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Isometries and Automorphisms of the Spaces of Spinors

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ABSTRACT. The relationships between the JB*-triple structure of a complex spin factor $\mathcal{S}$ and the structure of the Hilbert space $\mathcal{H}$ associated to $\mathcal{S}$ are discussed. Every surjective linear isometry L of $\mathcal{S}$ can be uniquely represented in the form $L(x)=\mu U(x)$ for some conjugation commuting unitary operator U on $\mathcal{H}$ and some $\mu\in\mathbb{C}$, $|\mu|=1$. Automorphisms of $\mathcal{S}$ are characterized as those linear maps (continuity not assumed) that preserve minimal tripotents in $\mathcal{S}$ and the orthogonality relations among them.

§0. INTRODUCTION

The spaces of spinors were introduced by E. Cartan in [1] to solve the problem of analytic classification of bounded symmetric domains in $\mathbb{C}^n$, and they also arise in the quantization of free fermionic fields [9, p. 104]. More recently, these spaces have been considered in various problems in the context of infinite dimensional holomorphy by Harris, Kaup and others. In this note, an arbitrary spinor space $\mathcal{S}$ is considered and the relationship between the «triple structure» of $\mathcal{S}$ and the structure of its underlying Hilbert space $\mathcal{H}$ is discussed. In §2, we prove that any surjective linear isometry L of $\mathcal{S}$ can be represented in the form $L=\mu U$ for some $\mu\in\mathbb{C}$, $|\mu|=1$ and some conjugation commuting surjective linear isometry of $\mathcal{H}$. Since surjective linear isometries of $\mathcal{S}$, and conjugation commuting unitary operators on $\mathcal{H}$, are the same as automorphisms of the corresponding structures of $\mathcal{S}$ and $\mathcal{H}$, our result can be rephrased by saying that, except for an automorphism, the spaces of spinors are uniquely determined by their underlying Hilbert space. This is a result that anyone could expect, though the authors have found no precise reference for the statement and proof. In §3, the sets $\mathrm{Min}(\mathcal{S})$ and

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$\mathrm{Extr}(\mathscr{S})$ of minimal and maximal tripotents of $\mathscr{S}$ are discussed. In §4 we prove that any linear mapping (continuity is not assumed) that preserves minimal tripotents and the orthogonality relations among them is an automorphism of $\mathscr{S}$, and that any holomorphic automorphism of the unit ball of $\mathscr{S}$ is uniquely determined by its values at the set $\mathrm{Min}(\mathscr{S})\cup\{0\}$.

§1. NOTATION AND PRELIMINARY RESULTS

Let H be a complex Hilbert space with $\dim(H)>1$. We recall ([2] p. 16, [4] p. 358) that a Cartan factor of type IV, also called a spinor or a spin factor, is a norm closed selfadjoint complex subspace $\mathscr{S}$ of $\mathscr{L}(H)$ such that $\{a^2 \mid a\in\mathscr{S}\}\subset\mathbb{C}1_H$, where $\mathscr{L}(H)$ is the C^*-algebra of bounded linear operators on H and 1_H is the identity operator. For a and b in $\mathscr{S}$, there is a unique complex number $(a|b)$ such that

(1.1) $$ab^*+b^*a=2(a|b)1_H$$

and the mapping $(a,b)\to(a|b)$ defines an inner product on $\mathscr{S}$ whose associated norm, denoted by $\|.\|$, is equivalent to the usual operator norm $\|\,.\,\|_\infty$ induced by $\mathscr{L}(H)$ on $\mathscr{S}$. Let us define

$$\chi=:\{a\in\mathscr{S}\mid a=a^*\}.$$

The norms $\|.\|$ and $\|\,.\,\|_\infty$ coincide on χ, and we have the topological direct sum decomposition

$$\mathscr{S}=\chi\oplus i\chi$$

In particular, $\mathscr{S}$ is a complex Hilbert space with conjugation $a\to\bar{a}=:a^*$, $a\in\mathscr{S}$, the hilbertian norm and the operator norm being related by

$$\|a\|_\infty^2=\|a\|^2+[\|a\|^4-|(a|a^*)|^2]^{1/2}$$

On the other hand, $\mathscr{S}$ is a J^*-algebra of operators, i.e., $\mathscr{S}$ is a norm closed complex subspace of $\mathscr{L}(H)$ such that the triple product

(1.2) $$\{a\,b^*\,c\}=:\frac{1}{2}(ab^*c+cb^*a)$$

is in $\mathscr{S}$ whenever a, b, and c are in $\mathscr{S}$. The J^*-algebra structure and the Hilbert space structure are linked by the formula ([4] p. 358)

(1.3) $$\{a\,a^*\,a\}=2(a|a)\,a-(a|a^*)\,a^*\qquad(a\in\mathscr{S})$$

An alternative introduction of the spaces of spinors is the following: Let K be a complex Hilbert space with conjugation $^-$ and inner product $(.|.)$. Define a triple product by

$$\{x y^* z\} = :(x|y)z - (z|\bar{x})\bar{y} + (z|y)x.$$

Then K with the conjugation $^-$, the triple product $\{...\}$ and the norm $\|.\|_\infty$ given by

$$\| x \|_\infty^2 = \|x\|^2 + [\, \|x\|^4 - |(x|\bar{x})|^2\,]^{1/2}$$

is a Cartan factor of type IV. The norms $\|.\|$ and $\|.\|_\infty$ are referred to as the Hilbert and the Lie norm on K, their unit balls being denoted by B and B_∞.

Some other basic facts on J^*-algebras are needed in the sequel. Let $L: \mathscr{A} \to \mathscr{B}$ be a vector space isomorphism (continuity not assumed) between the J^*-algebras $\mathscr{A}$ and $\mathscr{B}$. Then L commutes with the triple product, i.e.,

$$L\{a b^* c\} = \{L(a) L(b)^* L(c)\} \qquad (a, b, c \in \mathscr{A}),$$

if and only if L is an isometry. In that case, L is said to be a J^*-isomorphism. An element $a \in \mathscr{S}$ is said to be a tripotent if $a \neq 0$ and $\{a a^* a\} = a$, and in that case $\| a \|_\infty = 1$. The tripotent a is said to be minimal if, to each $x \in \mathscr{S}$, there exists $\lambda_x \in \mathbb{C}$ such that

$$\{a x^* a\} = \lambda_x a.$$

J^*-isomorphisms preserve tripotents and minimal tripotents. The set $\mathrm{Min}(\mathscr{S})$ of minimal tripotents of $\mathscr{S}$ is given by ([3] p. 179)

$$\mathrm{Min}(\mathscr{S}) = \{ a \in \mathscr{S} \mid a^2 = 0 \}.$$

§2. ISOMETRIES OF THE SPACES OF SPINORS

In this section, $\mathscr{S}$ and χ stand for a fixed space of spinors and its selfadjoint part.

Lemma. *Let $L: \mathscr{S} \to \mathscr{S}$ be any surjective linear $\|.\|_\infty$-isometry of $\mathscr{S}$. Then there exists a complex number $\lambda \in \mathbb{C}$, $|\lambda| = 1$, such that*

(2.1) $$L(a)^* = \lambda\, L(a) \qquad (a \in \chi).$$

As a consequence, $\| La \| = \| a \|_\infty$ holds for all a in χ.

Proof: Let $a \in \chi$, $a \neq 0$, be given. By (1.3)

$$\{a a^* a\} = \| a \|^2 a.$$

Thus applying L

$$L\{a a^* a\} = \| a \|^2 L(a)$$

whence again by (1.3)

$$\text{(2.2)} \qquad L\{a a^* a\} = \{L(a) L(a)^* L(a)\} = 2\| L(a) \|^2 L(a) - (L(a) \mid L(a)^*) L(a)^*$$

If $(L(a) \mid L(a)^*) = 0$ then by (1.1) and (1.3)

$$2 L(a)^2 = 2 (L(a) \mid L(a)^*) 1_H = 0$$

i.e., $L(a)$ is a minimal tripotent; but then a is also a minimal tripotent and so $a^2 = 0$ which, together with $a \in \chi$, implies $a = 0$, a contradiction. From (2.2)

$$\text{(2.3)} \qquad L(a)^* = \lambda L(a) \qquad (a \in \chi)$$

where $|\lambda| = 1$ and

$$\lambda = \frac{2 \| L(a) \|^2 - \|a\|^2}{(L(a) \mid L(a)^*)} \qquad (a \in \chi).$$

We claim that λ does not depend on $a \in \chi$. Indeed, let $b \in \chi$ be given so that a, b are linearly independent (this is possible since by assumption dim $\mathcal{S} > 1$). By (2.3) there are unitary numbers $\lambda, \mu, \nu \in \mathbb{C}$, such that

$$L(a) = \bar{\lambda} L(a)^*, \qquad L(b) = \bar{\mu} L(b)^*, \qquad L(a+b) = \bar{\nu} [L(a+b)]^*$$

whence by the linearity of L and the independence of $L(a)$ and $L(b)$, we get $\lambda = \mu = \nu$. Using (2.3) and the expression of λ,

$$\|L(a)\|^2 = (L(a) \mid L(a)) = (L(a) \mid \bar{\lambda} L(a)^*) = \lambda (L(a) \mid L(a)^*) = 2 \|L(a)\|^2 - \|a\|^2$$

whence, by the coincidence of $\| \, . \, \|_\infty$ and $\| \, . \, \|$ on χ, we get

$$\|L(a)\|^2 = \|a\|^2 = \|a\|_\infty^2$$

Theorem. *Let $\mathcal{S}$ be any spin factor, and let $L : \mathcal{S} \to \mathcal{S}$ be any surjective vector space isomorphism. Then the following statements are equivalent:*

1. *L is an $\| \, \|_\infty$-isometry of $\mathcal{S}$.*

2. *There is a unitary operator U on the Hilbert space $\mathcal{S}$ with $U(a^*)=U(a)^*$ for all a in $\mathcal{S}$, and there is a number $\mu \in \mathbb{C}$ with $|\mu|=1$ such that*

$$L(a)=\mu\, U(a) \qquad (a \in \mathcal{S}).$$

In particular, any $\|\ \|_\infty$-isometry of $\mathcal{S}$ is an isometry for the underlying Hilbert space.

Proof: "2⟹1" is immediate, so we show "1⟹2". By the previous lemma

$$L(a)^* = \lambda\, L(a) \qquad (a \in \chi)$$

for some $\lambda \in \mathbb{C}$, $|\lambda|=1$. Let $\mu \in \mathbb{C}$ be such that $\mu^2=\lambda$, and define $U{:}\,\mathcal{S} \to \mathcal{S}$ by $U =: \mu L$. Then, for $a \in \chi$ we have

$$U(a)=\mu L(a)=\bar{\mu} L(a)^* = [(\mu L)(a)]^* = U(a)^*$$

i.e., $U(\chi) \subset \chi$, and so

$$U(b^*)=[U(b)]^* \qquad (b \in \mathcal{S}).$$

Since $L=\bar{\mu}U$, in order to prove the theorem it suffices to show that U is an isometry for the hilbertian norm $\|.\|$ on $\mathcal{S}$. But this is a consequence of the sesquilinearity of the scalar product and the fact that U is an $\|.\|$-isometry on the selfadjoint part χ of $\mathcal{S}$.

Note. The authors thank Prof. Rodríguez Palacios for simplifying their original proof of this theorem.

§3. EXTREME POINTS OF THE UNIT LIE BALL OF $\mathcal{S}$

A tripotent e of $\mathcal{S}$ is said to be regular if the operator $\xi \to \{e e^* \xi\}$, $\xi \in \mathcal{U}$, is invertible in $\mathcal{L}(\mathcal{U})$, and this occurs ([7] p. 190) if and only if e is a (real or complex) extreme point of the unit Lie ball of $\mathcal{S}$, whose set is denoted by $\mathrm{Extr}(\mathcal{S})$. We have ([9] p. 37)

Theorem. *If $\mathcal{S}$ is any spin factor, then the following equalities hold:*

$\{e \in \mathcal{S} \mid e$ *is a unitary operator*$\} = \{e \in \mathcal{S} \mid e$ *is a normal operator and* $\|e\|_\infty = 1\}$

$= \{e \in \mathcal{S} \mid e$ *is a normal operator and* $\|e\|=1\} = \{e \in \mathcal{S} \mid e$ *is a real extreme point of* $B\} = \{e \in \mathcal{S} \mid \|e\|_\infty = 1$ *and* $e^* = \lambda e,\ \lambda \in \mathbb{C}\ |\lambda|=1\}$.

Proof: Let us denote by S_k, $1 \leq k \leq 5$, the sets above. The inclusion $S_1 \subset S_2$ is clear, and $S_2 \subset S_3$ follows by (1.1). We now prove $S_3 \subset S_4$. Let $e \in S_3$; from (1.1), the assumption $\|e\| = 1$ and the normality of e we get

$$ee^* = 1_H = e^*e \tag{3.1}$$

whence by composing with e, $\{e e^* e\} = e$ which shows that e is a tripotent. From (3.1) $\{e e^* x\}1 = \frac{1}{2}(ee^* x + xe^*e) = x$ for $x \in \mathscr{S}$, which shows the regularity of e, hence $e \in \mathrm{Extr}(\mathscr{S})$

We now prove "$S_4 \subset S_5$". Let $e \in S_4$, hence in particular

$$e = \{ee^*e\} = ee^*e$$

If here we compose with e and use the fact that $e^2 = \alpha 1_H$ for some $\alpha \in \mathbb{C}$ (recall the definition of $\mathscr{S}$), we obtain $\alpha(ee^*\text{-}1_H) = 0$. But $\alpha \neq 0$ as otherwise $e^2 = 0 = e^{*2}$ and so

$$\{e e^* e^*\} = \frac{1}{2}(ee^{*2} + e^{*2} e) = 0$$

which contradicts the regularity of e. Thus $ee^* = 1_H$ and multiplying by e on the left, $\alpha e^* = e$, where $|\alpha| = 1$ since $e \in \mathrm{Extr}(\mathscr{S})$ entails $\|e\|_\infty = 1$. The inclusion $S_5 \subset S_1$ is trivial.

§4. BOUNDARY BEHAVIOUR OF AUTOMORPHISMS

In this section, $\mathscr{S}$ denotes an arbitrary JB^*-triple of finite rank ([8], p. 5.4), $\|.\|_\infty$ denotes its unique JB^*-norm, B_∞ is the unit ball of $\mathscr{S}$, and $\mathcal{G} = \mathrm{Aut}(B_\infty)$ is the group of all holomorphic automorphisms of B_∞. It is known ([5], prop. 3.2) that each $g \in \mathcal{G}$ extends to a holomorphic mapping on a neighbourhood of $\overline{B}_\infty$, and that g maps the boundary ∂B_∞ of B_∞ onto itself. It is also known that ([8], p. 3.10) that each $x \in \mathscr{S}$, $x \neq 0$, admits a spectral representation of the form

$$x = \sum_{k=1}^{n} \lambda_k e_k \tag{4.1}$$

for some pairwise orthogonal minimal tripotents e_k and some uniquely determined scalars λ_k, $1 \leq k \leq n$ with

$$\lambda_1 \geq \lambda_2 \geq \ldots . > 0, \qquad \|x\|_\infty = \max\{\lambda_k \mid 1 \leq k \leq n\} \tag{4.2}$$

Theorem. *Let $\mathcal{S}$ be any finite rank* JB*-*triple, and let $L:\mathcal{S}\to\mathcal{S}$ be any linear mapping (continuity not assumed). Then the following statements are equivalent:*

1. *L is a J*-automorphism of $\mathcal{S}$.*

2. *L maps Min ($\mathcal{S}$) onto itself and preserves the orthogonality relations on* Min ($\mathcal{S}$),

$$L[\mathrm{Min}(\mathcal{S})]=\mathrm{Min}(\mathcal{S}) \qquad [a,\ b\in \mathrm{Min}(\mathcal{S}),\ a\perp b]\Rightarrow L(a)\perp L(b). \tag{4.3}$$

Proof: We show that "$2\Rightarrow 1$" as the converse is trivial. Let $x\in\mathcal{S}$ and let (4.1) be its spectral decomposition. Then

$$L(x)=\sum_{k=1}^{n}\lambda_k L(e_k)$$

where by (4.3), $L(e_k)$, $1\leq k\leq n$, are pairwise orthogonal minimal tripotents and so, by the properties of the spectral representation and (4.2),

$$\|L(x)\|_\infty=\|x\|_\infty.$$

Besides, L is surjective. Indeed, if

$$y=\sum_{1}^{m}\mu_j e_j\in\mathcal{S},$$

by (4.3) there are tripotensts f_j, $1\leq j\leq m$ (orthogonality is not needed now), such that $L(f_j)=e_j$, hence $x=\sum_{1}^{m}\mu_j f_j$ satisfies $L(x)=y$. Thus L is a J^*-automorphism.

Corollary. *Any J*-automorphism L of a finite rank* JB*-*triple $\mathcal{S}$ is uniquely determined by its values at the set* Min ($\mathcal{S}$).

Corollary. *A holomorphic automorphism of the unit ball B_∞ of a finite rank* JB*-*triple is uniquely determined by its values at the set* $\{0\}\cup\mathrm{Min}(\mathcal{S})$.

Proof: Let f and g in Aut (B_∞) be such that $f(0)=g(0)=a$ and $f(e)=g(e)$ for all $e\in\mathrm{Min}(\mathcal{S})$. Take any $h\in\mathrm{Aut}(B_\infty)$ such that $h(a)=0$. Then $L=:(hg)^{-1}\circ(hf)$ fixes the origin, hence by Cartan's uniqueness theorem, L is linear. Since L fixes any minimal tripotent of $\mathcal{S}$, we have $L=Id_{\mathcal{S}}$ and $f=g$.

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