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On a Nonlinear Stationary Problem in Unbounded Domains

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ABSTRACT. We study existence and some properties of solutions of the nonlinear elliptic equation

$$N(x, a(u))\, Lu = f$$

in unbounded domains. The above model is not a variational problem. Our techniques involve fixed point arguments and Galerkin method.

1. INTRODUCTION

In this paper we present some results on the following problem:

$$(1.1)\qquad \left|\begin{array}{l} N(x, a(u))\, Lu = f \quad \text{in} \quad \Omega \\ u = 0 \quad \text{on} \quad \partial\Omega = \Gamma \end{array}\right.$$

where Ω is an open subset of $\mathbf{R}^N$, $N \geq 1$ with boundary Γ, N is a real valued function with domain $\Omega \times [0, +\infty)$ and L is a differential operator defined by

$$Lu = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + a_o u$$

and

$$a(u) = \sum_{i,j=1}^{N} \int_\Omega a_{ij}(x) \frac{\partial u}{\partial x_i}\, \frac{\partial u}{\partial x_j}\, dx + \int_\Omega a_o(x) u^2(x)\, dx$$

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The origin of this problem can be found in Lions [6] where the following evolution problem connected with nonlinear vibrations was studied

$$
(1.2) \qquad \left| \begin{array}{l} u_{tt} - M(\|u\|^2)\, \Delta u = f \text{ in } Q = \Omega \times (0, T) \\ u = 0 \quad \text{on} \quad \Sigma = \partial\Omega \times (0, T) \\ u(0) = u_o \qquad u'(0) = u_1 \end{array} \right.
$$

In (1.2) M is a continuous real valued function which satisfies $M(\lambda) \geq m_o > 0$, $\forall \lambda \in \mathbf{R}$ and Ω is an open bounded set of $\mathbf{R}^N$ with smooth boundary $\Gamma = \partial\Omega$.

Menzala [7] solved the problem (1.2) when $\Omega = \mathbf{R}^N$ and the initial data are smooth functions.

Ebihara-Miranda-Medeiros [3] solved the problen (1.2) when Ω is an open bounded set of $\mathbf{R}^N$ with smooth boundary, M is a nonnegative continuously differentiable real function and the initial data are smooth.

Recently Crippa [2] generalized the above paper using «Hilbertien Integral» methods.

In Lions [6] the following problem was also proposed:

$$
(1.3) \qquad \left| \begin{array}{l} u_{tt} - M(x, \|u\|^2)\, \Delta u = f \text{ in } Q \\ u = 0 \quad \text{on} \quad \Sigma \\ u(0) = u_o \qquad u'(0) = u_1 \end{array} \right.
$$

This problem was solved in Rivera [8] when Ω is a smooth open bounded set of $\mathbf{R}^N$ and

$$
M(x, \lambda) \geq m_o > 0, \qquad \forall x \in \Omega \qquad \forall \lambda \in \mathbf{R}
$$

He used Galerkin method and the discrete spectrum of the Laplacian operator in bounded domains.

When Ω is a general unbounded open set of $\mathbf{R}^N$, in Vasconcellos [9] the existence and uniqueness of the solution for the problem (1.3) was proved. There, fixed point arguments together with energy methods were used.

The elliptic problem (1.1) can not be defined like a variational problem, that is, one can not find a nonlinear functional J defined in a suitable Sobolev space, such that the equation of problem (1.1) is its Euler equation. Then, to solve this problem we shall use a consequence of Brouwer fix point theorem and Galerkin methods.

In the section 2 we prove an abstract existence theorem (Theorem 2.1). Some properties about the set of solutions are given.

In section 3 we apply the Theorem 2.1 to solve the problem (1.1) when Ω is a general open subset of $\mathbf{R}^N$.

Our assumptions about the mapping N and the operator L are described in §3.

2. AN ABSTRACT EXISTENCE THEOREM

Let H be a real separable Hilbert space with inner product denoted by $(\cdot|\cdot)$ and norm $|\cdot|$.

We consider a linear operator A in H, with the following properties:

(2.1) $D(A)$, the domain of A, is dense in H.

(2.2) A is a self-adjoint operator and there is a constant $C_0>0$ such that $(Au/u)\geq C_o|u|^2$, for each u in H.

(2.3) The Hilbert space $V=D(A^{1/2})$, with norm denoted by $\|v\|=|A^{1/2}v|$, for v in V, is compactly embeded on H.

Now, we denote by $\mathcal{L}_s(H)$ the space of the symmetric linear bounded operators in H with the whole space as domain and we define the mapping $M:[0,+\infty)\to\mathcal{L}_s(H)$ such that:

(2.4) For each u in H the mapping $u\to M(\lambda)u$, is continuous.

(2.5) There is $m_o>0$ such that $(M(\lambda)u|u)\geq m_o|u|^2$ for each u in H and $\lambda\geq 0$.

Theorem 2.1 (abstract existence theorem).

Under above hypotheses, if f belongs to H, then there exists u in $D(A)$ such that:

(2.6) $$M(\|u\|^2)Au=f, \text{ in } H.$$

To prove this theorem, it is sufficient to show that, for f in H there exists v in V such that:

(2.7) $$(M(|v|^2)A^{1/2}v|A^{1/2}\omega)=(f|A^{1/2}\omega), \quad \forall\omega\in V$$

In fact, if $v \in V$ satisfies (2.7), we define $u = A^{-1/2} v$. Then, u belongs to $D(A)$ and $\|u\|^2 = |v|^2$. Moreover, from (2.7) we obtain that:

$$(M(\|u\|^2) Au | A^{1/2} \omega) = (f | A^{1/2} \omega), \qquad \forall \omega \in V$$

Since, $\omega = A^{-1/2} z$ belongs to V for every z in H we obtain (2.6).

Now, let us prove (2.7).

By (2.5) we obtain that

$$(2.8) \quad (M(|v|^2) A^{1/2} v | A^{1/2} v) - (f | A^{1/2} v) \geq m_o \|v\|^2 - |f| \, \|v\| \quad \forall v \in V$$

By (2.3), we observe that, the operator A has discrete spectrum, i.e., there exists a sequence of real numbers $\{\lambda_\nu\}_\nu$ and a sequence $\{\omega_\nu\}_\nu$ of elements in $D(A)$ such that:

$$(2.9) \qquad 0 < \lambda_\nu \leq \lambda_{\nu+1}, \quad v = 1, 2, \ldots$$

$$(2.10) \qquad \lim_{\nu \to +\infty} \lambda_\nu = +\infty$$

$$(2.11) \qquad A\omega_\nu = \lambda_\nu \omega_\nu$$

(2.12) $\{\omega_1, \omega_2, \ldots\}$ is a orthonormal complete set in H.

We denote by V_ν the space generated by $\{\omega_1, \omega_2, \ldots, \omega_\nu\}$, $\nu = 1, 2, \ldots$ and we define $P: \mathbf{R}^\nu \to \mathbf{R}^\nu$ by $P(\alpha_1, \ldots, \alpha_\nu) = (\beta_1, \ldots, \beta_\nu)$, where

$$\beta_j = \left(M(|v_\nu|^2) A^{1/2} v_\nu | A^{1/2} \frac{\omega_j}{\sqrt{\lambda_j}} \right) - \left(f | A^{1/2} \frac{\omega_j}{\sqrt{\lambda_j}} \right), \quad j = 1, \ldots, \nu$$

and

$$v_\nu = \sum_{j=1}^{\nu} \alpha_j \lambda_j^{-1/2} \omega_j.$$

By (2.4) and (2.8) we obtain that: P is a continuous function which satisfies

$$(2.13) \qquad (P\alpha, \alpha) \geq 0 \ \forall \alpha \in \mathbf{R}^\nu \text{ such that } |\alpha| = \frac{|f|}{m_o}$$

where $|\alpha| = \{\sum_{j=1}^{\nu} \alpha_j^2\}^{1/2}$ and $(\beta . \alpha)$ is a dot product in $\mathbf{R}^\nu$.

Now, we recall the following lemma:

Lemma 2.1 *Let P: $\mathbf{R}^{\nu} \to \mathbf{R}^{\nu}$ be a continuous functions which satisfies: $\exists r>0$ such that $(P\alpha.\alpha)\geq 0$, $\forall \alpha \in \mathbf{R}^{\nu}$, $|\alpha|=r$. Then there exists $\alpha_o \in \mathbf{R}^{\nu}$, $|\alpha_o| \leq r$ such that $P(\alpha_o)=0$.*

This lemma is a consequence of the Brouwer point fixed theorem and one can find the proof, for example, in Lions [4] page 53.

So, by (2.13) and Lemma 2.1 we obtain for each $\nu \in \mathbf{N}$ a vector v_ν in V_ν such that:

$$\|v_\nu\| \leq \frac{1}{m_o}|f|, \quad \nu=1,2,\dots . \tag{2.14}$$

$$\left(M(|v_\nu|^2)A^{1/2}v_\nu | A^{1/2}\frac{\omega_j}{\sqrt{\lambda_j}}\right) = \left(f | A^{1/2}\frac{\omega_j}{\sqrt{\lambda_j}}\right) \; j=1,\dots,\nu \tag{2.15}$$

By (2.14) and assumption (2.3) there is a subsequence, also denoted by $\{v_\nu\}$, and v belonging to V such that:

$$\lim_{\nu\to+\infty} v_\nu = v \text{ weak in } V. \tag{2.16}$$

$$\lim_{\nu\to+\infty} v_\nu = v \text{ strong in } H. \tag{2.17}$$

By (2.16), (2.17) and (2.4) we fix j and passing the limit in (2.15) we obtain (2.7).

Remark 2.1 If u satisfies (2.6) it follows, by (2.5) that

$$|Au| \leq \frac{|f|}{m_o} \tag{2.18}$$

and therefore by (2.2)

$$\|u\| \leq \frac{|f|}{\sqrt{c_o}\, m_o} \tag{2.19}$$

Theorem 2.2 (Properties of the solutions).

Under the hypotheses of the Theorem 2.1, if f belongs to H then:

(2.20) *The set $\chi_f = \{u \in D(A): M(\|u\|^2)Au=f\}$ is a compact set of V.*

(2.21) $\lim_{f \to f_o} d_V(\chi_f, \chi_{f_o}) = 0$ *where* $d_V(\chi_f, \chi_{f_o}) = \inf\{\|u-u_o\| : u \in \chi_f,\ u_o \in \chi_{f_o}\}$.

(2.22) *If* $M(\lambda) = \psi(\lambda) I_H$, *where* I_H *is the identity operator on* H *and* ψ: $\mathbf{R} \to [m_o, +\infty)$ *is a continuous increasing function, then for each* $f \in H$ *the set* χ_f *is a singleton.*

To prove (2.20) we consider $\{u_\nu\}$ a sequence in χ_f, then by (2.19), there is a subsequence of $\{u_\nu\}$, also denoted by $\{u_\nu\}$ such that:

$$\lim_{\nu \to +\infty} Au_\nu = Au \text{ weak in } H, \text{ where } u \text{ belongs to } D(A).$$

Hence, by (2.3),

$$\lim_{\nu \to +\infty} u_\nu = u \text{ strong in } V.$$

Since that $(M(\|u_\nu\|^2) Au_\nu | v) = (f | v)$ for all v in H and for each $\nu \in \mathbf{N}$, then by assumptions about M we obtain that u belongs to χ_f.

To prove (2.21), let f_o be in H and we consider $\{f_\nu\}$ a sequence in H such that:

$$\lim_{\nu \to +\infty} f_\nu = f_o \text{ strong in } H.$$

For each $\varepsilon > 0$ we define $P_\varepsilon = \{\nu \in \mathbf{N} : d_V(\chi_{f_\nu}, \chi_{f_o}) \geq \varepsilon\}$. We claim that P_ε is a finite set.

In fact, suppose that P_ε is an infinite set, then there is σ: $\mathbf{N} \to \mathbf{N}$ strictly increasing function such that $\sigma(\mathbf{N}) = P_\varepsilon$.

We denote by $g_\nu = f_{\sigma(\nu)}$, $\forall v \in \mathbf{N}$ thus we obtain, for each $\nu \in \mathbf{N}$, v_ν in χ_{g_ν} and therefore by (2.19) $\{Av_\nu\}$ is a bounded sequence in H.

Hence there is a subsequence of $\{v_\nu\}$ (also denoted by $\{v_\nu\}$) and v in $D(A)$ such that:

(2.23) $$\lim_{\nu \to +\infty} Av_\nu = Av \text{ weak in } H.$$

(2.24) $$\lim_{\nu \to +\infty} v_\nu = v \text{ strong in } V.$$

Since $\big(Av_\nu \,|\, M(\|v_\nu\|^2)\omega\big) = (g_\nu | \omega)$, for all ω in V and $v \in \mathbf{N}$ passing to limit we obtain that:

$$\big(Av \,|\, M(\|v\|^2)\omega\big) = (f_o | \omega), \quad \forall \omega \in V.$$

Therefore, v belongs to χ_{f_0} and moreover $d_V(\chi_{g_\nu}, \chi_{f_0}) \leq \|v_\nu - v\|$, $\nu \in \mathbf{N}$. This proves (2.21) by contradiction.

To prove (2.22) we consider φ belonging to V and we define the following functional:

$$J(\varphi) = \frac{1}{2}\int_0^{\|\varphi\|^2} \psi(\lambda)d\lambda - (f|\varphi). \tag{2.25}$$

where f belongs to H.

Let u be in χ_f then by (2.25) we obtain that:

$$\lim_{t\to 0} \frac{J(u+t(\varphi-u)) - J(u)}{t} = 0, \text{ for all } \varphi \text{ in } V. \tag{2.26}$$

Now, since that $g(t) = J(u+t(\varphi-u))$ is a differentiable convex function on $\mathbf{R}$ we obtain that:

$$\frac{g(t)-g(0)}{t} < g(1)-g(0), \quad 0<t<1$$

therefore by (2.26)

$$0 = g'(0) \leq g(1) - g(0)$$

so, $J(u) \leq J(\varphi)$, $\forall \varphi \in V$ that is, if $u \in \chi_f$, we prove that

$$J(u) = \min\{J(\varphi); \varphi \in V\}.$$

Since J is strictly convex functional we have (2.22).

3. APPLICATION TO A NONLINEAR PROBLEM

First, we need to present a compact immersion result for unbounded domains, in weighted Sobolev spaces.

Let Ω be an unbounded set of $\mathbf{R}^N$ with the cone property and $q: \Omega \to \mathbf{R}$ is positive measurable function which satisfies the following properties

$$q \in L^\infty_{\mathrm{loc}}(\overline{\Omega}) \tag{3.1}$$

$$\lim_{|x|\to+\infty} q(x) = 0 \tag{3.2}$$

Let $L^2(\Omega, q)$ be the Hilbert space of measurables functions $u: \Omega \to \mathbf{R}$ such that $\int_\Omega |u(x)|^2 q(x)\,dx < +\infty$ endowed with the norm

$$|u| = \{\int_\Omega |u(x)|^2 \, q(x)\,dx\}^{1/2} \tag{3.3}$$

Remark 3.1 Since the linear application $\sigma: L^2(\Omega, q) \to L^2(\Omega)$ defined by $\sigma(u) = u\sqrt{q}$ is a isometric isomorphism, then $L^2(\Omega, q)$ is a separable Hilbert space with inner product $(u|v) = \int_\Omega u(x)v(x)q(x)\,dx \; u, v \in L^2(\Omega, q)$.

Proposition 3.1 (compact immersion).

The Sobolev space $H^1(\Omega)$ is compactly embeded in $L^2(\Omega, q)$.

This proposition is proved in Benci-Fortunato [1] corollary 2.10.

Remark 3.2 It is possible to consider, in the above proposition, Ω a general open unbounded set of $\mathbf{R}^N$, if we replace the assumption (3.2) by

$$\lim_{x \to x_o} q(x) = 0 \quad x_o \in \partial\Omega \text{ and } \lim_{|x| \to +\infty} q(x) = 0. \tag{3.4}$$

In this case we can see the corollary 2.9 in Benci-Fortunato [1].

Let $N: \Omega \times [0, +\infty) \to \mathbf{R}$ be a function satisfying the following assumptions:

(3.5) $N(x, \cdot): [0, +\infty) \to \mathbf{R}$ is a continuous function a.e. in $x \in \Omega$.

$$N(\cdot, \lambda): \Omega \to \mathbf{R} \text{ belongs to } L^\infty(\Omega),\ \forall \lambda \geq 0. \tag{3.6}$$

$$N \in L^\infty_{\text{loc}}([0, +\infty).\ L^\infty(\Omega)). \tag{3.7}$$

(3.8) There exists $m_o > 0$ such that $N(x, \lambda) \geq m_o$, $\forall (x, \lambda) \in \Omega \times [0, +\infty)$.

We consider $H = L^2(\Omega, q)$ where q satisfies (3.1) and (3.2) (or (3.4)).

Now, we define, $\forall \lambda \geq 0$, the operator $M(\lambda)$ in H by

$$M(\lambda)u = N(\cdot, \lambda)u, \qquad u \in H. \tag{3.9}$$

Then, $\forall \lambda \geq 0$, $M(\lambda) \in \mathscr{L}_s(H)$ and by (3.5)–(3.8), the mapping M: $[0, +\infty) \to \mathscr{L}_s(H)$ satisfies (2.4) and (2.5).

We consider a_{ij}, $i, j=1,\dots,n$ and a_o elements of $L^\infty(\Omega)$ which satisfy:

(3.10) There is $c_o>0$ such that $\sum_{i,j=1}^{n} a_{ij}(x)\varsigma_i\varsigma_j \geq c_o(\varsigma_1^2+\dots+\varsigma_n^2)$ a.e. in Ω and $a_o(x)\geq c_o$ a.e. in Ω.

(3.11) $$a_{ij}=a_{ji} \quad i, j=1,\dots,n.$$

We define the differential operator $Lu=-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right)+a_o u$, u in $H_o^1(\Omega)$ and the bilinear form a: $H_o^1(\Omega)\times H_o^1(\Omega)\to\mathbf{R}$ by

$$a(u,v)=\sum_{i,j=1}^{n}\int_\Omega a_{ij}(x)\frac{\partial u}{\partial x_j}\,\frac{\partial v}{\partial x_i}\,dx+\int_\Omega a_o(x)u(x)v(x)dx.$$

By (3.10) and (3.11) $a(\cdot,\cdot)$ is a symmetric, coercive, continuous bilinear form.

We define $D(A)=\{u\in H_o^1(\Omega): \frac{1}{q}Lu\in L^2(\Omega,q)\}$. Then, by (3.10), (3.11) and Proposition 3.1, the operator $A: D(A)\subset H\to H$ defined by $Au=\frac{1}{q}Lu$, $u\in D(A)$, satisfies (2.1), (2.2) and (2.3), where $V=D(A^{1/2})=H_o^1(\Omega)$ with the norm $\|v\|=|A^{1/2}v|=[a(v,v)]^{1/2}=[a(v)]^{1/2}$.

Now, by Theorem 2.1, if f belongs to $H=L^2(\Omega,q)$, there exists u in $H_o^1(\Omega)$ such that:

(3.12) $$N(x,a(u))Lu=q\,f(x) \text{ a.e. in } \Omega.$$

Therefore, if f belongs to $L^2(\Omega,\frac{1}{q})$, by (3.12) we obtain that, there exists $u\in H_o^1(\Omega)$ such that:

(3.13) $$N(x,a(u))Lu=f(x) \text{ a.e. in } \Omega.$$

Remark 3.2 If we consider $q\in L^\infty(\overline{\Omega})$ we obtain that $L^2(\Omega,\frac{1}{q})$ is contained in $L^2(\Omega)$. In general, if Q is the set of q: $\Omega\to\mathbf{R}$ which belong to $L^\infty(\Omega)$ and satisfy (3.2) (or (3.4)) then $X=\bigcup_{q\in Q}L^2(\Omega,\frac{1}{q})$ is a dense subset of $L^2(\Omega)$.

Remark 3.3 If Ω is an general open bounded subset of $\mathbf{R}^N$, we consider $H=L^2(\Omega)$ and the problem (1.1) has a solution u in $H_o^1(\Omega)$ for each f in $L^2(\Omega)$.

Remark 3.4 If we consider $N: [0, +\infty) \to [m_o, +\infty)$ increasing continuous function, then by Theorem 2.2, the problem

$$\left| \begin{array}{l} N(a(u))\,Lu = f \quad \text{in} \quad \Omega \\ \\ u = 0 \quad \text{on} \quad \partial\Omega \end{array} \right.$$

has an unique solution u in $H_o^1(\Omega)$, for each $f \in H$.

Where $H = L^2(\Omega, \frac{1}{q})$ if Ω is an open unbounded set or $H = L^2(\Omega)$ if Ω is an open bounded set.

Remark 3.5 All the solutions of the problem (1.1) have the properties given by the Theorem 2.2.

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