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On a Formula for the Jumps in the Semi-Fredholm Domain

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ABSTRACT. In this paper we prove some properties of the lower s-numbers and derive asymptotic formulae for the jumps in the semi-Fredholm domain of a bounded linear operator on a Banach space.

1. INTRODUCTION AND PRELIMINARIES

In this note X, Y, Z and W are complex Banach spaces, and $B(X, Y)$ $(B(X))$ the set of all bounded linear operators from X into Y (on X). Let $K(X, Y)$ denote the set of compact linear operators from X into Y Let U denote the closed unit ball of X. Let $T \in B(X, Y)$ and

$$m(T) = inf\{\|Tx\| \quad : \quad \|x\| = 1\}$$

be the minimum modulus of T, and let

$$q(T) = sup\{\varepsilon \geq 0 \quad : \quad TU \supset \varepsilon U\}$$

be the surjection modulus of T. Recall that both $m(T)$ and $q(T)$ are positive if and only if T is invertible, and in this case $m(T) = q(T) = \|T^{-1}\|^{-1}$.

For each $r = 1, 2, ..., \infty$ we define the following lower analogues of the approximation numbers [8]:

$$m_r(T) = sup\,\{m(T+F)\text{: rank } F < r\},$$

$$q_r(T) = sup\,\{q(T+F)\text{: rank } F < r\},$$

$$g_r(T) = max\{m_r(T),\ q_r(T)\}.$$

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If M is a subspace of X, then $T_{|M}$ will denote the restriction of T to M. T is a semi-Fredholm operator if either the null space $N(T)$ is finite-dimensional and the range $R(T)$ is closed, or the codimension of $R(T)$ is finite. For such operators the index defined by

$$\text{ind}\,(T)=\dim N(T)-\text{codim}\,R(T),$$

and the minimum index by

$$\text{min. ind}\,(T)=\min\{\dim N(T),\ \text{codim}\,R(T)\},$$

which is always finite. It was shown in [12, Theorem 8.3] that

$$s(T)=\lim_k g_\infty(T^k)^{1/k}$$

is the semi-Fredholm radius of T, i.e. the supremum of all $\varepsilon\geq 0$ such that $T-\lambda I$ is semi-Fredholm for $|\lambda|<\varepsilon$. It is well known that the function min. ind $(T-\lambda I)$ is constant everywhere in the disk $|\lambda|<s(T)$ except possibly for a discrete subset G. We denote by $n(T)$ this constant, and call it the *stability index* of the semi-Fredholm operator T[8]. A point ω in G is called a jumping point of the minimum index in the semi-Fredholm domain. For ω in G we have min. ind $(T-\omega I)>n(T)$, and X decomposes into the direct sum of two closed T-invariant subspaces Y_ω and Z_ω, where Z_ω is finite-dimensional and $T-\omega I$ is nilpotent on it, while the restriction on $T-\lambda I$ to Y_ω has constant minimum index on a neighbourhood of ω[3, Theorem 4]. Consistently with the matrix case we define the (algebraic) *multiplicity of the jumping point* ω to be dim Z_ω [8, pp. 232]. Thus the point in G can be ordered in such a way that

$$|\omega_1(T)|\leq|\omega_2(T)|\leq\ldots<s(T),$$

where each jump appears consecutively according to its multiplicity. If there are only $p\,(=0,1,2,\ldots)$ such jumps, we put $|\omega_{p+1}(T)|=|\omega_{p+2}(T)|=\ldots=s(T)$. Recall that [8, Theorem 1.1] if T is a semi-Fredholm operator, then for each $r=1,2,\ldots$ we have

$$(1)\qquad |\omega_r(T)|=\lim_k g_{kn+r}(T^k)^{1/k},$$

where $n=n(T)$ is the stability index of T.

In this note we prove (1) when the stability index of T is zero, and we believe that in this case the proof is simpler than the mentioned one in the general case. Further, we use a restriction techniques and show how this particular case is related to general case.

2. RESULTS

In the following lemma we list some properties of the lower s-numbers.

Lemma 2.1. *Let* $T \in B(X, Y)$. *Then*

(i) $0 \leq m_1(T) \leq m_2(T) \ldots \leq m_\infty(T) \leq \sup_{K \in K(X, Y)} m(T+K) \leq \inf_{K \in K(X, Y)} \|T+K\|$,

(ii) $m_n(S+T) \leq m_n(S) + \|T\|$ *for* $S, T \in B(X, Y)$,

(iii) $m_n(RST) \geq m(R) m_n(S) m(T)$ *for* $T \in B(X, Y)$, $S \in B(Y, Z)$ *and* $R \in B(Z, W)$,

(iv) *If* $\dim X \geq n$, *then* $m_n(I) = 1$,

(v) $m_{n+m-1}(ST) \geq m_n(S) m_m(T)$ *for* $T \in B(X, Y)$ *and* $S \in B(Y, Z)$,

(vi) $m_n(T) > 0 \Leftrightarrow \dim N(T) < n$, $R(T)$ *is closed and* $\operatorname{ind}(T) \leq 0$.

Proof. *(i)* By the definition and [6. pp. 389].

(ii) Let $F \in B(X, Y)$ and rank $F < n$. By [1, Lemma 2.2] we have

$$m(S+T+F) \leq m(T+F) + \|S\| \leq m_n(T) + \|S\|,$$

and hence $m_n(S+T) \leq m_n(T) + \|S\|$

(iii) Let $F \in B(Y, Z)$ and rank $F < n$. Now, $RFT \in B(X, W)$, rank $RFT < n$ and by [1, pp. 21] we have

$$m_n(RST) \geq m(R(S+F)T) \geq m(R) m(S+F) m(T).$$

Further, it follows that $m_n(RST) \geq m(R) m_n(S) m(T)$.

(iv) It is clear that $m_n(I) \geq 1$. If $m_n(I) > 1$, then there is an $F \in B(X)$ and rank $F < n$, such that $m(I+F) > 1$. Since $m(F) = 0$, it follows that $m(I+F) \leq m(F) + \|I\| = 1$, which is a contradiction. Hence $m_n(I) = 1$.

(v) Let $F_1 \in B(X, Y)$, rank $F_1 < n$, $F_2 \in B(Y, Z)$ and rank $F_2 < m$. Then $(S+F_2)(T+F_1) \in B(X, Z)$, $(S+F_2)(T+F_1) = ST + SF_1 + F_2(T+F_1) \in B(X, Z)$ and rank $[SF_1 + F_2(T+F_1)] < n+m-1$. Thus $m_{n+m-1}(ST) \geq m[(S+F_2)(T+F_1)] > m(S+F_2) m(T+F_1)$, which proves *(v)*.

(vi) Suppose that $m_n(T)>0$, rank $F<n$ and dim $N(T)\geq n$. Now codim $N(F)<n$, and it follows that $N(T)\cap N(F)\neq\{0\}$. Thus $m(T+F)=0$, i.e., $m_n(T)=0$, whence a contradiction. Thus $m_n(T)>0$ implies dim $N(T)<n$. That $R(T)$ is closed and ind $(T)\leq 0$ follows by elementary properties of semi-Fredholm operators [9]. Conversely, if $R(T)$ is closed, dim $N(T)<n$ and ind $(T)\leq 0$, then by [11, Theorem 3.9 (2)] there is an operator $F\in B(X)$ such that rank $(F)<n$ and $m(T+F)>0$. This implies that $m_n(T)>0$.

This completes the proof of the lemma.

Theorem 2.2. *Let $T\in B(X)$ be a semi-Fredholm operator with the stabilitty index of T equal to zero and min. ind$(T-\lambda I)=$dim $N(T-\lambda I)$ in the disk $|\lambda|<s(T)$ except possibly for the jumps $\omega_r(T)$, $r=1, 2, \ldots$. Then for each $r=1, 2, \ldots$ we have*

$$|\omega_r(T)|=\lim_k m_r(T^k)^{1/k}.$$

Proof. We have to prove two things. First

$$|\omega_r(T)|\leq \lim_k \inf m_r(T^k)^{1/k}. \tag{2}$$

and second

$$\lim_k \sup m_r(T^k)^{1/k}\leq|\omega_r(T)|, \tag{3}$$

Note that $\omega_1(T)=\lim_k m_1(T^k)^{1/k}$ [4, Theorem 3], and it is clear that (2) and (3) are true for $r=1$. To show the induction step for (2), take the least q such that $\omega_{n-q}(T)\neq\omega_n(T)$. (If such a q does not exist, then (2) is obvious since $|\omega_n(T)|=|\omega_1(T)|$ in that case). Let Z be the direct sum of the finite-dimensional parts in the Kato decompositions corresponding to the points $\omega_1(T), \ldots, \omega_{n-q}(T)$ [3, Theorem 4]. Now dim $Z=n-q$. Let Y be the intersection of the corresponding Kato complements to the finite-dimensional parts in the Kato decompositions corresponding to the points $\omega_1(T), \ldots, \omega_{n-q}(T)$. Thus the space X decomposes into a direct sum of two closed subspace Y and Z. These subspaces are T-invariant. Let F be the removing operator from the proof of [12, Theorem 7.1], i.e., F is zero on Y and $\mu_1 I$ on Z; μ_1 is any complex number with $|\mu_1|>\|T\|+s(T)$. By the proof of [12, Theorem 7.1] and [4, Theorem 3] we have that

$$\lim_k m((T+F)^k)^{1/k}=|\omega_{n-q+1}(T)|.$$

Further for each $k=1, 2, \ldots$ we have

$$m_n(T^k)\geq m_{n-q+1}(T^k)\geq m((T+F)^k),$$

and so the proof of (2) is complete.

Now we turn to prove the inequality (3). Let W be the direct sum of the finite-dimensional parts in the Kato decompositions corresponding to the points $\omega_1(T), \ldots, \omega_n(T)$ [3, Theorem 4]. Now dim $W\geq n$. Let V be the intersetion of the corresponding Kato complements to the finite-dimensional parts in the Kato decompositions corresponding to the points $\omega_1(T), \ldots, \omega_n(T)$. Thus the space X decomposes into a direct sum of two closed subspaces W and V. These subspaces are T-invariant. Let $F\in B(X)$ and rank $F<n$. Hence, there is a vector $h\in W\cap N(F)$ such that $h\neq 0$. Let P be the projection of X onto W along V. Then

$$\|(T+F)h\| = \|Th\| = \|TPh\| \leq \|T_{|W}\| \; \|P\| \; \|h\|.$$

Thus, $m(T+F)\leq \|P\| \; \|T_{|W}\|$. It is easy to see that for each $k=1, 2, \ldots$ we have $m(T^k+F)\leq \|P\| \; \|T^k{}_{|W}\|$. Consequently $m_n(T^k)\leq \|P\| \; \|T^k{}_{|W}\|$, and since the spectral radius of $T_{|W}$ is equal to $|\omega_n(T)|$, it follows that

$$\lim_k \sup m_n(T^k)^{1/k}\leq |\omega_n(T)|.$$

This proves (3), and the proof of the theorem is complete.

Remark 2.3. *Let us mention that if in Theorem 2.2 we have that $\omega_1(T)\neq 0$, then we can prove (3) in the following way (we use the same notations as in the proof of Theorem 2.2):* Now $T_{|W}: W\to W$ is invertible and since dim $W\geq n$ we have by Lemma 2.1 *(iv)* that $m_n(T^k(T_{|W}{}^{-k}))=1$, $k=1, 2, \ldots$. Thus by Lemma 2.1 *(v)* we have $1\geq m_n(T^k)m((T_{|W}{}^{-1})^k)$, and so

$$m_n(T^k)\leq 1/m((T_{|W}{}^{-1})^k)= \|T_{|W}{}^k\|.$$

Since the spectral radius of $T_{|W}$ is equal to $|\omega_n(T)|$ we conclude that

$$\lim_k \sup m_n(T^k)^{1/k}\leq |\omega_n(T)|,$$

whence the result.

Next we state properties of $q_n(T)$ and the dual result of Theorem 2.2. They can be proved similarly, so we leave out details.

Lemma 2.4. *Let $T \in B(X, Y)$. Then*

(i) $0 \leq q_1(T) \leq q_2(T) \ldots \leq q_\infty(T) \leq \sup_{K \in K(X, Y)} q(T+K) \leq \inf_{K \in K(X, Y)} \|T+K\|$.

(ii) $q_n(S+T) \leq q_n(S) + \|T\|$ *for* $S, T \in B(X, Y)$,

(iii) $q_n(RST) \geq q(R) q_n(S) q(T)$ *for* $T \in B(X, Y)$, $S \in B(Y, Z)$ *and* $R \in B(Z, W)$,

(iv) *If* $\dim X \geq n$, *then* $q_n(I) = 1$,

(v) $q_{n+m-1}(ST) \geq q_n(S) q_m(T)$ *for* $T \in B(X, Y)$ *and* $S \in B(Y, Z)$,

(vi) $q_n(T) > 0 \Longleftrightarrow \operatorname{codim} R(T) < n$, *and* $\operatorname{ind}(T) \geq 0$,

(vii) If $m_n(T) > 0$ *and* $q_n(T) > 0$, *then* $m_n(T) = q_n(T)$ *and* $\operatorname{ind}(T) = 0$.

Proof. We shall prove only *(vii)*. From *(vi)* and Lemma 2.1 *(vi)*, it follows that $\dim N(T) < n$, $R(T)$ is closed, $\operatorname{codim} R(T) < n$ and $\operatorname{ind}(T) = 0$. Let $F \in B(X, Y)$ and rank $F < n$. If $m(T+F) > 0$, then $\dim N(T+F) = 0$, and it follows that $\operatorname{codim} R(T+F) = 0$. Thus, $m(T+F) = q(T+F) \leq q_n(T)$, and we have that $m_n(T) \leq q_n(T)$. In a similar way, we can prove that $q_n(T) \leq m_n(T)$, and the proof is complete.

Theorem 2.5. *Let $T \in B(X)$ be a semi-Fredholm operator with the stability index of T equal to zero and min. $\operatorname{ind}(T-\lambda I) = \operatorname{codim} R(T-\lambda I)$ in the disk $|\lambda| < s(T)$ except possibly for the jumps $\omega_r(T)$, $r = 1, 2, \ldots$. Then for each $r = 1, 2, \ldots$ we have*

$$|\omega_r(T)| = \lim_k q_r(T^k)^{1/k}.$$

Proof. By Lemma 2.4 and Theorem 2.2.

For T in $B(X)$ set $N(T^\infty) = \cup N(T^n)$ and $R(T^\infty) = \cap R(T^n)$. If T is a semi-Fredholm, then it is well known ([5, Theorem 4.1] see also [7, Theorem 5.2] for general case) that the function $\lambda \to N((T-\lambda)^\infty) + R((T-\lambda)^\infty)$ is constant, say W everywhere in the disk $|\lambda| < s(T)$. Let us remark that W is closed, hence Banach subspace in X (see ([5, pp. 517, Corollary 3.2] and [10, Proposition 1.10]) or ([7, Remark 5.3] and [2, Lemma 3.6 (a), Theorem 3.8])) The restriction of T to the subspace W has been studied in [2], [5], [7] and [10]. Now we have

Theorem 2.6. *Let $T \in B(X)$ be a semi-Fredholm operator, and $\omega_r(T)$, $r=1, 2, \ldots$ are as above. Then for each $\omega_r(T)$, $r=1, 2, \ldots$ we have*

$$|\omega_r(T)| = \lim_k q_r((T_{|W})^k)^{1/k}.$$

Proof. By [5, Theorem 4.1] and [3, Theorem 4] we know that everywhere in the disk $|\lambda| < s(T)$ we have that $W = R((T-\lambda)^\infty) \oplus N_\lambda$, where N_λ is finite dimensional subspace T-invariant and $(T-\lambda)_{|N_\lambda}$ is nilpotent on it (see also [7, Remark 5.3]). Thus by [2, Theorem 3.4] we have that $(T-\lambda)(W) = (T-\lambda)(R((T-\lambda)^\infty) \oplus N_\lambda) = R((T-\lambda)^\infty) \oplus (T-\lambda)(N_\lambda)$. Thus, $(T-\lambda)_{|W}$ is semi-Fredholm, dim $W/R((T-\lambda)_{|W}) < \infty$ and the stability index of $T_{|W}$ is zero ([5, Proposition 2.6]). Let us remark that $\omega_r(T)$, $r=1, 2, \ldots$ are jumps (with the same multiplicity) in the semi-Fredholm region of $T_{|W}$. Now the proof of the theorem follows by Theorem 2.5.

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