REVISTA MATEMATICA de la
Universidad Complutense de Madrid
Volumen 5, números 2 y 3; 1992.
http://dx.doi.org/10.5209/rev_REMA.1992.v5.n2.17889

Splitting of the $\bar{\partial}$-Complex in Weighted Spaces of Square Integrable Functions

MICHAEL LANGENBRUCH

ABSTRACT. *The splitting of the $\bar{\partial}$-complex in weighted spaces of (locally) square* integrable functions (defined on $\Omega \subset \mathbb{C}^N$ by means of an (increasing) weight system $\{W_n \mid n \geq 1\}$) is characterized by the following criterion on the existence of certain plurisubharmonic (psh.) functions: For any $t \in \Omega$ there are psh. functions Φ_t on Ω and for any $n \geq 1$ there are $I(n) \geq n$ and $A(n) \geq 0$ such thar for any $n \geq 0$ and any $z, t \in \Omega$:

$$\Phi_t(z) - \Phi_t(t) \leq W_{I(n)}(z) - W_n(t) + A(n). \qquad (*)$$

This is applied to the generation of weighted algebras of holomorphic functions and to the existence of extension operators for holomorphic functions defined on strongly interpolating varieties. A systematic study of (*) is given in [11].

The splitting of the $\bar{\partial}$-complex is closely related to the existence of extension and interpolation operators for holomorphic functions and to the existence of continuous linear right inverses for partial differential operators ([10 - 23)]. It has been studied by many authors: B. Mitjagin and G. Henkin [18] used Hilbert scales to solve related questions in spaces without growth conditions. V. P. Palamodov [22] has shown that in the space $C^\infty(\Omega)$ the $\bar{\partial}$-operator has a continuous linear right inverse when acting from $(0, k)$—forms to $(0, k+1)$—forms for $k \geq 1$, while it has none for $k = 0$. Then B. A. Taylor [23] used the theory of analytically uniform spaces to solve the problem affermatively in the space of C^∞-functions of exponential type. R. Meise and B. A. Taylor used methods from the structure theory of power series spaces to characterize the weights W, such that the $\bar{\partial}$-complex splits in an associated weighted space of distributions (growing on $\mathbb{C}^N$ at most like $\exp(nW)$, see

1991 Mathematics Subject Classification: H32F20, 32F05, 30D15.

Editorial Complutense. Madrid, 1992.

[17]). Recently, S. Momm [19-21] solved the splitting problem for radially symmetric weight systems on the disc, obtaining explicit estimates for the right inverses of the $\bar\partial$-operator. Langenbruch [10] studied similar weighted spaces as R. Meise and B. A. Taylor and used tame splitting theory ([24]) to obtain right inverses with tame continuity estimates. However the details were rather complicated and the use of distributions (necessary to apply the splitting theory) seemed to prevent optimal estimates. From the point of view of L. Hörmander's solution of the weighted $\bar\partial$-problem ([7]) it seemed more natural to treat the problem entirely in the framework of weighted L^2-spaces, and it is this ansatz which will be used in this paper.

The paper is divided into two parts: In the first section the splitting of the $\bar\partial$-complex is studied in the weighted spaces

$$L^2(\mathfrak{B},\Omega):=\{f\in L^2_{\mathrm{loc}}(\Omega)\mid \|f\|_n^2:=\int|f(z)|^2e^{-2W_n(z)}dz<\infty \text{ for some } n\geq 1\}.$$

Here $\Omega\subset\mathbb{C}^N$ is a pseudoconvex set and $\mathfrak{B}:=\{W_n\mid n\geq 1\}$ is an increasing system of weights satisfying some mild technical conditions (see (1.1)-(1.3)). So the problem is considered in the most general setting. It is shown that the splitting of the $\bar\partial$-complex on $L^2(\mathfrak{B},\Omega)$ is equivalent to the following condition: For any $t\in\Omega$ there are plurisubharmonic (psh.) functions Φ_t and for any $n\geq 1$ there are $I(n)\geq n$ and $A(n)\geq 0$ such that for any $n\geq 1$, $t\in\Omega$ and $z\in\Omega$:

$$\Phi_t(t)\geq 0 \text{ and } \Phi_t(z)\leq W_{I(n)}(z)-W_n(t)+A(n) \qquad (*)$$

Moreover we obtain explicit continuity estimates for the right inverses R of the $\bar\partial$-operator. These estimates cannot be improved essentially (see Remark 1.8). Condition (*) can easily be evaluated in many cases ([11]).

To prove the sufficiency of (*), we first solve the splitting problem locally by induction, using a suitable C^∞-resolution of the identity. Here (*) is needed to define intermediate weighted Hilbert spaces, where Hörmander's solution of the $\bar\partial$-problem can be applied, and which locally serve as a simultaneous substitute for the (LF)-topology of $L^2(\mathfrak{B},\Omega)$. So the local «projections» can be shown to converge in the (LF)-topology. To prove the necessity of (*), we use the fact, that the Koszul complex for $(z-t)$ splits for $t\in\Omega$, if the $\bar\partial$-complex is split.

In passing we notice, that the weight system $\mathfrak{B}$ is equivalent to a system $\tilde{\mathfrak{B}}$ consisting of psh. functions, if the $\bar\partial$-complex is split for $L^2(\mathfrak{B},\Omega)$.

For a decreasing weight system $\mathfrak{B}:=\{V_n\mid n\geq 1\}$ the splitting of the $\bar\partial$-complex for the (F)-spaces $L^2(\mathfrak{B},\Omega)$ is equivalent to the following variant of

(*): For any $t\in\Omega$ there are psh. functions Ψ_t and for any $n\geq 1$ there are $I(n)\geq n$ and $A(n)\geq 0$ such that for any $n\geq 1$ and any $z,\ t\in\Omega$:

$$\Psi_t(t)\geq 0 \quad \text{and} \quad \Psi_t(z)\leq W_n(z)-W_{I(n)}(t)+A(n) \qquad (^{**})$$

In the second section we give some applications of the results of the first section. We first consider weighted algebras $H^2(\mathfrak{B},\Omega)$ of holomorphic functions, which are generated by a finite set of functions $G_1,\dots,G_m$. By a linear version of the proof of L. Hörmander [6] it is shown, that this generation can be linearized, that is, there are continuous linear operators T_i in $H^2(\mathfrak{B},\Omega)$ such that

$$f=\sum_{i\leq m} T_i(f)\,G_i \text{ for any } f\in H^2(\mathfrak{B},\Omega),$$

if (*) holds.

We then follow the idea of C. Berenstein, R. Meise and B. A. Taylor ([2, 3, 17]) to prove the existence of extension operators for holomorphic functions defined on strongly interpolating varieties, if (*) holds. The assumptions are more general than in the papers above and again explicit continuity estimates can be given.

The author wants to thank S. Momm (Düsseldorf) for valuable discussions and remarks on the subject of this paper.

1. SPLITTING OF THE $\bar{\partial}$-COMPLEX

In this section we will consider the $\bar{\partial}$-complex in certain spaces of square integrable functions, which are defined by inductive weight conditions. We will show that the splitting of the $\bar{\partial}$-complex is equivalent to the existence of certain plurisubharmonic (psh.) functions (see (1.4)), if the weight system satisfies some mild technical conditions (see (1.1)–(1.3)), weaker than those normally used in the literature.

Let $\Omega\subset\mathbb{C}^N$ be open and pseudoconvex and let $\mathfrak{B}:=\{W_n \mid n\geq 1\}$ be an increasing system of locally bounded Lebesgue measurable functions on Ω. Let $d(z):=\operatorname{dist}(z,\partial\Omega):=\inf\{\,|z-\xi|_\infty \mid \xi\in\partial\Omega\,\}$ with $|\xi|_\infty:=\max\{\,|\xi_i| \mid i\leq N\}$ for $\xi\in\mathbb{C}^N$. Let r be defined on Ω such that

$$0<r(z)<\min(1,d(z)). \tag{1.0}$$

r is connected with $\mathfrak{B}$ by the following conditions:

For any $n\geq 1$ there are $I_j(n)\geq n$ and $A_j(n)\geq 0$ such that for any $z\in\Omega$:

$$\sup\{ W_n(z+\xi) \mid |\xi|_\infty \leq r(z)\} \leq \inf\{ W_{I_1(n)}(z+\xi) \mid |\xi|_\infty \leq r(z)\} + A_1(n) \tag{1.1}$$

$$W_n(z) + (2N+1)\ln(1+|z|^2) \leq W_{I_2(n)}(z) + A_2(n) \tag{1.2}$$

$$W_n(z) + 4(N+1)N\ln(1/r(z)) \leq W_{I_3(n)}(z) + A_3(n) \tag{1.3}$$

In the literature mainly $r=1$ and $\Omega=\mathbb{C}^N$ is considered (see e.g. [17]). This means that the weight system $\mathfrak{B}$ (or rather the spaces defined by $\mathfrak{B}$) are invariant under shifts. Smaller functions r are certainly needed for $\Omega\neq\mathbb{C}^N$ (see [15. 16]), but they can also be useful for shift invariant systems $\mathfrak{B}$ to improve on the continuity estimates for the projections in Theorem 1.3. A convenient choice for r (and $\Omega=\mathbb{C}^N$) is $r(z)=(1+|z|^2)^{-d}$, $d\in\mathbb{N}$, since then (1.3) and (1.2) almost coincide.

The essential condition now is the following:

For any $t\in\Omega$ there are psh. functions Φ_t and for any $n\geq 1$ there are $I_4(n)\geq n$ and $A_4(n)\geq 0$ such that for any $n\geq 1$ and $t.z\in\Omega$:

$$\begin{aligned} &\Phi_t(t)\geq 0 \\ &\Phi_t(z)\leq W_{I_4(n)}(z) - W_n(t) + A_4(n) \end{aligned} \tag{1.4}$$

(1.4) is the only condition to be evaluated in concrete situations, while (1.1)–(1.3) are trivially satisfied in most cases. Moreover I_1 -I_3 are often «small», while I_4 can be «large» ([11]).

We will consider the following weighted L^2-spaces:

$$L^2(\mathfrak{B}) := L^2(\mathfrak{B},\Omega) := \{f\in L^2_{\mathrm{loc}}(\Omega) \mid \|f\|_n^2 = \int |f(z)|^2 e^{-2W_n(z)}dz < \infty \text{ for some } n\geq 1\}.$$

Let $L^2_{(0,k)}(\mathfrak{B},\Omega)$ be the k-forms in $d\bar{z}$ with coefficients in $L^2(\mathfrak{B},\Omega)$, that is, the set of

$$f=\sum_{J\in Q_k} f_J d\bar{z}_J,$$

where $f_J\in L^2(\mathfrak{B},\Omega)$ and $Q_k := \{J=(J_1,..,J_k) \mid 1\leq J_1 < .. < J_k \leq N\}$, endowed with the «norms» $\|f\|_n := \| |f| \|_n$, where $|f(z)|^2 := \sum |f_J(z)|^2$.

Let $H^2(\mathfrak{B}) := H^2(\mathfrak{B},\Omega)$ denote the holomorphic functions in $L^2(\mathfrak{B},\Omega)$.

We consider the $\bar\partial$-complex

$$0 \longrightarrow H^2(\mathfrak{B}) \longrightarrow \tilde{L}^2_{(0,0)}(\mathfrak{B}) \xrightarrow{\bar\partial=\bar\partial_0} \tilde{L}^2_{(0,1)}(\mathfrak{B}) \xrightarrow{\bar\partial_1} \cdots$$
$$\xrightarrow{\bar\partial_{N-1}} \tilde{L}^2_{(0,N)}(\mathfrak{B}) \xrightarrow{\bar\partial_N=0} 0 \qquad (1.5)$$

where $\bar\partial=(\partial/\partial\bar z_1,\ldots,\partial/\partial\bar z_N)$ is the Cauchy-Riemann system and

$$\tilde{L}^2_{(0,k)}(\mathfrak{B}) := \{f\in L^2_{(0,k)}(\mathfrak{B}) \mid \bar\partial_k f\in L^2_{(0,k+1)}(\mathfrak{B})\}$$

endowed with the «norms»

$$p_n(f) := \left(\|f\|_n^2 + \|\bar\partial_k f\|_n^2\right)^{1/2} \text{ for } 0\le k\le N.$$

Since we want to keep track of the several choices of seminorms and the use of conditions (1.1) – (1.4), the following convention is used to simplify the notation:

1.1. Convention. *We often delete the number n counting the seminorms (e.g. $I_k=I_k(n)$) and indicate compositions with the functions I_k only in the index (e.g. $I_2(I_3(n))=I_{23}$).*

For a locally bounded function f defined on Ω let f^* be its upper regularization, that is.

$$f^*(z) := \lim_{\eta\to z} \sup f(\eta)$$

As a first and simple consequence of (1.4) we notice:

1.2. Remark. *Let $\mathfrak{B}$ satisfy (1.1) – (1.4). Then there is a weight system $\tilde{\mathfrak{B}}$ which is equivalent to $\mathfrak{B}$ and consists of psh. functions. (1.5) is algebraically exact.*

Proof. a) Let $\tilde W_n := \left(\sup_{t\in\Omega} \Phi_t(z)+W_n(t)\right)^*$. $\tilde W_n$ exists by (1.4), is psh. ([14], Theorem 1.26) and dominates W_n again by (1.4). Using also (1.1) we get:

$$\tilde W_n(z)\le \sup\left\{\Phi_t(z+\xi)+W_n(t) \mid t\in\Omega,\ |\xi-z|_\infty\le r(z)\right\}$$
$$\le \sup\left\{W_{I_4}(z+\xi)+A(n) \mid |\xi-z|_\infty\le r(z)\right\}\le W_{I_{14}}(z)+A'(n)$$

b) The range of $\bar\partial_k$ is contained in $\tilde{L}^2_{(0,k+1)}(\mathfrak{B})$ since $\bar\partial_{k+1}\bar\partial_k=0$. $\bar\partial_k$ is surjective onto the kernel of $\bar\partial_{k+1}$ by Hörmander's solution of the weighted $\bar\partial$-problem ([7]), use also a) and (1.2)).

The weighted space $H^2(\mathfrak{B},\Omega)$ can always be given by a system of psh. weights. Since we will see in Theorem 1.7, that (1.4) is a necessary condition for the splitting of the $\bar\partial$-complex (1.5), Remark 1.2 implies, that in this case also $L^2(\mathfrak{B},\Omega)$ can be given by a system of psh. weights. For instance, if the weights in $\mathfrak{B}$ are radial, then a necessary condition for the splitting of (1.5) is, that $\mathfrak{B}$ is equivalent to a radial weight system of logarithmically convex functions. So we could assume without loss of generality, that $\mathfrak{B}$ consists of psh. functions, as far as the splitting problem is concerned. Of course, this is not the main feature of condition (1.4) (see the remarks below Theorem 1.3). For a psh. function ψ let $\mathfrak{B}_\psi:=\{W_n+\psi \mid n\geq 1\}$. The sufficiency part of the main result of this section is now contained in the following theorem:

1.3. Theorem. *Let $\mathfrak{B}$ satisfy (1.1)–(1.4) and let Ω be pseudoconvex. Let ψ be psh. Then (1.5) is split for $\mathfrak{B}_\psi$, more precisely, for $N\geq k\geq 0$ there are continuous linear projections π_k in $\tilde{L}^2_{(0,k)}(\mathfrak{B}_\psi,\Omega)$ onto*

$K_k:=\tilde{L}^2_{(0,k)}(\mathfrak{B}_\psi,\Omega)\cap\ker\bar\partial_k$ satisfying the following estimate

$$p_{F(n)}(\pi_k(f))=\|\pi_k(f)\|_{F(n)}\leq D_n\,p_n(f)\ \text{for}\ f\in\tilde{L}^2_{(0,k)}(\mathfrak{B}_\psi,\Omega), \tag{1.6}$$

where $F(n)\leq I_{214}JJI_3J$ and J is the $2(N+1)$-fold composition of I_1.

The proof of Theorem 1.3 will be obtained in several steps. First we will construct suitable cut off functions (in Lemma 1.4) such that the problem can be treated locally. Then by (1.4) we can define suitable weighted Hilbert spaces of square integrable functions, which locally serve as a simultaneous substitute for the (LF)-topology of $L^2(\mathfrak{B},\Omega)$, and where Hörmander's solution of the weighted $\bar\partial$-problem can be applied. The choice of these spaces is the main meaning of condition (1.4). The splitting problem is then solved locally by induction (in Lemma 1.5) and finally we show, that the local solutions converge and have the properties stated in Theorem 1.3.

Since the functions in $\mathfrak{B}$ are locally bounded, we know by (1.3), that

$$\text{for any } K\subset\subset\Omega \text{ there is } \varepsilon>0 \text{ such that } r(z)\geq\varepsilon \text{ for } z\in K. \tag{1.7}$$

Let $r_0(z):=r(z)$ and for $k\geq 1$ let r_k be defined by

$$r_k(z):=\inf\{r_{k-1}(\eta)\mid |\eta-z|_\infty\leq r(\eta) \text{ or } |\eta-z|_\infty\leq r(z)\}.$$

r_k is positive by (1.7).

1.4. Lemma. *There is a sequence $z_j \in \Omega$ such that the following holds:*

a) The balls

$$b_j := \{ \xi \mid |\xi - z_j|_\infty < r(z_j)/2 \}$$

are an open covering of Ω, and any $z \in \Omega$ is contained in at most $(8/r_2(z))^{2N}$ different balls

$$B_j := \{ \xi \mid |\xi - z_j|_\infty < r(z_j) \}.$$

b) The set $M_j := \{ m \mid B_m \cap B_j \neq \emptyset \}$ contains at most $(8/r_3(z))^{2N}$ elements.

c) There is a C^∞-resolution of the identity $\{ h_j \mid j \in \mathbb{N} \}$ subordinate to $\{ B_j \mid j \in \mathbb{N} \}$ such that

$$\| grad\, h_j \|_\infty \leq C(1/r_3(z_j))^{2N+1}$$

Proof. The proof is parallel to that of Lemma 1.4.9 in [8], however we do not use the slowly varying condition (1.4.5) in [8].

a) i) By (1.7) we can choose a maximal sequence $(z_j)_{j \in \mathbb{N}}$ such that

$$|z_j - z_k|_\infty \geq r_1(z_k)/2 \text{ for } k \neq j. \tag{1.8}$$

For any $z \in \Omega$ we have:

$$|z - z_k|_\infty < r_1(z_k)/2 \leq r(z_k)/2 \text{ or } |z - z_k|_\infty < r_1(z)/2 \leq r(z_k)/2$$

by the definition of r_1. So the balls b_j cover Ω.

ii) Let $z \in B_j \cap B_k$ for $k \neq j$ and let $\beta_j := \{ \xi \mid |\xi - z_j|_\infty < r_2(z)/4 \}$. The balls $\{ \beta_j \mid z \in B_j \}$ are contained in $\{ \xi \mid |\xi - z|_\infty \leq 2 \}$ and they are disjoint, since otherwise

$$r_2(z)/2 \leq r_1(z_j)/2 \leq |z_j - z_k|_\infty < r_2(z)/2.$$

So at most $(8/r_2(z))^{2N}$ different balls B_j can contain z.

b) As in a) ii) we see, that the balls $\tilde{\beta}_m := \{ \xi \mid |\xi - z_m|_\infty < r_3(z_j)/4 \}$ are disjoint for $m \in M_j$. So b) follows.

c) We choose $\varphi_j \in D(B_j)$ such that $0 \leq \varphi \leq 1, \varphi_j = 1$ on b_j and

$$\|\text{grad}\, \varphi_j\|_\infty \leq C/r(z_j).$$

Then $h_j := \varphi_j(1-\varphi_1)\dots(1-\varphi_{j-1})$ is a resolution of the identity as desired.

In each step of the induction proving Lemma 1.5 we will have to work with different psh. weights ψ_{jk}. These and the intermediate spaces E_{jk} and F_{jk} will now be introduced.

For $j\in\mathbb{N}$ let $V_{jN} := B_j$ and for $k<N$ let V_{jk} be the union of all sets $V_{m,k+1}$ such that $m\in M_j := \{m \mid B_m \cap B_j \neq \emptyset\}$.

For $N\geq k\geq 0$ let

$\Phi_{jk} := \left(\sup\{\Phi_t \mid t\in V_{jk}\}\right)^* + \psi$ and $\psi_{jk} := \Phi_{jk} + (2N-k+1)\ln(1+|z|^2)$ and

$$|f|_{jk}^2 := \int |f(z)|^2 e^{-2\psi_{jk}(z)}\, dz \text{ for } k\text{-forms } f.$$

ψ_{jk} is psh. by (1.1). For $m\in\mathbb{N}$ define

$$E_{mk} := \{f\in L^{2,\mathrm{loc}}_{(0,k)}(\Omega) \mid q_{mk}(f)^2 := |f(1+|\cdot|^2)^{-1}|^2_{m,k+1} + |\bar\partial_k f|^2_{m,k+1} < \infty\}$$

$$F_{mk} := \{f\in L^{2,\mathrm{loc}}_{(0,k+1)}(\Omega) \mid |f|_{m,k+1} < \infty \text{ and } \bar\partial_{k+1} f = 0\}$$

Then E_{mk} and F_{mk} are Hilbert spaces and $K_{mk} := E_{mk}\cap \ker\bar\partial_k$ is closed in E_{mk}, since $\bar\partial_k$ is continuous from E_{mk} into F_{mk}. Since ψ_{jk} is psh., we know from Hörmander's solution of the weighted $\bar\partial$-problem ([7]), that for any $g\in F_{mk}$ there is $f\in E_{mk}$ such that

$$\bar\partial_k f = g \text{ and } |f(1+|\cdot|^2)^{-1}|_{m,k+1} \leq |g|_{m,k+1}. \tag{1.9}$$

Let π_{mk} be the orthogonal projection in E_{mk} onto $K_{mk}^{\perp}$ and let

$$r_{mk}\colon F_{mk}\to E_{mk} \text{ be defined by } r_{mk}(g) := \pi_{mk}(f)$$

with f from (1.9). r_{mk} is well defined and linear.

$$\bar\partial_k r_{mk}(g) = \bar\partial_k \pi_{mk}(f) = \bar\partial_k f = g \tag{1.10}$$

Since π_{mk} is an orthogonal projection we obtain from (1.9)

$$|r_{mk}(g)(1+|\cdot|^2)^{-1}|_{m,k+1} \leq q_{mk}(\pi_{mk}(f)) \leq q_{mk}(f) \leq 2|g|_{m,k+1} \tag{1.11}$$

For $f\in\tilde L^2_{(0,k)}(\Omega)$ with compact support let $\pi_N(f) := f$ and for $k<N$ let

$$\pi_k(f) := f - \sum_m r_{mk}\left(\pi_{k+1}(h_m\bar\partial_k f)\right) \tag{1.12}$$

with h_m from Lemma 1.4. This definitions is justified in the following lemma:

1.5. Lemma. *(1.12) defines linear mappings π_k for $N \geq k \geq 0$ having the following properties for any $j \in \mathbb{N}$:*

$$\pi_k(\bar\partial_{k-1} f) = \bar\partial_{k-1} f \text{ for } f \in \tilde{L}^2_{(0,k-1)}(\Omega) \text{ with compact support.} \tag{1.13}$$

$$\pi_k(f) \in F_{j,k-1} \text{ for } f \in \tilde{L}^2_{(0,k)}(\Omega) \text{ with supp } f \subset B_j \text{ and} \tag{1.14}$$

$$|\pi_k(f)|_{jk} \leq C_k\, D_{jk}\, p^k(f) \tag{1.15}$$

with $D_{jk} := (1+|z_j|^2)^{-N-1} r_{l(k)}(z_j)^{-(4N+1)(N-k)}$, $l(k) := 2(N-k)+3$, *and*

$$p^k(f)^2 := \|fe^{-\psi}\|_2^2 + \|\bar\partial_k fe^{-\psi}\|_2^2.$$

Proof. i) For $k=N$ and $\pi_N = \mathrm{id}$ (1.13) and (1.14) are trivial, and (1.15) follows from the first part of (1.4), since $V_{jN} = B_j \supset \operatorname{supp} f$ and

$$\psi_{jN} \geq \psi + (N+1)\ln(1+|z_j|^2) + C \text{ on } B_j. \tag{1.16}$$

ii) Let $k<N$ and let Lemma 1.5 be proved for $k+1$. Then π_k is defined by (1.14). Fix $j \in \mathbb{N}$ and $f \in \tilde{L}^2_{(0,k)}(\Omega)$ with supp $f \subset B_j$. Then

$$h_m \bar\partial_k f = 0, \text{ if } m \notin M_j = \{m \mid B_m \cap B_j \neq \emptyset\}.$$

Since $\Phi_{jk} \geq \Phi_{m,k+1}$ for $m \in M_j$, we get by (1.11). Lemma 1.5 and Lemma 1.4b)

$$\begin{aligned} |\pi_k(f) - f|_{jk} &\leq \sum_{m\in M_j} |r_{mk}(\pi_{k+1}(h_m \bar\partial_k f))(1+|\cdot|^2)^{-1}|_{m,k+1} \\ &\leq 2 \sum_{m \in M_j} |\pi_{k+1}(h_m\, \bar\partial_k f)|_{m,k+1} \\ &\leq C_k r_3(z_j)^{-2N} \sup\{D_{m,k+1} \mid m \in M_j\} p^{k+1}(h_m\, \bar\partial_k f) \end{aligned} \tag{1.17}$$

By Lemma 1.4c) we have for $m \in M_j$

$$\begin{aligned} p^{k+1}(h_m\, \bar\partial_k f)^2 &\leq (1 + C_1\, \|\nabla h_m\|_\infty^2) \|\bar\partial_k fe^{-\psi}\|_2^2 \\ &\leq C_2 r_3(z_m)^{-4N-2} \|\bar\partial_k fe^{-\psi}\|_2^2. \end{aligned} \tag{1.18}$$

since $\bar\partial_{k+1}(h_m\, \bar\partial_k f) = \sum_i \partial/\partial\bar z_i h_m d\bar z_i \wedge \bar\partial_k f$. Also, we have for $m \in M_j$

$$D_{m,k+1} r_3(z_j)^{-2N}\, r_3(z_m)^{-2N-1} \leq C_k'\, D_{jk} \tag{1.19}$$

Combining (1.17) – (1.19) we get

$$|\pi_k(f) - f|_{jk} \leq C_k''\, D_{jk}\, \|\bar\partial_k fe^{-\psi}\|_2$$

The corresponding estimate for f follows from (1.16), since $\psi_{jk} \geq \psi_{jN}$.

So (1.15) is proved. (1.13) is obvious from the definition of π_k. (1.14) follows for k from (1.10) and (1.13) for $k+1$: For compactly supported $f \in \tilde{L}^2_{(0,k)}(\Omega)$ we have

$$\bar{\partial}_k (f - \pi_k(f)) = \Sigma\, \pi_{k+1}(h_m(\bar{\partial}_k f)) = \pi_{k+1}(\bar{\partial}_k f) = \bar{\partial}_k f \tag{1.20}$$

Until now, we essentially only have used (1.7) and the first part of condition (1.4). The connection of Φ_t and $\mathfrak{B}$ as stated in the second part of (1.4) is now used to complete the

Proof of Theorem 1.3: Let $\pi_k(f)$ be defined by (1.12) for $f \in \tilde{L}^2_{(0,k)}(\mathfrak{B}_\psi, \Omega)$.

i) The theorem obviously holds for $k = N$ and $\pi_N = \mathrm{id}$.

ii) π_k satisfies (1.6) for $k < N$.

Proof. Let $L(n)$ be the $[2(N-k)-1]$-fold composition of I_1. By the definition of $\psi_{j,k+1}$ and (1.1)–(1.4) we obtain for any n

$$-W_{I_{214}L}(z) - \psi(z) \leq -\psi_{j,k+1}(z) - \inf\{W_L(z) \mid z \in V_{j,k+1}\} - ln(1+|z|^2) + D_n$$
$$\leq -\psi_{j,k+1}(z) - W_n(z_j) - ln(1+|z|^2) + D_n.$$

Here and in the following D_n changes from line to line. (1.11), Lemma 1.5 and (1.18) now imply:

$$\begin{aligned}\|\pi_k(f) - f\|_{I_{214}L} &\leq D_n \sum_j |r_{jk}(\pi_{k+1}(h_j \bar{\partial}_k f))\,(1+|\cdot|^2)^{-1}|_{j,k+1}\, e^{-W_n(z_j)} \\ &\leq D_n \sum_j |\pi_{k+1}(h_j \bar{\partial}_k f)|_{j,k+1}\, e^{-W_n(z_j)} \\ &\leq D_n \sum_j D_{j,k+1}\, r_3(z_j)^{-2N-1}\, e^{-W_n(z_j)}\, \|\bar{\partial}_k f e^{-\psi}\|_{L_2(B_j)}.\end{aligned} \tag{1.21}$$

Let K be the $(2N+1)$-fold composition of I_1. Then (1.3) implies for $z \in B_j$:

$$W_n(z) + 4(N+1)N\, ln(1/r_{2N+1}(z_j)) \leq W_{KI_3KI_1}(z_j) + D_n$$

From (1.21) we now get for $F(n) = I_{214}JJI_3J$:

$$\|\pi_k(f) - f\|_{F(n)}$$

$$\le D_n \sum_j (1+|z_j|^2)^{-N-1} r_3(z_j)^{4N} \exp\left(-W_{KI_3KI_1}(z_j)\right) r_{2N+1}(z_j)^{-4(N+1)N}$$
$$\times \|\bar\partial_k f e^{-\psi}\|_{L_2(B_j)} \le D_n \sum_j (1+|z_j|^2)^{-N-1} r_3(z_j)^{4N} p_n(f).$$

This proves (1.6), since by Lemma 1.4a)

$$\sum_j r_3(z_j)^{4N} (1+|z_j|^2)^{-N-1} \le C\int \sum_j r_3(z_j)^{2N} \chi_{B_j}(z)\,(1+|z|^2)^{-N-1} dz$$
$$\le C'\int \max\{r_3(z_j)^{2N} \mid z\in B_j\} r_2(z)^{-2N}(1+|z|^2)^{-N-1} dz$$
$$\le C'\int (1+|z|^2)^{-N-1} dz < \infty.$$

iii) Since π_k is continuous in $\tilde L^2_{(0,k)}(\mathfrak{B}_\psi)$ by ii), the equation $\bar\partial_k \pi_k = 0$ follows from (1.20), since the compactly supported k-forms are dense in $\tilde L^2_{(0,k)}(\mathfrak{B}_\psi)$. Since also $\pi_k = \mathrm{id}$ on K_k, π_k is a projection onto K_k and this completes the proof of Theorem 1.3.

To prove that (1.4) is also necessary for the splitting of the $\bar\partial$-complex we will use the Koszul-complex ([6, 9]). Let $L_{0k} := L^2_{(0,k)}(\mathfrak{B})$ and for $s\in\mathbb{N}$ and $k\in\mathbb{N}_0$ let L_{sk} denote the set of all skew symmetric mappings from

$\Gamma_s := \{I = (i_1, .., i_s) \mid 1 \le i_j \le N\}$ into $L^2_{(0,k)}(\mathfrak{B})$. L_{sk} is topologized by the 'norms'

$$\||f|\|_n := \|\,|f|\,\|_n \text{ with } |f|^2 := \Sigma |f_I|^2, \text{ where the sum runs over } I\in\Gamma_s.$$

$\bar\partial_k$ acts componentwise on the elements of L_{sk}. For $G := (G_1 .., G_m) \in H(\Omega)^m$ and $f\in L_{s+1,k}$ let $P_G(f)$ be the interior product of f wich G, that is,

$$(P_G(f))_I := \sum_{i\le m} G_i f_{(I,i)} \text{ for all } I\in\Gamma_s.$$

Let also $P_G(f) := 0$ for $f\in L_{0k}$.

For $0\le k\le N$ and $0\le s\le m-1$ let $\delta_{sk} := \min\{2(N-k)+1,\ 2(m-s)-1\}$ and $\gamma_{sk} := \min\{N-k,\ m-1-s\}$. With $G_-(z) := \min\{1, |G(z)|\}$ and $|DG(z1)| := \left(\sum_{i,j} |\partial/\partial z_i\, G_j(z)|\right)^{1/2}$ we define

$$L^G_{sk} := \{f\in L_{sk} \mid \bar\partial_k f = 0,\ P_G(f) = 0,$$
$$\int |f(z)|^2 e^{-2W_n(z)} G_-(z)^{-2\delta_{sk}} (1+|DG(z)|)^{2\gamma_{sk}} dz < \infty \text{ for some } n\in\mathbb{N}\}.$$

$G\in H(\Omega)^m$ is called a multiplier in $H^2(\mathfrak{B}, \Omega)$, if the following holds:

For any $n \geq 1$ there are $I_5(n) \geq n$ and $A_5(n) \geq 1$ such that

$$W_n(z) + ln|G(z)| \leq W_{I_5(n)}(z) + A_5(n) \tag{1.22}$$

The following lemma is a linearized variant of Theorem 7 in [6] and Theorem 2.6 in [9], which is stated in the generality needed in this paper.

1.6. Lemma. *Suppose that $\mathfrak{B}$ satisfies (1.1)–(1.3) and that $\bar\partial_k$ has a continuous linear right inverse R_k in the complex (1.5) for $0 \leq k \leq N$, which satisfies*
$\| R_k(f) \|_{F(n)} \leq CB(n) \|f\|_n$ for $n \geq 1$ and $f \in \tilde{L}^2_{(0,k+1)}(\mathfrak{B}) \cap \ker \bar\partial_{k+1} =: K_{k+1}$. Let $G \in H(\Omega)^m$ be a multiplier in $H^2(\mathfrak{B}, \Omega)$. Then for $0 \leq s \leq m-1$ and $0 \leq k \leq N$ there are continuous linear mappings

$$T^G_{sk}\colon L^G_{sk} \to L_{s+1,k} \text{ such that } P_G\, T^G_{sk} = id \text{ and } \bar\partial_k\, T^G_{sk} = 0$$

and such that the following estimate holds for $n \geq 1$:

$$\| T^G_{sk}(f) \|_{J(n)} \leq C_1 A(n) \Big(\int |f(z)|^2\, G_-(z)^{-2\delta_{sk}} (1 + |DG(z)|)^{2\gamma_{sk}}\, e^{-2W_n}(z)\, dz\Big)^{1/2}, \tag{1.23}$$

where J is the γ_{sk}-fold composition of $I_5 F$.

Proof. Fix $k \leq N$ and $s \leq m=1$. Let $f \in L^G_{sk}$.

i) Let $H^G_{sk}(f)$ be the exterior product of f by $\bar{G}|G|^{-2}$, that is,

$$H^G_{sk}(f)_I := \sum_{j \leq s+1} (-1)^{s+1-j}\, \bar{G}_{i_j} |G|^{-2} f_{I_j},$$

where $I \in \Gamma_{s+1}$ and I_j is obtained from I by deleting i_j. Since $\delta_{sk} \geq 1$, we have

$$\| H^G_{sk}(f) \|_{J(n)} \leq C_1 \|f\|_{n,G} \text{ and } P_G\, H^G_{sk}(f) = f, \tag{1.24}$$

where $\| \ \|_{n,G}$ denotes the norms in L^G_{sk}.

If $k = N$ or $s = m-1$, then $\bar\partial H^G_{sk} = 0$. For $s = m-1$ this follows from the injectivity of P_G and (1.25) below. So the lemma is proved in this case (with $J(n) = n$).

ii) Let $k < N$ and $s < m-1$. We use the ansatz

$$T^G_{sk}(f) := H^G_{sk}(f) - P_G\, R_k\, T^G_{s+1,k+1}\, \bar\partial H^G_{sk}(f),$$

where R_k is a right inverse for $\bar\partial$ from K_{k+1} into $\tilde{L}^2_{(0,k)}(\mathfrak{B})$, existing by assumption. Since $\bar\partial T^G_{s+1,k+1}=0$ by the induction hypothesis, we only have to show, that

$$\bar\partial H^G_{sk}(f)\in L^G_{s+1,k+1}.$$

We obviously have the following:

$$\bar\partial(\bar\partial H^G_{sk}(f))=0 \text{ and } 0=\bar\partial f=\bar\partial(P_G H^G_{sk}(f))=P_G(\bar\partial H^G_{sk}(f)). \tag{1.25}$$

To estimate $\bar\partial H^G_{sk}(f)$, we use formula (2.1) of [9], which holds for forms g with $\bar\partial g=0$ and $g|G|^{-2}\in L^2_{\text{loc}}(\Omega)$, namely

$$\bar\partial(|G|^{-2}\overline{G}_i g)=|G|^{-4}\sum_{j\leq m} G_j\big(\overline{G}_j\,\bar\partial\overline{G}_i - \overline{G}_i\,\bar\partial\overline{G}_j\big)\wedge g$$

Let J, δ and γ be chosen for $(s+1,k+1)$.

$$\|T^G_{sk}(f)-H^G_{sk}(f)\|_{I_5FJ}\leq D_n\|R_k\,T^G_{s+1,k+1}\,\bar\partial H^G_{sk}(f)\|_{FJ}$$

$$\leq D_n\|\,T^G_{s+1,k+1}\,\bar\partial H^G_{sk}(f)\|_J$$

$$\leq D_n\Big(\int|\bar\partial H^G_{sk}(f)(z)|^2\,G_-(z)^{-2\delta}(1+|DG(z)|)^{2\gamma}\,e^{-2W_n(z)}dz\Big)^{1/2}$$

$$\leq D_n\Big(\int|f(z)|^2\,G_-(z)^{-2(\delta+2)}(1+|DG(z)|)^{2(\gamma+1)}\,dz\Big)^{1/2}$$

D_n is changing from line to line. Together with (1.24) this shows the desired continuity estimate.

$$\bar\partial T^G_{sk}=\bar\partial H^G_{sk}-P_G\,\bar\partial R_k\,T^G_{s+1,k+1}\,\bar\partial H^G_{sk}=\bar\partial H^G_{sk}-P_G\,T^G_{s+1,k+1}\,\bar\partial H^G_{sk}=0$$

$$P_G\,T^G_{sk}=P_G\,H^G_{sk}=\text{id}$$

by (1.24). The lemma is proved.

We will apply Lemma 1.6 in this section to the functions

$$G_a(z):=(z_1-a_1,\dots,z_N-a_N) \text{ for } a\in\Omega$$

(see section 2 for further applications). Here the dependence of I_5, A_5 and $A(n)$ on the parameter a is important. We have $m=N$ and $I_5=I_2$ and

$A_5(n)=A_2(n)(1+|a|)$ and $|DG_a(z)|=C$. We therefore have for the constant in Lemma 1.6

$$A(n)=A_n(1+|a|)^{\gamma_{sk}} \qquad \text{for some } A_n. \tag{1.26}$$

1.7. Theorem. *Let $\mathfrak{B}$ satisfy (1.1)—(1.3). The following are equivalent:*

i) *$\mathfrak{B}$ satisfies (1.4).*

ii) *$\mathfrak{B}$ satisfies (1.4) with $\Phi_t:=ln|g_t|$ for some $g_t\in H(\Omega)$.*

iii) *The ∂-complex (1.5) is exact and split.*

iv) *The $\bar\partial$-complex on $L^2(\mathfrak{B}_\psi,\Omega)$ is exact and split for any psh. function ψ on Ω.*

Proof. «i)$\Rightarrow$iv)» This follows from Remark 1.2 and Theorem 1.3.

«iv) $\Rightarrow$iii)» and «ii)$\Rightarrow$i)» These implications are trivial.

«iii) $\Rightarrow$ii)» Since the complex (1.5) is exact and split, there is a continuous linear right inverse R_k for $\bar\partial_k$ in (1.5). So Lemma 1.6 is applicable. For $a\in\Omega$ choose $\varphi\in D(B_a)$ such that

$$0\leq\varphi\leq 1, \varphi(z)=1 \text{ for } |z-a|_\infty\leq r(a)/2 \text{ and } \|\nabla\varphi\|_\infty\leq C_1/r(a).$$

Let R_0 be a right inverse for the Cauchy-Riemann operator and define

$$\varphi(z)-\sum_{j\leq N}(z_j-a_j)\,R_0\,(T^a_{0,1})_j\,(\bar\partial\varphi)(z)=:\,g_a(z)=:\varphi(z)-h_a(z)$$

where $T^a_{0,1}=T^G_{0,1}$ is taken from Lemma 1.6 for $G=G_a$.

$$\begin{aligned}\bar\partial g_a(z)&=\bar\partial\varphi(z)-\sum_j(z_j-a_j)\;\bar\partial R_0(T^a_{0,1})_j(\bar\partial\varphi)(z)\\ &=\bar\partial\varphi(z)-P_{G_a}T^a_{0,1}(\bar\partial\varphi)(z)=0\end{aligned}$$

So g_a is holomorphic on Ω and obviously $g_a(a)=1$. The mean value property of g_a and (1.1)—(1.3) imply that

$$\begin{aligned}|g_a(z)|&\leq C_1\,r(z)^{-N}\Big(\int_{|\eta|\leq r(z)}|g_a(z+\eta)|^2\,d\eta\Big)^{1/2}\\ &\leq C_1\,A_n\,\|g_a(1+|\cdot|^2)^{-N}\|_{\tilde H(n)}\exp\big(W_{H(n)}(z)\big)\end{aligned} \tag{1.27}$$

with $\tilde{H}=FJI_{123}$ and $H=I_{123}FJI_{123}$, where J is the $(N-1)$-fold composition of I_2F by Lemma 1.6. By (1.26) and again by (1.1)−(1.3) we obtain

$$\|\varphi(1+|\cdot|^2)^{-N}\|_{\tilde{H}(n)}\leq A_n e^{-W_n(a)}$$

$$\|h_a(1+|\cdot|^2)^{-1}\|_{\tilde{H}(n)}\leq C_1(1+|a|)\sup_{j\leq N}\|R_0(T^a_{0,1})_j(\bar\partial\varphi)\|_{\tilde{H}(n)}$$

$$\leq A_n(1+|a|)\;\|T^a_{0,1}(\bar\partial\varphi)\|_{JI_{123}}$$

$$\leq A_n(1+|a|)^N\big(\textstyle\int|\bar\partial\varphi(z)|^2|z-a|^{-2(2N-1)}\exp(-2W_{I_{123}}(z))\,dz\big)^{1/2}$$

$$\leq A_n(1+|a|)^N\,r(a)^{-2N}\big(\textstyle\int_{\operatorname{supp}\bar\partial\varphi}\exp(-W_{I_{123}}(z))\,dz\big)^{1/2}$$

$$\leq A_n(1+|a|)^N\,r(a)^{-N}\exp(-W_{I_{23}}(a))\leq B_n\exp(-W_n(a))$$

Here A_n may change in the inequalities. Together with (1.27) this proves ii) with $I_4=H$.

The proofs of Theorem 1.3 and 1.7 also show, that the continuity estimates obtained in Theorem 1.3 cannot be improved in general:

1.8. Remark. *Let $\mathfrak{B}$ consist of psh. functions and satisfy (1.1)−(1.3). If $\mathfrak{B}$ satisfies (1.4), then we get from Theorem 1.3 right inverses R_k for $\bar\partial_k$ with module of continuity $\tilde{F}(n)=F(I_2(n))$, which is essentially $I_4(n)$ (modulo finite compositions with the «small» functions $I_1,\dots,I_3$). Conversely, if right inverses R_k for $\bar\partial_k$ exist with module of continuity F, then (1.4) holds with the N-fold composition of $\tilde{F}$ taken as I_4 (again modulo $I_1,\dots,I_3$). So the estimates are optimal in the case of one variable, for radial weight systems $\mathfrak{B}$ and also in the case of several variables, if $\mathfrak{B}$ consists of positive functions of the form*

$$W_n(z)=\Sigma\, W_{i,n}(z_i)$$

and Ω is the product of open sets in $\mathbb{C}$, since these cases can easily be reduced to the first case.

The splitting of the $\bar\partial$-complex (including continuity estimates) in weighted spaces of distributions as considered by *R.* Meise and B. A. Taylor [17] can be obtained from the splitting of the $\bar\partial$-complex in weighted L^2-spaces by the homotopy argument in the proof of Proposition 1.9 in [17] (for $\Omega=\mathbb{C}^N$ and shift invariant spaces). Also the results of S. Momm ([19-21]) for the Cauchy-Riemann operator on the polydisc and the former results of the author ([10]) can be obtained and improved from Theorems 1.3 and 1.5 (see [11]).

In spite of Theorem 1.7 the complex (1.5) may change from nonsplitting to splitting by adding a psh. function to the weight system: Let $\mathfrak{B}:=\{n\, ln(1+|z|^2)\,|\, n\geq 1\}$. Then $\mathfrak{B}$ satisfies (1.1)$-$(1.3) for $r=1$ on $\Omega=\mathbb{C}$. (1.5) is not split in this case. Indeed, if (1.5) splits, the dual space $H^2(\mathfrak{B},\mathbb{C})'_b$ has a continuous norm, since it is a (complemented) subspace of $L^2(\mathfrak{B},\mathbb{C})'_b$, which obviously has a continuous norm. But $H^2(\mathfrak{B},\mathbb{C})$ is the space of all polynomials for the above choice of $\mathfrak{B}$ and the dual is isomorphic to the space ω of all sequences, which certainly has no continuous norm. On the other hand take $\mathfrak{B}_\psi:=\{n\, ln(1+|z|^2)+(ln(1+|z|^2))^2\,|\, n\geq 1\}$. Then $\mathfrak{B}_\psi$ satisfies (1.1)$-$(1.3) for $r=1$ on $\Omega=\mathbb{C}$ and $\mathfrak{B}_\psi$ is equivalent to $\tilde{\mathfrak{B}}:=\{(ln(1+|nz|^2))\,|\, n\geq 1\}$ and (1.5) splits for $\tilde{\mathfrak{B}}$ (see [11]).

In the context of the implicit function theorem of Nash and Moser it is important to know, whether there is a tame or linear tame splitting for (1.5). Here a mapping T is called tame (resp. linearly tame) if the following holds: There is c such that for any $n\geq 1$

$$\|T(f)\|_{s(n)}\leq C_n\,\|f\|_n \text{ with } s(n)=n+c \text{ (and } s(n)=cn+c. \text{ resp.).}$$

So a module of continuity is $(n+c)$ (and $(cn+c)$, resp.). From the above remarks it is clear, that Theorem 1.7 also holds in the tame (resp. linear tame category).

1.9. Corollary. *Let $\mathfrak{B}$ satisfy (1.1)$-$(1.3) with a tame (resp. linear tame) choice of I_j. The following are equivalent:*

i) $\mathfrak{B}$ satisfies (1.4) with a (linear) tame choice of I_4.

ii) $\mathfrak{B}$ satisfies (1.4) with a (linear) tame choice of I_4 and $\Phi_t=ln|g_t|$ with $g_t\in H(\Omega)$.

iii) The $\bar\partial$-complex (1.5) is (linear) tamely exact and splits (linear) tamely.

iv) The $\bar\partial$-complex on $L^2(\mathfrak{B}_\psi,\Omega)$ is (linear) tamely exact and splits (linear) tamely for any psh. function ψ on Ω.

1.9iii) means, that there are (linear) tame projections onto ker $\bar\partial_k$ and (linear) tame right inverses for $\bar\partial_k$ for any k.

We finally notice a variant of Lemma 1.6, which uses Theorem 1.3 to improve on the choice of J in 1.6. Let $G_+(z) := \max\{1, |G(z)|\}$ and define

$$\tilde L_{sk} := \{ f \in L_{sk} \mid \int |f(z)|^2 G_+(z)^{-2\gamma_{sk}} e^{-2W_n(z)} dz < \infty \text{ for some } n \geq 1 \}.$$

Recall that $\gamma_{sk} := \min\{N-k, m-s-1\}$ and $\delta_{sk} := \min\{2(N-k)+1, 2(m-s)-1\}$.

1.10. Remark. *Let $\mathfrak{B}$ consists of psh. functions and satisfy (1.1)–(1.4). Let $G \in H(\Omega)^m$ be a multiplier in $H^2(\mathfrak{B}, \Omega)$. Then for $0 \leq k \leq N$ and $0 \leq s \leq m-1$ there are linear mappings.*

$$T^G_{sk} : L^G_{sk} \to \tilde L_{s+1,k} \text{ such that } P_G T^G_{sk} = id \text{ and } \bar\partial_k T^G_{sk} = 0$$

and such that the following estimate holds for $n \geq 1$:

$$\left(\int |T^G_{sk}(f)(z)|^2 G_+(z)^{-2\gamma_{sk}} e^{-2W_{J(n)}(z)} (1+|z|^2)^{-2\gamma_{sk}} dz\right)^{1/2}$$

$$\leq CA(n)\left(\int |f(z)|^2 G_-(z)^{-2\delta_{sk}} (1+|DG(z)|^{2\gamma_{sk}} e^{-2W_n(z)} dz\right)^{1/2},$$

where F is the γ_{sk}-fold composition of F from Theorem 1.3.

Proof. We formally use the same ensatz as in Lemma 1.6 with a different choice of R_k. The cases $k=N$ and $s=m-1$ are treated as in the proof of 1.6. In part ii) we then estimate P_G by G_+ and define a right inverse R_k for $\bar\partial_k$ on ker $\bar\partial_{k+1} \cap \tilde L^2_{(0,k+1)}(\mathfrak{B} + \gamma \ln[G_+((z)(1+|z|^2)], \Omega)$ with $\gamma = \gamma_{sk}$ as follows:

$$R_k(f) := g - \pi_k(g),$$

where π_k is a projection in $\tilde L^2_{(0,k)}(\mathfrak{B} + (\gamma-1) \ln G_+ + \gamma \ln(1+|\cdot|^2))$ onto ker $\bar\partial_k$ existing by Theorem 1.3. g is chosen by Hörmander's theorem ([7]) such that $\bar\partial_k g = f$ and such that

$$\int |g(z)|^2 (1+|z|^2)^{-2\gamma} G_+(z)^{-2\gamma-2} e^{-2W_n(z)} dz$$

$$\leq \int |g(z)|^2 (1+|z|^2)^{-2\gamma+2} G_+(z)^{-2\gamma-2} e^{-2W_n(z)} dz$$

Notice that the module of continuity for π_k is independent of γ by the proof of Theorem 1.3. Using this choice of R_k the desired estimates now follow similar as in the proof of Lemma 1.6.

The case of weighted (F)-spaces $L^2(\mathfrak{B}, \Omega)$ defined by a decreasing system $\mathfrak{B} := \{ V_n \mid n \geq 1 \}$ can be trated as before. We assume, that $\mathfrak{B}$ satisfies the variant of $(1.1)-(1.3)$ for decreasing weight systems and obtain:

1.11. Remark. *Let $\mathfrak{B}$ be a decreasing weight system as above. Then the $\bar{\partial}$-complex (1.5) is split for $L^2(\mathfrak{B}, \Omega)$ if and only if the following holds: For any $t \in \Omega$ there are psh. functions ψ_t and for any $n \geq 1$ there are $I(n) \geq n$ and $A(n) \geq 0$ such that for any $n \geq 1$ and any $z, t \in \Omega$:*

$$\psi_t(t) \geq 0 \text{ and } \psi_t(z) \leq V_n(z) - V_{I(n)}(t) + A(n).$$

2. INTERPOLATION IN WEIGHTED SPACES OF HOLOMORPHIC FUNCTIONS

In this section we will use the results of the first section to linearize the solution of some problems in weighted spaces of holomorphic functions. This extends and improves on results of L. Hörmander [6]. C. A. Berenstein and B. A. Taylor [1-3], R. Meise and B. A. Taylor [17], G. Marino, P. Pietramala and *D*, Struppa [15] and B. A. Taylor [23].

The first result deals with the generation of an algebra $H^2(\mathfrak{B}, \Omega)$ and is just a reformulation of Remark 1.10 in a special case. Let $\mathfrak{B}$ satisfy the following condition: $\mathfrak{B}$ consists of positive functions and for any $k \geq 1$ and $n \geq 1$ there are $I(k, n)$ and $A(k, n)$ such that

$$(3N+1) W_k(z) + W_n(z) \leq W_{I(k,n)}(z) + A(k,n) \tag{2.1}$$

(2.1) implies that $H^2(\mathfrak{B}, \Omega)$ is an algebra. The choice of the constant before W_k comes from the proof of the following theorem:

2.1. Theorem, *Let $\mathfrak{B}$ satisfy $(1.1)-(1.4)$ and (2.1) on a pseudoconvex open set $\Omega \subset \mathbb{C}^N$. The following are equivalent:*

i) *$G = (G_1, \ldots, G_m) \in H^2(\mathfrak{B}, \Omega)$ generates $H^2(\mathfrak{B}, \Omega)$ as an algebra.*

ii) *There is $k \geq 1$ such that*

$$|G(z)| \geq C\, e^{-W_k(z)} \text{ for any } z \in \Omega.$$

iii) *There are continuous linear opertors T_j in $H^2(\mathfrak{B},\Omega)$ such that*

$$f = \Sigma\, T_j(f)\, G_j \text{ for any } f \in H^2(\mathfrak{B},\Omega).$$

Proof. «i)$\Longrightarrow$ii)» Take $g_i \in H^2(\mathfrak{B},\Omega)$ such that $\Sigma\, G_j g_j = 1$.

«ii)$\Longrightarrow$iii)» We may assume that $\mathfrak{B}$ consists of psh. functions. Let $T_j := (T^G_{0,0})_j$ with $T^G_{0,0}$ from Remark 1.10. There are C and κ such that

$$|G(z)| \leq C e^{W_\kappa}(z) \text{ for any } z \in \Omega.$$

This implies that

$$|DG(z)| \leq C_1 \exp\left(W_{I_{31}(\kappa)}(z)\right)$$

and T_j satisfies

$$\| T_j(f)\|_{H(n)} \leq A_n \|f\|_n$$

where $H(n) = I_2\left(I\{k_1,\ J(I\{I_{31}(k_1),\ n\})\}\right)$, $k_1 := \max\{k,\ \kappa\}$ and J is the $\min\{N,\ m-1\}$-fold composition of F from Theorem 1.3.

The equivalence of 2.1.i) and ii) for systems $\mathfrak{B} = \{nW \mid n \geq 1\}$ is due to Hörmander [6]. More general weight systems were recently considered by G. Marino, P. Pietramala and D. Struppa [15] for one variable. The assumptions of Hörmander [6] mean in our notation (and for the general weight systems considered here), that we could take $r(z) = \exp\left(-W_k(z) - C\right)$ for some k and C. Following C. Berenstein and B. A. Taylor [2.3] we now consider extension problems for holomorphic functions defined on strongly interpolating submanifolds of Ω.

2.2. Definition. *Let $\Omega \subset \mathbb{C}^N$ be pseudoconvex and let V be a complex submanifold of Ω of complex dimension p. Let $\mathfrak{B}$ satisfy (1.1)–(1.3). V is called strongly interpolating for $\mathfrak{B}$, if there are a psh. positive function κ on Ω and $G = (G_1, \dots, G_m) \in H(\Omega)^m$ such that the following holds:*

i) *For any $n \geq 1$ there are $I_j(n)$ and $A_j(n)$ such that*

$$N\left(\ln(1+|z|^2) + \kappa(z)\right) + W_n(z) \leq W_{I_5(n)}(z) + A_5(n) \text{ for any } z \in \Omega \quad (2.2)$$

$$8(N+1)^3\left(\ln(1/r(z)) + \kappa(z)\right) + 2(N+1)^2 \ln(1+|z|^2) + W_n(z)$$
$$\leq W_{I_6(n)}(z) + A_6(n) \qquad \text{on } V \quad (2.3)$$

$$sup\{\kappa(z+\xi) \mid |\xi|_\infty < r(z)\} \leq C_l + \kappa(z) \tag{2.4}$$

ii) $V = \{z \in \Omega \mid G(z) = 0\}$ *and*

$$\Delta(z) := (\Sigma |\Delta_{L,M}(z)|^2)^{1/2} \geq e^{-\kappa(z)} \text{ on } V \tag{2.5}$$

$$|G(z)| \leq e^{\kappa(z)} \text{ for any } z \in \Omega. \tag{2.6}$$

The sum in (2.5) runs over all determinants $\Delta_{L,M}$ *of the* $(N-p)\times(N-p)$ *submatrices of* $DG(z) = (\partial G_j(z)/\partial z_i)_{i,j}$.

The constants in (2.2) and (2.3) are chosen uniformly for p and m to fit into Theorem 2.3 below. For small m and large p they can be improved. Of course (2.2), (2.3), (1.2) and (1.3) are not independent.

We do not assume that the underlying space $H^2(\mathfrak{B}, \Omega)$ is an algebra. In the case of algebras $H^2(nW, \mathbb{C}^N)$ the definition coincides with the definition of C. Berenstein and B. A. Taylor [2,3], if W is psh. However the assumptions

$$r = 1 \text{ and } W(z+\xi) \leq CW(z) + C \text{ for } |\xi| \leq 1$$

are used in [2.3] instead of (2.4) and (1.1)–(1.3). Let

$$H(\mathfrak{B}, V) := \{f \in H(V) \mid \|f\|_{V,n} := \sup\{|f(z)|e^{-W_n(z)} \mid z \in V\} < \infty \text{ for some } n \geq 1\}.$$

2.3. Theorem. *Let* $\mathfrak{B}$ *satisfy (1.1)–(1.4). Let* $ln(1/r(z))$ *be psh. and let*

$$r(z) \leq Cr(z+\xi) \text{ for } |\xi|_\infty \leq r(z). \tag{2.7}$$

Let V *be strongly interpolating for* $\mathfrak{B}$. *Then there is a continuous linear operator* $E\colon H(\mathfrak{B}, V) \to H^2(\mathfrak{B}, \Omega)$ *such that* $E(f)|_V = f$ *for any* $f \in H(\mathfrak{B}, V)$ *and*

$$\|E(f)\|_{H(n)} \leq CB_n \|f\|_{V,n}$$

with $H(n) \leq I_5(J(I_6(n)))$, *where* J *is the min* $\{N, m\}$*-fold composition of* F *from Theorem 1.3.*

Proof. Since $ln(1/r(z))$ is psh. and satisfies (2.7), we can use Remarque 6, p. 99 of Demailly [4] with $\varphi_2 = \varphi_3 = \kappa$ and $\chi = ln(A_0/r(z))$. The conditions (57)–(59) in [4] are then satisfied and we obtain from Théorème 5 in [4] a holomorphic retraction

$$\pi\colon U := \{z \in \Omega \mid |G(z)| < C_1/\psi(z)\} \to V$$

such that

$$|\pi(\xi)-\xi|\leq r(\xi)/A_0 \text{ on any component } U' \text{ of } U \text{ with } U'\cap V\neq\emptyset. \quad (2.8)$$

Here

$$\psi(z)=\exp\left(4(N+1)^2\left[\kappa(z)+\ln(A_0/r(z))\right]+(N+1)\,ln(1+|z|^2)\right).$$

We now use the idea of the proof of Theorem 2.2 in [2]. Let

$$b_\xi=\left\{z\in\Omega \mid |z-\xi|_\infty\leq 1/(C_2\psi_l(\xi))\right\} \text{for } \psi_l:=\psi e^{\kappa+ln(A_0/r)}$$

and let U_1 be the union of all components U_1' of

$\left\{z\in\Omega \mid |G(z)|<C_1/(C_2\psi(z))\right\}$ such that $U_1'\cap V\neq\emptyset$. Since

$$|DG(z)|\leq C_3 e^{\kappa(z)+ln(1/r(z))} \text{ for any } z\in\Omega,$$

we can choose C_2 so large that

$$\left\{b_\xi \mid b_\xi\cap U_1\neq\emptyset\right\}\subset U. \quad (2.9)$$

$|\ |_z:=|\ |_2\psi_1(z)$ is a slowly varying metric ([7]. Definition 1.4.7) by (2.4) and (2.7). Therefore are $\varepsilon>0$ and a cut off function $\chi\in C^\infty(U)$ such that

$$\chi(z)=1 \text{ for any } z\in U_1 \text{ and } \operatorname{supp}\chi\subset U \quad (2.10)$$

$$|\bar{\partial}\chi(z)|\leq C_4\psi_1(z) \quad (2.11)$$

G is a multiplier in $H^2(\mathfrak{B},\Omega)$ by (2.2) and (2.6). Let $T^G_{0,1}:=\left((T^G_{0,1})_j\right)_{j\leq m}$ be chosen from Remark 1.10 and set

$$E(f):=\chi(f\circ\pi)-\Sigma\, G_j R(T_{0,1})_j\left(\bar{\partial}\chi(f\circ\pi)\right).$$

where R is a right inverse for the Cauchy-Riemann system on $L^2(\mathfrak{B}+\gamma_{01}[ln\, G_+ + ln(1+|z|^2)],\Omega)$, existing by Theorem 1.3. We obviously have

$$\bar{\partial}E=0 \text{ and } E(f)|_V=f \text{ for any } f\in H(\mathfrak{B},V)$$

The continuity estimate follows easily from (2.8)−(2.11) and Remark 1.10.

For systems $\mathfrak{B}=\left\{nW \mid n\geq 1\right\}$ Theorem 1.3 was proved by R. Meise and B. A. Taylor ([17], Theorem 2.2) without explicit continuity estimates (and for $\Omega=\mathbb{C}^N$ and $r=1$).

To prove Theorem 2.3 it is sufficient to know (2.2) and (2.3) for $\mathfrak{W}$ and (1.1)–(1.4) for some weight system $\tilde{\mathfrak{W}}$ with some radius function $\tilde{r}$ such that

$$\tilde{\mathfrak{W}} \leq \mathfrak{W} \text{ and } \tilde{r} \leq r \text{ on } \Omega, \text{ while } \mathfrak{W} \geq \tilde{\mathfrak{W}} \text{ on } V.$$

Here $\tilde{\mathfrak{W}} \leq \mathfrak{W}$ means as usual, that any $\tilde{W} \in \tilde{\mathfrak{W}}$ is bounded by $C + W$ for some $C \geq 0$ and some $W \in \mathfrak{W}$. Thus $\mathfrak{W}$ should be equivalent on V to some «good» weight system $\tilde{\mathfrak{W}}$, which is globally dominated by $\mathfrak{W}$.

Condition (2.5) is necessary for the surjectivity of the restriction mapping ρ: $H^2(nW, \mathbb{C}^N) \to H(nW, V)$, if V is discrete and generated by slowly decreasing functions $G \in H(nW, \mathbb{C}^N)^N$ with $\det DG(a) \neq 0$ for any $a \in V$ ([2], Theorem 4.4). For discrete strongly interpolating varieties (and special weight systems) the existence of an extension operator E as in Theorem 2.3 is equivalent to (1.4) being satisfied only for $t \in V$ (see [13]).

Again Theorem 2.3 holds in the (linear) tame category:

2.4. Corollary. *Let $\mathfrak{W}$ satisfy (1.1)–(1.4) with (linear) tame choices of I_j and let $\ln(1/r(z))$ be psh. Let V be strongly interpolating for $\mathfrak{W}$ with (linear) tame choice of I_5 and I_6. Then there is a (linear) tame extension operator*

$$E: H(\mathfrak{W}, V) \to H^2(\mathfrak{W}, \Omega).$$

References

[1] C. A. BERENSTEIN, B. A. TAYLOR: *A new look at interpolation theory for entire functions of one variable*, Adv. Math. **33** (1979), 109-143.

[2] C. A. BERENSTEIN, B. A. TAYLOR: *Interpolation problems in $\mathbb{C}^N$ with applications to harmonic analysis*, J. d'Analyse Math. **38** (1980), 188-254.

[3] C. A. BERENSTEIN, B. A. TAYLOR: *On the geometry of interpolating varieties*, Lecture Notes in Math., vol. 919, Springer, Berlin/New York 1982, pp. 1-25.

[4] J.-P. DEMAILLY: *Scindage holomorphe d'un morphisme de fibrés vectoriels semi-positifs avec estimation L^2*, Lecture Notes in Math., vol. 919, Springer, Berlin/New York 1982, pp. 77-107.

[5] P. B. DJAKOV, B. S. MITIAGIN: *The structure of polynomial ideals in the algebra of entire functions*, Studia Math. **68** (1980), 85-104.

[6] L. HÖRMANDER: *Generators for some rings of analytic functions*, Bull. Amer. Math. Soc. **73** (1967), 943-949.

[7] L. HÖRMANDER: *An introduction to complex analysis in several variables*, North Holland/American Elsevier, New York 1973.

[8] L. HÖRMANDER: *The analysis of linear partial differential operators I+II*, Grundlehren 256+257, Springer, Berlin/New York 1983.

[9] J. J. KELLEHER, B. A. TAYLOR: *Finitely generated ideals in rings of analytic functions*, Math. Ann. **193** (1971), 225-237.

[10] M. LANGENBRUCH: *Tame right inverses for partial differential equations,* in: Advances in the theory of Frechet spaces, T. Terziogiu (ed.), NATO ASI Series C, vol. **287** (1989), 79-114.

[11] M. LANGENBRUCH: *The splitting condition for the weighted $\bar{\partial}$-complex.* Results in Math. **22** (1992), 560-597.

[12] M. LANGENBRUCH: *Convolution operators admitting a continuous linear right inverse,* manuscript.

[13] M. LANGENBRUCH: S. MOMM: *Complemented submodules in weighted spaces of analytic functions.* Math. Nachr. **157** (1992), 263-276.

[14] P. LELONG, L. GRUMAN: *Entire functions in several complex variables,* Grundlehren 286, Springer, Berlin/New York 1986.

[15] G. MARINO, P. PIETRAMALA, D. STRUPPA: *Factorization of solutions of convolution equations,* Kobe J. Math. **6** (1989), 23-36.

[16] R. MEISE: *Sequence space representations for (DFN)-algebras of entire functions modulo closed ideals,* J. Reine Angew. Math. **363** (1985), 59-95.

[17] R. MEISE, B. A. TAYLOR: *Splitting of closed ideals in (DFN)-algebras of entire functions and the property (DN).* Trans. Amer. Math. Soc. **302** (1987), 341-370.

[18] B. MITIAGIN, G. HENKIN: *Linear problems in complex analysis,* Russian Math. Surveys **26** (1971), 99-164.

[19] S. MOMM: *Ideale in gewichteten Algebren holomorpher Funktionen auf dem Einheitskreis,* Dissertation Düsseldorf 1988.

[20] S. MOMM: *Linear topological invariants and splitting of closed ideals in weighted algebras of analytic functions on the disc,* Results in Math. **17** (1990), 128-139.

[21] S. MOMM: *Partial differential operators of infinite order with constant coefficients on the space of analytic functions on the polydisc,* Studia Math, **96** (1990), 52-71.

[22] V. P. PALAMODOV: *On a Stein manifold the Dolbeault complex splits in positive dimensions,* Math. USSR-Sb. **17** (1972), 289-316.

[23] B. A. TAYLOR: *Linear extension operators for entire functions.* Mich. Math. J. **29** (1982), 185-197.

[24] D. VOGT: *Tame spaces and power series spaces.* Math. Z. **196** (1987), 523-536.

Mathematisches Institut
Einsteinstr, 62
D-4400 Münster
Bundesrepublik Deutschland

Recibido: 1 de abril de 1991