb{R} \times \mathbb{R}) \]
morover, for any \(s \geq 0, t \geq 0; f(x,s,t) \geq 0 \text{ and } g(x,s,t) \geq 0 \)

\((H2)\) 
There are nonnegative constants:

\[ \alpha > 0, \beta > 0, p_i, q_i \text{ (i = 1,2)} \]
\[ a_j, b_j \text{ (j = 1,...,6)} \text{ where } a_i > 0, a_j > 0, b_i > 0, b_j > 0 \text{ satisfying} \]
\((H2)_a\) and \((H2)_b:\)

\[
\frac{\alpha+1}{p} + \frac{\beta+1}{p} < 1
\]
\[(H2)_a:\]
\[
1 < p_1 < p_2; \quad 0 < q_1 - 1 < \frac{q}{p^*} \\
0 < p_2 - 1 < \frac{p}{q^*}; \quad 1 < q_2 < q
\]

We have the following existence theorem:

**Theorem 2.1.** Under hypotheses \((H1)\) and \((H2)\), \((P)\) has a nontrivial solution \((u^*, v^*)\) in \(X \cap [L^\infty(\Omega)]^2\).

**Example:** existence result for \(f(x,s,t) = a(x)s^{\alpha-1}t^{\beta+1} - a_2 s^{\alpha-1}t^{\beta+1} + a_3 s^{\alpha-1}t^{\beta+1} + a_4 \) and \(g(x,s,t) = b(x)s^{\alpha-1}t^{\beta+1} + b_3 s^{\alpha-1}t^{\beta+1} + b_4 \).

**Corollary 2.1.** Let \(f\) and \(g\) be as above where \(a\) and \(b\) are
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nonnegative continuous functions and assume that \( \alpha > 0 \) and \( \beta > 0 \) are such that

\[
\frac{\alpha+1}{np} + \frac{\beta+1}{nq} < 1; \quad \frac{\alpha+1}{p} + \frac{\beta+1}{q} < 1.
\]

Then, the corresponding problem \((P)\) has a nontrivial solution in \( X \cap [L^\infty(\Omega)]^2 \).

The proof of Theorem 2.1 is in three steps.

1st step: Construction of sub-supersolutions of \((P)\).

Definition 2.1. A pair \([(u^0,v^0),(u^0,v^0)]\) is said a weak sub-super solution for the Dirichlet problem \((P)\) if the following conditions are satisfied:

\[
\begin{align*}
(1): & \quad (u^0,v^0) \in (W^{1,p}(\Omega) \times W^{1,q}(\Omega)) \cap [L^\infty(\Omega)]^2 \\
& \quad (u^0,v^0) \in (W^{1,p}(\Omega) \times W^{1,q}(\Omega)) \cap [L^\infty(\Omega)]^2
\end{align*}
\]

\[
\begin{align*}
(2.1): & \quad -\Delta_p u^0 f(x;u^0,v^0) \leq 0 \leq -\Delta_q v^0 g(x;u^0,v^0) \quad \text{in} \ \Omega \\
& \quad \forall v \in [v^0,v^0]
\end{align*}
\]

\[
\begin{align*}
(2): & \quad u^0 \leq u^0 \quad \text{in} \ \Omega \\
& \quad v^0 \leq v^0 \quad \text{in} \ \Omega \\
& \quad u^0 \leq 0 \leq u^0 \quad \text{on} \ \partial \Omega \\
& \quad v^0 \leq 0 \leq v^0 \quad \text{on} \ \partial \Omega
\end{align*}
\]

Similar definitions can be found in Díaz-Hernández [8], Díaz-Herrero [9], Hernández [16].

Proposition 2.1. Assume \((H2)\) and

\[
\frac{\alpha+1}{p} + \frac{\beta+1}{q} < 1;
\]
then, for any $M > 0$, the problem $(P)$ admits a pair $[(u_0,v_0),(u^0,v^0)]$ of sub-super solution satisfying $u_0(x) \leq M \leq u^0(x)$, $v_0(x) \leq M \leq v^0(x)$ in $\Omega$.

**Proof.** a) Construction of $(u^0,v^0)$

Consider $R > 0$ such that $\Omega \subset B(0; R)$. We seek for $u^0, v^0$ in the following forms:

$$
\begin{align*}
u^0(x) &= \varphi^0(r) = ar^p + b \\
v^0(x) &= \psi^0(r) = cr^q + d
\end{align*}
$$

with: $b > 0$ and $d > 0$

$$
|x| = r.
$$

(2.2)

We fix a real $M > 0$ and choose

$$
(2.3) \quad a = \frac{b-M}{R^p}, \quad c = \frac{d-M}{R^q},
$$

we have, for $b$ and $d$ greater than $M$

$$
(2.4) \quad M \leq u_0(x); \quad M \leq v_0(x) \quad \forall x \in \Omega.
$$

and for each point $x$ in $\Omega$, we have:

$$
(2.5) \quad \Delta_p u_0(x) = (p-1)\|\varphi'(r)\|^{p-2} \varphi''(r) + \frac{n-1}{r} \|\varphi'(r)\|^{p-2} \varphi'(r) = -npa|a|^{p-1} = np\left(\frac{b-M}{R^p}\right)^{p-1} .
$$

For $u \leq u^0$, $v \leq v^0$ and $a < 0; e < 0$ we have
Let \( k > 0, \ b = k^{1/\rho} \) and \( d = k^{1/\sigma} \). Comparing, the growth of the different terms in (2.6) for large \( k \), we obtain

\[
\begin{align*}
\Delta_\nu \lambda^0 + f(x; \lambda^0, \nu) &\leq -n\eta + \frac{b-M}{R^{\rho^*}} + a_3 b^{g^1} d^{\beta-1} + a_6, \quad \forall \nu \leq \nu^0 \\
\Delta_\lambda \nu^0 + g(x; \nu^0) &\leq -n \eta + \frac{d-M}{R^{\sigma^*}} + b_3 b^{\sigma^1} d^{\beta} + a_6, \quad \forall \lambda \leq \lambda^0.
\end{align*}
\]

(2.7)

b) Construction of \((\lambda_0, \nu_0)\). Consider \( x_0 \in \Omega \), and \( R > 0 \) such that \( B(x_0; R) \subset \Omega \); we can assume \( 0 \in \Omega \).

As in [11], [26], we seek \((\lambda_0, \nu_0)\) in the following form

\[
\begin{align*}
\lambda_0(x) &= \psi_0(r) = \begin{cases} 
Ar^{\rho^*} + B & \text{for } 0 \leq r \leq \frac{nR}{n+1}, \\
C(R-r)^{\rho^*} & \text{for } \frac{nR}{n+1} \leq r \leq R, \\
0 & \text{for } R < r,
\end{cases}
\end{align*}
\]

(2.8)

\[
\begin{align*}
\nu_0(x) &= \psi_0(r) = \begin{cases} 
\tilde{A}r^{\sigma^*} + \tilde{B} & \text{for } 0 \leq r \leq \frac{nR}{n+1}, \\
\tilde{C}(R-r)^{\sigma^*} & \text{for } \frac{nR}{n+1} \leq r \leq R, \\
0 & \text{for } R < r
\end{cases}
\end{align*}
\]

(2.9)
Take
\[ A = -B \left( \frac{n+1}{n} \right)^{\nu - 1} \frac{1}{R^{\nu}}, \quad \bar{A} = -\bar{B} \left( \frac{n+1}{n} \right)^{\nu - 1} \frac{1}{\bar{R}^{\nu}} \]

(2.10)

By (2.10) \( u_0 \) and \( v_0 \) are in \( C^1(\Omega) \) and moreover they vanish on \( \partial \Omega \).

First consider \( x \) such that
\[ \frac{nR}{n+1} \leq r = ||x|| \leq R; \]

we have
\[
\begin{align*}
0 & \leq u_0(x) \leq C \left( R - \frac{nR}{n+1} \right)^{\nu} \\
0 & \leq v_0(x) \leq C \left( R - \frac{nR}{n+1} \right)^{\nu}
\end{align*}
\]

(2.11)

Consequently
\[
\Delta_p u_0(x) \geq \frac{p \cdot p^\nu \cdot C \cdot (n-1) \frac{R-r}{r}}{n}
\]

(2.12)

Whence for any \((u,v) \in [u_0, u_0'] \times [v_0, v_0']\) and for sufficiently small \( R \):
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\[
\begin{cases}
\Delta_p u_0 + f(x; u_0, v) \geq C_{p-1} \left\{ \frac{p}{n}^{p-1} - a_2 \left( \frac{R}{n+1} \right) \right\} \\
\Delta_q v_0 + g(x; u_0, v) \geq C_{q-1} \left\{ \frac{q}{n}^{q-1} - b_2 \left( \frac{R}{n+1} \right) \right\}
\end{cases}
\geq 0
\]

(2.13)

Now consider \( x \in \Omega \) such that:

\[ 0 \leq |x| \leq \frac{nR}{n+1} \]

We have in this case

\[ 0 \leq u_0(x) \leq B \text{ and } 0 \leq v_0(x) \leq \bar{B} \]

Moreover

\[ \Delta_p u_0(x) = -B^{(p-1)} \frac{n+1}{R^p P^{p-1}} \]

(2.14)

Using the hypothesis (H2), for any \((u, v) \in [u_0, v_0] \times [v_0, \bar{v}]\), we obtain

\[ \begin{cases}
-B^{p-1} \frac{n+1}{R^p (p+1)^{p-1}} + a_1 B^a \bar{B}^{p-1} \frac{1}{(n+1)^{a-1}} - a_2 B^{p-1} \leq \Delta_p u_0 + f(x; u_0, v) \\
-\bar{B}^{q-1} \frac{n+1}{R^q (q+1)^{q-1}} + b_1 B^a \bar{B}^{q-1} \frac{1}{(n+1)^{b-1}} - b_2 \bar{B}^{q-1} \leq \Delta_q v_0 + g(x; u_0, v)
\end{cases} \]

(2.16)

Hence the conclusion follows for \( B = D^{1/p}, \bar{B} = D^{1/q}, D > 0 \) sufficiently small.

2nd Step: The truncated problem \((\tilde{P})\) associated to \((P)\).

Following [7], we define a truncated problem \((\tilde{P})\), associated to \((P)\).
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\[ \begin{aligned}
\tag{\bar{P}}
\begin{cases}
\text{(1)} & -\Delta_p u = \tilde{f}(x;u,v) - \gamma_1(x,u) \quad \text{in } \Omega \\
\text{(2)} & -\Delta_q v = \tilde{g}(x;u,v) - \gamma_2(x,v) \quad \text{in } \Omega
\end{cases}
\end{aligned} \]

Where

\[
\gamma_1(x,u(x)) = -(u_0(x) - u(x))_{p+1} + (u(x) - u^0(x))_{p+1}
\]

\[
\gamma_2(x,v(x)) = -(v_0(x) - v(x))_{q+1} + (v(x) - v^0(x))_{q+1}
\]

(2.17)

\[
\tilde{f}(x;u(x),v(x)) = f(x,U(x),V(x))
\]

\[
\tilde{g}(x;u(x),v(x)) = g(x,U(x),V(x))
\]

With

\[
U(x) = u(x) + (u_0(x) - u(x))_+ - (u(x) - u^0(x))_+
\]

(2.18)

\[
V(x) = v(x) + (v_0(x) - v(x))_+ - (v(x) - v^0(x))_+
\]

For any \((u,v) \in X, (\bar{u},\bar{v}) \in X,\) we define:

\[
A(u,v) = \begin{pmatrix}
\Delta_p & 0 \\
0 & \Delta_q
\end{pmatrix}
\begin{pmatrix}
u \\
v
\end{pmatrix}
+ \begin{pmatrix}
\gamma_1(x,u) - \tilde{f}(x;u,v) \\
\gamma_2(x,v) - \tilde{g}(x;u,v)
\end{pmatrix}
\]

(2.19)

\[
\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left[ |\nabla u|^{\frac{p-2}{2}} \frac{\partial u}{\partial x_i} \right] + \begin{pmatrix}
\gamma_1(:,u) - \tilde{f}(x;u,v) \\
\gamma_2(:,v) - \tilde{g}(x;u,v)
\end{pmatrix}
\]

\[
\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left[ |\nabla v|^{\frac{q-2}{2}} \frac{\partial v}{\partial x_i} \right]
\]
\[ a[(u,v);(\tilde{u},\tilde{v})] = \int_{\Omega} A(u,v) W dx \]

with \( W = \begin{pmatrix} \tilde{w} \\ \tilde{v} \end{pmatrix} \)

We have

\[
a[(u,v);(\tilde{u},\tilde{v})] = \int_{\Omega} |\nabla u|^{r-2} \nabla u \nabla \tilde{u} dx + \int_{\Omega} |\nabla v|^{r-2} \nabla v \nabla \tilde{v} dx
\]

(2.20)

\[ -\int_{\Omega} f(x;u,v) \tilde{u} dx - \int_{\Omega} g(x;u,v) \tilde{v} dx + \int_{\Omega} \gamma_1(x,u,v) \tilde{u} dx + \int_{\Omega} \gamma_2(x,v) \tilde{v} dx. \]

**Lemma 2.1.** A is a bounded operator from \( X \) to \( X^* \).

**Proof [31].**

**Definition 2.2** (C.f [17]). An operator \( A : X \rightarrow X^* \) is called a calculus of variations operator, if it is bounded and if it can be represented in the form

(1) \[ A(u,v) = \mathcal{A}[(u,v);(\tilde{u},\tilde{v})] \]

where \( ((u,v),(\tilde{u},\tilde{v})) \rightarrow \mathcal{A}[(u,v);(\tilde{u},\tilde{v})] \) is an operator \( X \times X \rightarrow X^* \) which satisfies
Existence and Nonexistence of Nontrivial...

\[
\begin{align*}
\forall (u,v) \in X; (\hat{u},\hat{v}) &\rightarrow A[(u,v);(\hat{u},\hat{v})] \text{ is a hemicontinuous bounded operator } X \rightarrow X^* \text{ and} \\
\langle A[(u,v);(u,v)] - A[(\hat{u},\hat{v})],(u,v)-(\hat{u},\hat{v}) \rangle &\geq 0; \forall (u,v),(\hat{u},\hat{v}) \in X
\end{align*}
\]

For any \((\hat{u},\hat{v}) \in X, (u,v) \rightarrow A[(u,v);(\hat{u},\hat{v})] \text{ is a bounded hemicontinuous operator } X \rightarrow X^*.

\[\text{(3)}\]

If \((u_\mu,v_\mu) \rightharpoonup (u,v) \text{ weakly in } X \text{ and} \]
if \(\langle A[(u_\mu,v_\mu),(u_\mu,v_\mu)] - A[(u_\mu,v_\mu),(u_\mu,v_\mu)],(u_\mu-u,v_\mu-v) \rangle \rightarrow 0 \text{ (4)} \]
then, for any \((\hat{u},\hat{v}) \in X \text{ the sequence } A[(u_\mu,v_\mu),(\hat{u},\hat{v})] \text{ converges weakly to } A[(u,v),(\hat{u},\hat{v})] \text{ in } X^*.

\[\text{(5)}\]

In our problem, we define \(A\) by the following relation; for any \((u_1,v_1), (u_2,v_2),(\hat{u},\hat{v})\):

\[
\begin{align*}
\langle A [(u_1,v_1),(u_2,v_2);(\hat{u},\hat{v})],(u,v) \rangle &\rightarrow \int_{\Omega} |\nabla u_2|^{p-2} \nabla u_2 \nabla \hat{u} dx + \int_{\Omega} |\nabla v_2|^{q-2} \nabla v_2 \nabla \hat{v} dx \\
&- \int_{\Omega} f(x;u_1,v_1) \hat{u} dx - \int_{\Omega} \hat{g}(x;u_1,v_1) \hat{v} dx \\
&+ \int_{\Omega} \gamma_1(x,u_1) \hat{u} dx + \int_{\Omega} \gamma_2(x,v_1) \hat{v} dx
\end{align*}
\]

(2.21)

**Lemma 2.2.** \(A\) is a calculus of variations operator.

**Proof.** (c.f [31])

**Lemma 2.3.** Let \(V\) be a Banach space and let \(A\) be a coercive calculus of variations operator.
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Then, for any $f$ in $V^*$, the equation $A(u) = f$ has a solution $u$ in $V$.

**Proof** (c.f [17], proposition 2.6, theorem 2.7, p. 180-181).

**Lemma 2.4.** If the application $\tilde{f}$, $g$, $g_1$, and $g_2$ are defined as above, then the problem (P) has a solution $(\tilde{u}, \tilde{v})$ in $X$.

**3° Step:** Existence of a non-trivial solution for (P).

Now, we prove that $u_0 \leq \tilde{u} \leq u^0$, $v_0 \leq \tilde{v} \leq v^0$, in $\Omega$.

We show for example $\tilde{u} \leq u^0$.

Consider $\tilde{u} = (\tilde{u} - u^0)$, and $\tilde{v} = (\tilde{v} - v^0)$.

Multiplying (1) by $\tilde{u}$ and (2) by $\tilde{v}$, we have

\[
\int_\Omega |\nabla \tilde{u}|^{p-2} \nabla \tilde{u} \nabla \tilde{u} dx = \int_\Omega \tilde{f}(x;\tilde{u},\tilde{v}) \tilde{u} dx + \| (\tilde{u} - u^0) \|_{L^p(\Omega)} = 0
\]

but, according to the definition of $u^0$, $\forall v \in [v_0, v^0]$, we have

\[
\int_\Omega |\nabla \tilde{u}|^{p-2} \nabla \tilde{u} \nabla \tilde{u} dx = \int_\Omega \tilde{f}(x;u^0,\tilde{v}) \tilde{u} dx \geq 0
\]

Thus, combining (2.22) and (2.23), we obtain

\[
0 \geq \int_\Omega |\nabla \tilde{u}|^{p-2} \nabla \tilde{u} - |\nabla \tilde{u}|^{p-2} \nabla \tilde{u} \tilde{u}(\tilde{u} - u^0) \mathcal{V}(\tilde{u} - u^0) dx
\]

\[
+ \int_\Omega \left[ \tilde{f}(x;u^0,\tilde{v}) - \tilde{f}(x;\tilde{u},\tilde{v}) \right] \tilde{u} dx + \| (\tilde{u} - u^0) \|_{L^p(\Omega)}^p
\]

Take $v = \tilde{V}$ where $\tilde{V}$ is associated to $\tilde{v}$ as in (2.18). On the set $\{ x \in \Omega; \tilde{u}(x) - u^0(x) > 0 \}$, we have $\tilde{U}(x) = u^0(x)$. 

(2.27) \[
\int_\Omega (f(x;u^0,\bar{v}) - f(x;\bar{u},\bar{v}))(\bar{u} - u^0)(x)dx = \int_\Omega (f(x;\bar{u},\bar{v}) - f(x;\bar{u},\bar{v}))(\bar{u}^0 - u^0)(x)dx = 0
\]

By monotonicity of $-\Delta_p$, we get that $0 \geq \|(\bar{u} - u^0)_+\|_{p} \geq 0$.

Thus $\bar{u} \leq u^0$ on $\Omega$ and similarly $\bar{v} \leq v^0$ on $\Omega$.

3. EXISTENCE RESULTS VIA VARIATIONAL METHODS

3.0. Introduction. We present in this final section an existence result for the following problem $(P)$

\[
\begin{cases}
\text{Find } (u,v) \in X \text{ such that} \\
(1^*) \quad -\Delta_p u = \frac{\partial H}{\partial u}(x;u,v) \quad \text{in } \Omega \\
(2^*) \quad -\Delta_q v = \frac{\partial H}{\partial v}(x;u,v) \quad \text{in } \Omega
\end{cases}
\]

This result extends to a potential system those obtained by L. Nirenberg [18] and F. de Thélin [26], in the scalar case. Our existence result follows from an appropriate adaptation of the variational method given by Ambrosetti-Rabinowitz [2].

Recall that $X = W_{0}^{1,p}(\Omega) \times W_{0}^{1,q}(\Omega)$.

In the next section, we shall prove that in fact $(u,v) \in X \cap [L^{\infty}(\Omega)]^2$.

We make the following assumptions

(H1) $H \in C^1(\Omega \times \mathbb{R} \times \mathbb{R})$

(H2) There exist two positive real numbers $\delta, A$, with $\delta < A$ such that, for a partition of $\mathbb{R}^2$ in $D_1, D_2, D_3$, respectively defined by
We have:

\((H2)_a\) there exists a nonnegative constant \(C\) and

\[
\begin{align*}
p' &\in \left[\frac{np}{n-p}, \frac{nq}{n-q}\right], \\
qu &\in \left[\frac{np}{n-q}, \frac{nq}{n-p}\right],
\end{align*}
\]

such that \(0 \leq H(x; s, t) \leq C(\|s\|^p + \|t\|^q)\), for any \(x \in \Omega\) and for any pair \((s, t) \in D_j\).

\((H2)_b\) There exists a positive function \(a \in L^\infty(\Omega)\) such that \(H(x; s, t) = a(x) \|s\|^{p+} \|t\|^{q+}\) for any \(x \in \Omega\) and \((s, t) \in D_j\).

Remark. We are interested by the nonnegative solutions for the problem \((P)\), so we can add the following hypothesis

\[(H3)\) For any \(x \in \Omega, s \leq 0\) or \(t \leq 0;\)

\[
\frac{\partial H}{\partial s}(x; s, t) = 0 \quad \text{and} \quad \frac{\partial H}{\partial t}(x; s, t) = 0.
\]

For any \((u, v)\) in \(X\), we define:

\[
J(u, v) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx + \frac{1}{q} \int_{\Omega} |\nabla v|^q dx - \int_{\Omega} H(x; u, v) dx
\]

We shall use the Mountain-Pass Lemma to obtain an existence theorem for \((P)\). The nontrivial solution is obtained as a critical point of \(J\).
Theorem 3.1. We suppose that the hypotheses (H1) and (H2) are satisfied and that the real numbers $\alpha$ and $\beta$ in (H2), are such that

\[
\begin{align*}
1) & \quad \frac{(\alpha+1)\frac{n-p}{np}+(\beta+1)\frac{n-q}{nq}}{p} > 1 \\
2) & \quad \frac{\alpha+1}{p} + \frac{\beta+1}{q} > 1,
\end{align*}
\]

then, the problem (P) possesses a nontrivial solution $(u^*, v^*)$ in $X \cap [L^\infty(\Omega)]^2$.

Corollary 3.1. All the hypotheses of Theorem 3.1. are satisfied for $H(x; s, t) = a(x) |s|^{\alpha+1} |t|^{\beta+1}$.

If

\[
(\alpha+1)\frac{n-p}{np}+(\beta+1)\frac{n-q}{nq} < 1, \quad \frac{\alpha+1}{p} + \frac{\beta+1}{q} > 1,
\]

then, the corresponding problem possesses a nontrivial solution $(u^*, v^*)$ in $X \cap [L^\infty(\Omega)]^2$.

Proof of Corollary 3.1. Consider a truncature $\tilde{H}$ of the application $H$

\[
\tilde{H}(x; s, t) = \begin{cases} 
0 & \text{if } s \leq 0 \text{ or } t \leq 0 \\
H(x; s, t) & \text{otherwise}
\end{cases}
\]

$\tilde{H}$ satisfies the hypotheses (H1), (H2). For proving (H2), we write for any real $s$ and $t$

\[
(*) \quad |s|^{\lambda+1} |t|^{\mu+1} \leq C(|s|^{\lambda_0} + |t|^{\mu_0})
\]

Where $\lambda$ and $\mu$ are such that
Existence of a solution in $X$.

Lemma 3.1.1. If

$$\left(\frac{\alpha+1}{p^n} + \frac{\beta+1}{q^n}\right) < 1,$$

there exist $\gamma_1$ and $\gamma_2$ such that

$$\begin{cases}
\gamma_1 \in \left[1, \frac{n_p}{n-p}\right] \\
\gamma_2 \in \left[1, \frac{n_q}{n-q}\right]
\end{cases}$$

Moreover, if $(u_k,v_k)$ is bounded in $X$, the applications

$$x \mapsto u_k(x) \left| u_k(x) \right|^{\alpha-1} \left| v_k(x) \right|^{\beta-1} \text{ and } x \mapsto v_k(x) \left| v_k(x) \right|^{\beta-1} \left| u_k(x) \right|^{\alpha-1}$$

are bounded in $L^\infty(\Omega)$ and $L^\infty$ respectively.

Lemma 3.1.2. If

$$\frac{\alpha+1}{p^n} + \frac{\beta+1}{q^n} > 1,$$

$J$ satisfies the Palais-Smale (P.S) condition.

Proof. Let $\{u_k,v_k\}; k \in \mathbb{N}$ be a sequence in $X$ such that

there exist $M > 0, \ |J(u_k,v_k)| \leq M \ (P.S)$. 

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\[ J'(u_k, v_k) \to 0 \text{ strongly in } X^* \text{ as } k \text{ goes to } +\infty \ (P.S)_2. \]

We claim that this sequence is bounded in \( X \).

By contradiction, suppose that we can extract from \((u_k, v_k)\) a subsequence denoted again by \((u_k, v_k)\) such that \(\|(u_k, v_k)\|_X \to +\infty\).

Hereafter, we set
\[
e_k = \frac{1}{p} \int_{\Omega} |\nabla u_k|^p \, dx + \frac{1}{q} \int_{\Omega} |\nabla v_k|^q \, dx.
\]

The \((P.S)_1\) condition implies that
\[
(3.1.1) \quad \frac{M}{e_k} \leq 1 - \frac{1}{e_k} \int_{\Omega} |H(x; u_k, v_k)| \, dx \leq \frac{M}{e_k}.
\]

Let \( \Omega_{i,k} = \{ x \in \Omega : (u_i(x), v_i(x)) \in D_i \} \), for \( i = 1, 2, 3 \); we obtain
\[
(3.1.2) \quad -\frac{M}{e_k} \leq 1 - \frac{1}{e_k} \left( \int_{\Omega_{i,k}} a(x) u_k^{\alpha-1} v_k^{(\beta-1)/2} \, dx + \int_{\partial \Omega_{i,k}} H(x; u_k, v_k) \, dx \right) \leq \frac{M}{e_k}.
\]

On the other hand, by \((P.S)_2\) we have:
\[
-\varepsilon \|(u_k, v_k)\|_X \leq J'(u_k, v_k) \left( \frac{u_k}{p}, \frac{v_k}{q} \right) \leq \varepsilon \|(u_k, v_k)\|_X.
\]

That means
\[
-\varepsilon \|(u_k, v_k)\|_X \leq \frac{1}{p} \int_{\Omega} u_k \frac{\partial H}{\partial u}(x; u_k, v_k) \, dx - \frac{1}{q} \int_{\Omega} v_k \frac{\partial H}{\partial v}(x; u_k, v_k) \, dx.
\]
Then, taking the limit with respect to \( k \) in the inequalities (3.1.2) and (3.1.3), we obtain respectively

\[
\lim_{k \to +\infty} \frac{1}{\epsilon_k} \int_{\Omega} a(x) u_k^{\alpha+1} v_k^{\beta+1} \, dx = 1
\]

But, this contradicts the hypothesis

\[
\frac{\alpha+1}{p} + \frac{\beta+1}{q} > 1.
\]

Thus, there exist positive constants \( C_1 \) et \( C_2 \) such that: \( \|u_k\|_{L^p} \leq C_1 \) and \( \|v_k\|_{L^q} \leq C_2 \).

Denoting again by \( \{u_k; k \in \mathcal{N}\} \) and \( \{v_k; k \in \mathcal{N}\} \) the extracted subsequences, they converge strongly in the spaces \( L^p(\Omega) \) and \( L^q(\Omega) \) respectively; we claim that the subsequence \( \{(u_k, v_k); k \geq 0\} \) converges strongly in \( X \).

In fact, for any integer pair \( (m, l) \)

\[
\int_\Omega (F_p(\nabla u_m) - F_p(\nabla u_l)) \nabla (u_m - u_l) \, dx = A_{m,l}
\]

where
Existence and Nonexistence of Nontrivial...

\[ A_{m,j} = J'_{\rho,q}(u_m,v_m) - J'_{\rho,q}(u,v);(u_m-u_j,0) \bigg|_{X,X} \]
\[ + \int_{\Omega} \left( \frac{\partial H}{\partial u}(x;u_m,v_m) - \frac{\partial H}{\partial u}(x;u_j,v_j) \right) (u_m-u_j) \, dx \]

and

\[ B_{m,j} = \left< J'_{\rho,q}(u_m,v_m) - J'_{\rho,q}(u,v);(0,v_m-v_j) \right>_{X,X'} \]
\[ + \int_{\Omega} \left( \frac{\partial H}{\partial v}(x;u_m,v_m) - \frac{\partial H}{\partial v}(x;u_j,v_j) \right) (v_m-v_j) \, dx \]

By \((P,S)\), it is easy to remark that \( < J'_{\rho,q}(u_m,v_m) - J'_{\rho,q}(u,v);(u_m-u_j,0) >_{X,X} \) converges to 0 as \( m \) and \( l \) tend to \(+\infty\).

From the hypotheses \((H1)\) and \((H2)\), there exist two constants \( A_1 \) and \( A_2 \) such that for any \((s,t)\) in \( \mathbb{R}^2 \) and \( x \) in \( \Omega \)

\[ |v_m - v_j|^p \leq C \left[ (\nabla u_m - \nabla v_m)^p \right] \left( |\nabla u_m|^p + |\nabla u_j|^p \right)^{(1-\alpha)/2} \]

By use of Lemma 3.1.,

\[ \left| \int_{\Omega} \left( \frac{\partial H}{\partial u}(x;u_m,v_m) - \frac{\partial H}{\partial u}(x;u_j,v_j) \right) (v_m-v_j) \, dx \right| \leq A_1 \alpha |s|^\alpha |t|^{p-1} \]
Integrating (3.1.8) on $\Omega$ and using Hölder's inequality in the right hand side, we obtain

$$
(3.1.9) \quad \|u_m - u\|_{L^p} \leq C |A_m|^{1/2} \|u_m\|_{L^p}^{1/2} + \|u\|_{L^p}^{1/2} \|u_m\|_{L^p}^{1/2}
$$

and

$$
(3.1.10) \quad \|v_m - v\|_{L^q} \leq C |B_m|^{1/2} \|v_m\|_{L^q}^{1/2} + \|v\|_{L^q}^{1/2} \|B_m\|_{L^q}^{1/2}
$$

From the convergence results related above, these inequalities give strong convergence of \{(u_k,v_k); \ k \in \mathbb{N}\}.

**Lemma 3.1.3.** Under the hypotheses of Theorem 3.1.

1) There exist two positive real numbers $\rho$, $\nu_1$ and a neighborhood $V_0$ of the origin of $X$ such that for any element $(u,v)$ on the boundary of $V_0$, $J(u,v) \geq \nu_1 > 0$.

2) There exist $(\phi, \psi)$ in $X$ such that $J(\phi, \psi) < 0$.

**Proof.** 1) By $(H1)$ and $(H2)$

$$
\int_{\Omega} H(x;u,v)dx \leq C \int_{\Omega} |u|^\rho + |v|^\rho dx + \int_{\Omega} Bdx + \int_{\Omega} a(x)|u|^\nu_1 |v|^\nu_1 dx
$$

$$
(3.1.11) \quad \leq C \|u\|_{L^\rho}^{\nu_1} + \|v\|_{L^\rho}^{\nu_1} + b_\delta \int_{\Omega} |u|^\nu_1 |v|^\nu_1 dx + \int_{\Omega} a(x)|u|^\nu_1 |v|^\nu_1 dx
$$

By lemma 3.1.1., we obtain
Therefore, we get

\begin{align}
(3.1.13) \quad & \int_{\Omega} |u|^\alpha|v|^\beta \, dx \leq C(\|u\|_{1,p}^{\alpha+1} + \|v\|_{1,q}^{\beta+1} + (b_\delta + \|a\|_\infty)\{\|u\|_{1,p}^{\alpha+1} + \|v\|_{1,q}^{\beta+1}\}) \\
\end{align}

where $b_\delta$ is a positive constant $B = b_\delta \delta^{\alpha+\beta+1}$, $\delta$ fixed,

\[ r = 1 + \frac{p}{q} \frac{\beta+1}{\alpha+1} \quad \text{and} \quad r^* = 1 + \frac{q}{p} \frac{\alpha+1}{\beta+1}. \]

Denoting by $\theta$ and $\eta$ respectively $\|u\|_{1,p}$ and $\|v\|_{1,q}$, we therefore obtain the following minoration of $J$ for any $(u,v) \in X$,

\begin{align}
(3.1.14) \quad & J(u,v) \geq \theta \left[ 1 - C \delta^{\alpha+\beta+1} - (b_\delta + \|a\|_\infty)\delta^{\alpha+\beta+1} \right] + \eta \left[ 1 - C \delta^{\alpha+\beta+1} - (b_\delta + \|a\|_\infty)\delta^{\alpha+\beta+1} \right] \\
\end{align}

Whence,

\begin{align}
(3.1.15) \quad & J(u,v) \geq 0
\end{align}

2) Let $\phi \in W_0^{1,p}(\Omega)$ and $\psi \in W_0^{1,q}(\Omega)$ be positive in $\Omega$, for any $\sigma > 0$, we have

\begin{align}
J(\sigma^\phi \psi, \sigma^\phi \psi) = & \sigma \|\phi\|_{1,p}^{\alpha+1} + \sigma \|\psi\|_{1,q}^{\beta+1} - \int_{\Omega} H(x; \sigma^\phi \psi, \sigma^\phi \psi) \, dx \\
\end{align}

\begin{align}
= & \sigma \|\phi\|_{1,p}^{\alpha+1} + \sigma \|\psi\|_{1,q}^{\beta+1} - \int_{\Omega} H(x; \sigma^\phi \psi, \sigma^\phi \psi) \, dx - \sigma^\phi \sigma^\psi \sigma^{\alpha+\beta+1} \int_{\Omega} |\phi|^{\alpha+1} |\psi|^{\beta+1} \, dx
\end{align}

Taking $\sigma$ sufficiently large to have $|\Omega_\sigma| > 0$, we obtain
By the continuity for $J(\cdot,\cdot)$ on $X$, we find a pair $(\phi,\psi)$ in $X \setminus B_\rho(0)$ such that $J(\phi,\psi) < 0$.

**Proof of the theorem** 3.1. (1st part). By Mountain-Pass Lemma [2], there exist a pair $(u^*,v^*)$ in $X$ which is a critical point of $J$. This means that for any $(w_1,w_2) \in X$, $J'(u^*,v^*) \cdot (w_1,w_2) = 0$, i.e.

\[
\begin{align*}
-\Delta_p u^* &= \frac{\partial H}{\partial u}(x;u^*,v^*) \quad \text{in } \Omega \\
-\Delta_q v^* &= \frac{\partial H}{\partial v}(x;u^*,v^*) \quad \text{in } \Omega.
\end{align*}
\]

So, we have proved that $(P)$ possesses a nontrivial solution in $X$. The second part is devoted to prove that the solutions are bounded in $\Omega$.

Moreover, [26] (c.f the definition for $H$) ensure $u^* \geq 0$ and $v^* \geq 0$ in $\Omega$.

### 3.2. $L^\infty$-Estimate of the solution

**3.2.0. Introduction.** In this part, we use an iterative method to estimate the solution $(u^*,v^*)$ obtained in section 3.1. We prove here that in fact $(u^*,v^*) \in [L^\infty(\Omega)]^2$.

In this matter, the crucial point is the construction of two strictly increasing unbounded sequences $\{\lambda_k; k \geq 0\}$ and $\{\mu_k; k \geq 0\}$ such that $u^*$ and $v^*$ verify:
We shall present some properties deriving to the fact that \( u^* \) and \( v^* \) belong to \( L^\nu(\Omega) \) and \( L^\mu(\Omega) \) respectively. In a second step, we shall proceed to the appropriate construction for these sequences.

It is very important to note that this iterative scheme uses some regularity properties of \( u^* \) and \( v^* \), for example \( (u^*, v^*) \) belong to \( [C^2(\Omega) \cap C^1(\Omega)]^2 \). The study of regularized equations (cf. [20], [26]) allows us to suppose \( u^* \) and \( v^* \) smooth throughout all this part. Though we do not make extensive development about our iterative method, more detailed proofs are given in [31].

**Proposition 3.2.** Suppose that all the hypotheses of Theorem 3.1. are satisfied. Then, there exist sequences \( \{\lambda_k; k \geq 0\} \) and \( \{\mu_k; k \geq 0\} \) such that

1) For each \( k \), \( u^* \) and \( v^* \) belong respectively to \( L^\nu(\Omega) \) and \( L^\mu(\Omega) \).
2) There exist two real constants \( A_p \) and \( A_q \) be such that

\[
\lim_{k \to \infty} \|u^*\|_{L^\nu(\Omega)} \leq A_p
\]

\[
\lim_{k \to \infty} \|v^*\|_{L^\mu(\Omega)} \leq A_q
\]

**Lemma 3.2.1.** Let \( \pi_p \) (resp. \( \pi_q \)) be such that

\[
1 < \pi_p < \frac{np}{n-p} \quad \text{(resp.} \quad 1 < \pi_q < \frac{ng}{n-q} \text{),}
\]

and for any \( k \geq 0 \)
\[ a_k = \lambda_k \left( 1 - \frac{\alpha}{\lambda_k} - \frac{\beta + 1}{\mu_k} \right)^{-1} \quad (1)_k \]

\[ b_k = \mu_k \left( 1 - \frac{\alpha + 1}{\lambda_k} - \frac{\beta}{\mu_k} \right)^{-1} \quad (2)_k \]

Then there are some constants \( c \) and \( c' \) such that for any \( u^* \in L^{q'}(\Omega) \) and \( v^* \in L^p(\Omega) \) we have

\[
\int_{\Omega} |u^*|^{1 - \frac{\alpha}{q}} dx \leq \left( 1 + \frac{a_k}{p} \right)^{\frac{1}{p}} \theta_k^{1 - \frac{\alpha}{p}}, \quad \int_{\Omega} |v^*|^{1 - \frac{\beta}{q}} dx \leq c' \left( 1 + \frac{b_k}{q} \right)^{\frac{1}{q}} \Phi_k^{1 - \frac{\beta}{q}}
\]

where \( \theta_k \) and \( \Phi_k \) are defined as

\[
\theta_k = \int_{\Omega} \frac{\partial H}{\partial u}(x; u^*, v^*) u^* |u^*|^\alpha dx, \quad \Phi_k = \int_{\Omega} \frac{\partial H}{\partial v}(x; u^*, v^*) v^* |v^*|^\beta dx.
\]

**Proof of the Lemma 3.2.1.** Multiplying (1*) by \( u^* |u^*|^\alpha \) and integrating on \( \Omega \), we obtain

\[
(3.2.1) \quad \int_{\Omega} |\nabla u^*|^p |\nabla u^*|^q |u^*|^\alpha dx = \frac{\partial H}{\partial u}(x; u^*, v^*) u^* |u^*|^\alpha dx
\]

On the other hand, we have,

\[
(3.2.2) \quad \int_{\Omega} |\nabla u^*|^{1 - \frac{\alpha}{p}} dx = \left( 1 + \frac{a_k}{p} \right)^{\frac{1}{p}} \int_{\Omega} |u^*|^\alpha |\nabla u^*|^p dx
\]

Since, \( u^* \) is in \( C^1(\overline{\Omega}) \), so is \( \{u^*\}^{1 + \alpha \beta} \) and consequently \( \{u^*\}^{1 + \alpha \beta} \) belongs to \( W_0^{1,p}(\Omega) \). The continuous imbedding \( W_0^{1,p}(\Omega) \rightarrow L^p(\Omega) \) implies the existence of a constant \( c > 0 \) such that
(3.2.3) \[
\left( \int_{\Omega} |u^*|^{1+\frac{a_k}{p}} \, dx \right)^{\frac{p}{1+\frac{a_k}{p}}} \leq C \left( \int_{\Omega} \left| \nabla u^* \right|^{\frac{p}{2}} \, dx \right)^{\frac{1}{2}}
\]

Since \( a_k \) is nonnegative, (3.2.1), (3.2.2), (3.2.3) give,

\[
\left( \int_{\Omega} |u^*|^{1+\frac{a_k}{p}} \right)^{\frac{p}{1+\frac{a_k}{p}}} \leq C \left( 1 + \frac{a_k}{p} \right)^{\frac{p}{1+\frac{a_k}{p}}} \left( \int_{\Omega} |\nabla u^*|^p |u^*|^\alpha \, dx \right)^{\frac{p}{p}}
\]

(3.2.4)

\[
\leq C \left( 1 + \frac{a_k}{p} \right)^{\frac{p}{1+\frac{a_k}{p}}} \theta_k^{\frac{p}{1+\frac{a_k}{p}}}
\]

**Lemma 3.2.2.** Assume that

\[
\lambda_{k+1} \leq \left( 1 + \frac{a_k}{p} \right) \pi_p \quad (3.3)
\]

\[
\mu_{k+1} \leq \left( 1 + \frac{b_k}{q} \right) \pi_q \quad (4.3)
\]

Then, if \( u^* \in L^r(\Omega) \) and \( v^* \in L^s(\Omega) \), we have

\[
\|u^*\|_{L^{\lambda_{k+1}}(\Omega)}^2 \leq K_p^{-1} \left( 1 + \frac{a_k}{p} \right)^{\frac{1}{2}} \left[ A_1 \|u^*\|_{L^{\lambda_k}(\Omega)}^{\alpha_{k+1}} + A_2 \left( \|u^*\|_{L^{\lambda_k}(\Omega)}^{\alpha_{k+1}} \right)^{\frac{1}{2}} \left( \|v^*\|_{L^{\lambda_k}(\Omega)}^{1} + A_3 \right) \right]^{\frac{1}{p}}
\]

(3.2.5)

where \( A_i (i=1, 2, 3) \) are positive constants.
Proof. We first call (c.f (3.1.7)) that the hypotheses on $H$ imply the existence of positive constants $A_i (i=1;2)$ such that for any real numbers $s$ and $t$,

$$\frac{\partial H}{\partial s} (x;\xi,\tau) \leq A_1 + A_2|s|^{\alpha} |\tau|^{\beta+1}$$

Thus, by Hölder’s inequality we obtain

$$\int_{\Omega} \frac{\partial H}{\partial u} (x;u_*,v_*)u_*|u_*|^{\gamma} dx \leq A_1 \int_{\Omega} |u_*|^{\alpha_1} dx + A_2 \int_{\Omega} |u_*|^{\alpha_2} |v_*|^{\beta+1} dx$$

(3.2.6) \hspace{1cm} \leq A_1 \int_{\Omega} |u_*|^{\gamma} dx + A_2 \left( \int_{\Omega} |u_*|^{\alpha_1} dx \right)^{\alpha_2/\alpha_1} \left( \int_{\Omega} |v_*|^{\beta+1} dx \right)^{\beta/\beta+1} + A_3

That implies with (3.2.4),

(3.2.7)

$$\int_{\Omega} |u_*|^{\left(1 - \frac{\gamma}{\alpha_1}\right)} \leq C \left( 1 + \frac{a_2}{p} \right) \left[ \int_{\Omega} |\nabla u_*|^{p} |u_*|^{\alpha_2} dx \right]^{\gamma/p}$$

$$\leq C \left( 1 + \frac{a_2}{p} \right) \left[ A_1 \int_{\Omega} |u_*|^{\gamma} dx + A_2 \left( \int_{\Omega} |u_*|^{\alpha_1} dx \right)^{\alpha_2/\alpha_1} \left( \int_{\Omega} |v_*|^{\beta+1} dx \right)^{\beta/\beta+1} + A_3 \right]^{\gamma/p}$$

Now, by (3.4), $L^{(1+\alpha/p)\gamma}(\Omega)$ is continuously imbedded into $L^{\gamma}(\Omega)$, so there exists a constant $K_\rho$ such that
Combined with (3.2.7), we have

\[
\left( \int_\Omega |u^*|^{\lambda \gamma} \, dx \right)^{\frac{1}{\lambda \gamma}} \leq K_p \left( \int_\Omega |u^*|^{\frac{\alpha + \beta}{p}} \, dx \right)^{\frac{1}{\frac{\alpha + \beta}{p} \lambda \gamma}}.
\]

An analogous result is obtained for \( v^* \).

3.2.1. Definition and construction of sequences \( \{\lambda_k; \, k \in \mathbb{N}\} \) and \( \{\mu_k; \, k \in \mathbb{N}\} \). Here, we construct the sequences \( \{\lambda_k; \, k \in \mathbb{N}\} \) and \( \{\mu_k; \, k \in \mathbb{N}\} \). This construction requires similar tools as in [20], [26] or [27] use for the study of first eigenvalue, but here the problem is different from [27], because \( \alpha \) and \( \beta \) do not verify

\[
\frac{\alpha + 1}{p} + \frac{\beta + 1}{q} = 1.
\]

Here, the first terms of each sequence cannot be determined directly by using the Rellich-Kondrachov's continuous embedding result. So, we first construct Lebesgue spaces of exponents \( \lambda_k \) and \( \beta_k \) containing respectively \( u^* \) and \( v^* \). By an appropriate choice for \( k_0 \in \mathbb{N} \) and \( \beta_0 \), give the respective first terms of \( \{\lambda_k; \, k \geq 0\} \) and \( \{\mu_k; \, k \geq 0\} \). After that, we shall show that \( u^* \) and \( v^* \) are estimated independently to \( k \) by a same constant in every \( L^{\lambda}(\Omega) \) and \( L^p(\Omega) \) spaces respectively. This is not always the case when we are limiting us only to \( L^{\lambda}(\Omega) \) and \( L^p(\Omega) \) spaces.
a) Construction of $[\lambda_k; k > 0]$ and $[\mu_k; k > 0]$. We consider here $\alpha$ and $\beta$ satisfying the relations

$$\frac{\alpha + 1}{p} \left(\frac{n-p}{n}\right) + \frac{\beta + 1}{q} \left(\frac{n-q}{n}\right) < 1$$

(3.2.9)

So, we can find $C > 1$ and $(\lambda, \mu)$ such that

$$\begin{align*}
1 < & \frac{\lambda}{C} < \frac{n}{(n-q)C} \\
1 < & \frac{\mu}{C} < \frac{n}{(n-q)C} \\
\frac{\alpha + 1}{\lambda p} + \frac{\beta + 1}{\mu q} = & 1
\end{align*}$$

(3.2.10)

Now, we take $\lambda_k = \lambda p^k C^l$, $\mu_k = \mu q^k C^l$.

From (1) and (2), we get

$$\begin{align*}
\lambda_k = & \lambda^k p^l C^l \\
\mu_k = & \mu^k q^l C^l \\
\lambda_k = & \lambda^k p^l C^l \\
\mu_k = & \mu^k q^l C^l \\
\lambda_k = & \lambda^k p^l C^l \\
\mu_k = & \mu^k q^l C^l
\end{align*}$$

(3.2.11)

**Lemma 3.2.3.** For each $k \in \mathcal{K}$, $u^*$ and $v^*$ belong respectively to $L^{\lambda_k}(\Omega)$ and $L^{\mu_k}(\Omega)$.

**Proof.** We give a proof by induction.

By Sobolev imbedding Theorem, we have $u^* \in L^{\lambda_k}(\Omega)$; $v^* \in L^{\mu_k}(\Omega)$. 
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Then the Lemma is proved for \( k = 0 \). Suppose that it is true for all integer \( k' \) such that \( 0 \leq k' \leq k \in \mathcal{K} \).

Take \( \pi_p = \lambda p \hat{C} \) and \( \pi_q = \mu q \hat{C} \), and \( u^\ast \in L^b(\Omega) \). The relation:

\[
\left( 1 + \frac{a_k}{p} \right) \pi_p = \lambda^2 p \hat{C}^{k+1} + \lambda p \hat{C} - \lambda^2 p \hat{C} \geq \lambda p \hat{C}^{k+1} = \hat{h}_{k+1},
\]

and Lemma 3.2.1. give \( u^\ast \in L^b(\Omega) \) and \( v^\ast \in L^b(\Omega) \).

b) Construction of sequences \( \{ \lambda_k; k \in \mathcal{N} \} \) and \( \{ \mu_k; k \in \mathcal{N} \} \). Let

\[
C = \min \left( \frac{n}{n-p}, \frac{n}{n-q} \right), \quad \gamma = \frac{\alpha+1}{\lambda p}, \quad \beta = \frac{\beta+1}{\mu q}, \quad \delta = (M - (\gamma - 1))C,
\]

with \( M > \gamma - 1 \); we define the sequences \( \{ \lambda_k; k \in \mathcal{N} \} \) and \( \{ \mu_k; k \in \mathcal{N} \} \) by

\[
\lambda_k = pf_k, \quad \mu_k = qf_k,
\]

where \( f_k \) denotes the sequence

\[
(3.2.12) \quad f_k = \frac{C}{C - 1} [\delta C^{k-1} + (\gamma - 1)].
\]

Remark the sequences \( \{ \lambda_k; k \in \mathcal{N} \} \) and \( \{ \mu_k; k \in \mathcal{N} \} \) are strictly increasing and tend to \( +\infty \), furthermore, we have the iterative relation

\[
f_{k+1} = C[\hat{f}_k - (\gamma - 1)] \quad (5_k).
\]

**Proof of Proposition 3.2.** We proceed again by induction.

First, we use the fact that the sequences \( \hat{h}_k \) and \( \hat{h}_k \) are strictly increasing to establish the existence of an integer \( k_0 \) such that \( \lambda_0 \geq \hat{h}_0 \) and \( \mu_0 \geq \hat{h}_0 \); we obtain from Lemma 3.2.3. that \( u^\ast \in L^b(\Omega) \) and \( v^\ast \in L^b(\Omega) \).
Suppose that the proposition is true for $0 \leq k' \leq k$. Let $\pi_p = C p$ and $\pi_q = C q$, (1) and (2) give: $a_k = p(f_k \gamma)$ and $b_k = q(f_k \gamma)$.

So,

$$1 + \frac{a_k}{p} = 1 + f_k \gamma \leq \frac{C}{C-1} \left[ \frac{\delta}{C} + (\gamma - 1) \right] C^k.$$

Moreover by (5), we obtain

$$\lambda_{k+1} = \left( 1 + \frac{b_k}{q} \right) \pi_q.$$

and similarly

$$\mu_{k+1} = \left( 1 + \frac{b_k}{q} \right) \pi_q.$$

Then, we conclude with Lemma 3.2.2. that $u^* \in L^{\lambda_q}(\Omega)$, according to (3.1.6) and taking

$$A = \frac{C}{C-1} \left[ \frac{\delta}{C} + (\gamma - 1) \right],$$

$$\|u^*\|_{L^{\lambda_q}(\Omega)}^{\lambda_q} \leq C \left( 1 + \frac{a_k}{p} \right)^{\gamma} \left( A_1 \|u^*\|_{L^\lambda(\Omega)}^\lambda + A_2 \|u^*\|_{L^\lambda(\Omega)}^{\alpha_{q+1}} \left( \frac{\|v^*\|_{L^\lambda(\Omega)}^{\beta+1}}{\alpha^\beta} \right)^{\frac{\beta+1}{\beta}} \right).$$

(3.2.13)

$$\leq A^C C^{\kappa_p} \max \left( 1; \|u^*\|_{L^\lambda(\Omega)}^\lambda; \|v^*\|_{L^\lambda(\Omega)}^\beta \right)^C.$$

Considering the equality
we obtain an analogous inequality

\[
\|v^*\|_{L^{\infty}(\Omega)} \leq A C^{\frac{1}{2}} \max\left\{1; \|u^*\|_{L^{\infty}(\Omega)}; \|v^*\|_{L^{\infty}(\Omega)}\right\}^{\frac{1}{2}}
\]

(3.2.14)

As in [20], [26], [27], we obtain the iterative relation \( E_{k+1} \leq r_k + CE_k \), where

\[
E_k = \ln \max\left(\|u^*\|_{L^{\infty}(\Omega)}^{\frac{1}{2}}; \|v^*\|_{L^{\infty}(\Omega)}^{\frac{1}{2}}\)
\]

(3.2.15)

\[
r_k = ak + b \quad a = \ln C^{\max(p,q)}, \quad b = \ln(A)^C
\]

So, we get the iterative relation \( E_k \leq dC^{k-1} \), where \( d \) denotes a positive constant.

Thus,

\[
\|u^*\|_{L^{\infty}(\Omega)} \leq \exp\left(\frac{E_k}{\lambda_k}\right) \exp\left(\frac{d(C-1)}{pC\delta}\right)
\]

(3.2.16)

\[
\|v^*\|_{L^{\infty}(\Omega)} \leq \exp\left(\frac{d(C-1)}{qC\delta}\right)
\]

then, \( u^* \) and \( v^* \) are bounded in \( L^\infty(\Omega) \) and \( L^p(\Omega) \) independently of \( k \in \mathbb{N} \).
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