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On Vector Fields in $\mathbb{C}^3$ without a Separatrix

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ABSTRACT. A family of germs at 0 of holomorphic vector fields in $\mathbb{C}^3$ without separatrices is constructed, with the aid of the blown-up foliation $\tilde{\mathscr{F}}$ in the blown-up manifold $\tilde{\mathbb{C}}^3$. We impose conditions on the multiplicity and the linear part of $\tilde{\mathscr{F}}$ at its singular points (i.e. non-semisimplicity and certain nonresonancy), which are sufficient for the original vector field to be separatrix-free.

A separatrix of a germ of a holomorphic vector field W in $\mathbb{C}^3$ is a germ of an analytic curve $\mathscr{C}$ at 0 tangent to W. The aim of this work is to construct a locally closed algebraic subvariety

$$\mathscr{W} \subset \frac{m_{C^3,0}^{d+1} \cdot \Theta_{C^3,0}}{m_{C^3,0}^{d+3} \cdot \Theta_{C^3,0}}$$

such that every vector field of algebraic multiplicity $d+1$ whose $(d+2)$-jet lies in $\mathscr{W}$ does not have a separatrix.

The exposition and main ideas follow [4]. In a few lines, these are as follows: One considers a polynomial vector field W in $\mathbb{C}^3$, with non-vanishing homogeneous terms only in degrees $d+1$ and $d+2$, the foliation $\mathscr{F}$ it induces in a neighbourhood of $0 \in \mathbb{C}^3$, and the blown-up foliation $\mathscr{F}_{\tilde{W}}$ on $\tilde{\mathbb{C}}^3$ for which the exceptional divisor $E \simeq \mathbb{CP}^2$ is invariant and non-dicritical (i.e. $\mathscr{F}_{\tilde{W}}$ is tangent to, and not identically zero on E).

We fix a projective line $\mathscr{L} \subset E$ and a finite set of points $\{p_k\}_{k \in I}$ on it, and impose the following conditions on the coefficients of the vector field W:

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(i) $\mathscr{L}$ is tangent to $\mathscr{F}_{\tilde{w}}$.

(ii) Each p_k is an isolated singularity of $\mathscr{F}_{\tilde{w}}$.

(iii) $\sum_{p_k} \mu(\mathscr{F}_{\tilde{w}}; p_k) = c_2(L_d \otimes T\mathbb{CP}^2)$ (Where μ is the Milnor Number of the singularity).

(iv) At each p_k, the *linear part* $D\tilde{W}(p_k)$ of $\mathscr{F}_{\tilde{w}}$ has a non-zero double eigenvalue whose generalized eigenspace V is not tangent to E, and

(v) $D\tilde{W}(p_k)|_V$ is non-semisimple.

The terms of degree $d+2$ play a role only in (v).

It has been proved in [4] that conditions *(ii)* to (v) above are sufficient for a vector field W to be separatrix-free *if*, by blowing-up the foliation at each p_k, the two (and only two) arising singularities of the new foliation are *simple corners* (see section 3 below). The stability properties of simple corners under succesive blowing-up's (see [3], pp. 166) are then used to prove that all the separatrices of $\mathscr{F}_{\tilde{w}}$ are contained in E.

A section X of the bundle $L_d \otimes T\mathbb{CP}^2$ defines a foliation by curves $\mathscr{F}_X$, where L_d is the line bundle on $\mathbb{CP}^2$ with Chern class d. It is known that the number of isolated singularities (counted with multiplicities) of such an X equals the second Chern class of this bundle, which is d^2+3d+3. If X leaves a line $\mathscr{L}$ invariant then at least $d+2$ of these singularities must occur on $\mathscr{L}$.

From now on, we assume $d \geq 1$. We shall construct a family of such foliations $\mathscr{F}_X$ with a chosen invariant line, having the property that the singular set is concentrated on the origin and the $(d+1)$ — roots of unity in some local coordinate of $\mathscr{L}$. We show that $\mathscr{F}_X$ can be extended to a foliation $\mathscr{F}_{\tilde{w}}$ on the blown-up manifold $\tilde{\mathbb{C}}^3$ in such a way that conditions (i) to (v) above are verified. Finally, we prove that every separatrix of $\mathscr{F}_{\tilde{w}}$ is contained in the exceptional divisor E, taking care that no *resonances* are present (Section 3, **resonant case**). To that end, we use the arguments described above, and a *Normal Form* for singular vector fields in $\mathbb{C}^3$ having an invariant divisor (see the Appendix, Proposition A).

The extension of this plan to higher dimension, *i.e.* to impose conditions for non-semisimplicity at the singular points, depends on whether the dimension of the (projective) space of foliations of degree d in $\mathbb{CP}^n$ (the number of *variables*) is smaller than $c_n(L_d \otimes T\mathbb{CP}^n)$ (*i.e.* the number of *equations*: at least one repeated-eigenvalue condition for each singular point). For dimension $n=2$, these dimensions fit, for every degree d, but, for big enough dimension n and Chern class d, one has the opposite inequality. The present calculation in dimension $n=2$ will show that, for foliations having high-multiplicity singular points, less equations are needed to impose the repeated-eigenvalue conditions. It will also show that not all the variables are relevant for the construction. However, it allows us to expect that the above restriction is not an obstruction to push this ideas to get separatrix-free vector fields in higger dimensions.

From another point of view, this construction shows that some eigen-values-prescribed dynamics in $\mathbb{CP}^2$ are indeed realizable.

1. CONSTRUCTION

The construction of the family goes as follows: We choose the expression

$$X|_{\mathcal{L}} = az_2(z_2^{d+1}-1)\frac{\partial}{\partial z_2}$$

to preassign the singularities on the invariant line, and then we extend it, first to a section $X \in H^0(\mathbb{CP}^2, L_d \otimes T\mathbb{CP}^2)$, which has the form

$$X(z_2, z_3) = [az_2(z_2^{d+1}-1) + z_3(\mathbf{P}_0^d + z_2\mathbf{L}_d)]\frac{\partial}{\partial z_2}$$

$$+ z_3[az_2^{d+1} + z_3\mathbf{L}_d + \mathbf{Q}_0^d]\frac{\partial}{\partial z_3}$$

$$= X_2\frac{\partial}{\partial z_2} + X_3\frac{\partial}{\partial z_3} \tag{1.1}$$

where $\mathbf{L}_d = \sum_{i+j=d} l_{ij} z_2^i z_3^j$, $\mathbf{Q}_0^d = \sum_{i+j=0}^d q_{ij} z_2^i z_3^j$

and $\mathbf{P}_0^d = \sum_{i+j=0}^d p_{ij} z_2^i z_3^j$

(see [5], Proposition 1.23) and afterwards, to the blow-up $\tilde{W}$ of a $(d+1)$-homogeneous polynomial vector field W in $\mathbb{C}^3$, having local expression (1.1) when restricted to the invariant (and non-dicritical) exceptional divisor $E \simeq \mathbb{CP}^2$.

To be precise: Using the correspondence between $H^0(\mathbb{CP}^2, L_d \otimes T\mathbb{CP}^2)$ and the set of $(d+1)$-homogeneous polynomial vector fields in $\mathbb{C}^3$, one obtains that the homogeneous vector field in $\mathbb{C}^3$

$$W(x_1, x_2, x_3) = -[ax_2^{d+1} + x_3 \sum_{i+j=d} l_{i,j} x_2^i x_3^j]\frac{\partial}{\partial x_1}$$

$$+ [-ax_1^d x_2 + x_3 \sum_{i+j=0}^{d} p_{i,j} x_1^{d-(i+j)} x_2^i x_3^j]\frac{\partial}{\partial x_2}$$

$$+x_3\Big[\sum_{i+j=0}^{d} q_{i,j}\, x_1^{d-(i+j)} x_2^i x_3^j\Big]\frac{\partial}{\partial x_3}$$

$$=\sum_{k=1}^{3} W_k(x_1,x_2,x_3)\frac{\partial}{\partial x_k}$$

satisfies $D\Pi(W)=X$, where $\Pi:\mathbb{C}^3-\{0\}\to\mathbb{CP}^2$ is the natural projection. Let $\sigma:\tilde{\mathbb{C}}^3\to\mathbb{C}^3$ denote the blow-up map and chooose coordinates on $\tilde{\mathbb{C}}^3$ such that σ is given by

$$\sigma(z_1,z_2,z_3)=\quad(z_1,z_1z_2,z_1z_3)=\quad(x_1,x_2,x_3) \tag{1.2}$$

and E by $\{z_1=0\}$. The local expression of the blown-up vector field in this chart is

$$\tilde{W}(z_1,z_2,z_3)=\frac{1}{z_1^d}(D\sigma)^{-1}\; W(z_1,z_1z_2,z_1z_3)$$

$$=z_1\,W_1(1,z_2,z_3)\frac{\partial}{\partial z_1}+\sum_{k=2}^{3}[W_k-z_kW_1](1,z_2,z_3)\frac{\partial}{\partial z_k}$$

so that $\tilde{W}(0,z_2,z_3)=X(z_2,z_3)$.

For any d-homogeneous polynomial

$$H=\sum_{i+j+k=d} h_{i,j,k}\,x_1^i x_2^j x_3^k \tag{1.3}$$

the $(d+1)$-homogeneous vector field in $\mathbb{C}^3$

$$W^H=W+H\sum_{i=1}^{3} x_i\frac{\partial}{\partial x_i}$$

induces the same foliation $\mathscr{F}_{\tilde{W}}$ as $\tilde{W}$ on E with local expression in $\tilde{\mathbb{C}}^3$ given by

$$\tilde{W}^H=z_1[H+W_1](1,z_2,z_3)\frac{\partial}{\partial z_1}+\sum_{k=2}^{3}X_k\frac{\partial}{\partial z_k}$$
$$=\sum_{k=1}^{3}\tilde{W}_k^H\frac{\partial}{\partial z_k}$$

The linear part $D\,\tilde{W}^H(0,z_2,0)$ of $\mathscr{F}_{\tilde{W}}|_{\mathscr{V}}$ is

$$\begin{pmatrix} W_1^H(1,z_2,0) & 0 & 0 \\ 0 & (d+2)az_2^{d+1}-a & \dfrac{\partial X_2}{\partial z_3}(z_2,0) \\ 0 & 0 & az_2^{d+1}+\sum_{i=0}^{d} q_{i,0}\, z_2^i \end{pmatrix} \tag{1.4}$$

where

$$W_1^H(1, z_2, 0) = \sum_{i=0}^{d} h_{i,d-i,0}\, z_2^{d-i} - a z_2^{d+1}$$

$$\frac{\partial X_2}{\partial z_3}(z_2, 0) = z_2 [\mathbf{L_d} + \frac{\partial \mathbf{L_d}}{\partial z_3}](z_2, 0) + \mathbf{P_0^d}(z_2, 0)$$

By construction, the foliation $\mathscr{F}_{\tilde{W}}$ has $(d+2)$ singularities, located at the points $p_0 = (0,0,0)$, and $p_k = (0, \omega^k, 0)$, $k = 1, \ldots, d+1$, where $\omega = e^{2\pi i/(d+1)}$.

The vector space of $(d+1)$-homogeneous polynomial vector fields in $\mathbb{C}^3$ has dimension $(3d^2 + 15d + 18)/2$ and so, its associated projective space, where most of the time we will work, has dimension $(3d^2 + 15d + 16)/2$. The family just constructed has codimension $2(d+2)$ there.

Our goal now is to impose conditions on the coefficients of W so that $\mathscr{F}_{\tilde{W}}$ has simple singularities at the points $p_1, \ldots, p_{d+1}$ and a singular point at p_0 of multiplicity $1 + (d+1)^2$. we want this because

$$c_2(L_d \otimes T\mathbb{CP}^2) = d^2 + 3d + 3 = d + 1 + (1 + (d+1)^2)$$

Observe in (1.4) that the matrix is triangular: The eigenvalues of $D\,\tilde{W}^H$ at each singular point lie in the main diagonal. Since the lower right 2×2 block is $DX(z_2, 0)$, the linear part of $\mathscr{F}_X$ on $\mathscr{L}$, we shall say that the other eigenvalue is *normal* to E.

1.1 Multiplicities

This calculation involves only the $(d+2)^2$ coefficients of X defined on (1.1), which are regarded as variables (*i.e.,* does not depend on the $h_{i,j,k}$ variables).

From (1.4), we see that the points $\{p_k : k = 1, \ldots, d+1\}$ on $\mathbb{CP}^2$ are simple singularities of $\mathscr{F}_{\tilde{W}}|_E$ if the correspondent eigenvalues are non-zero:

$$(d+1)\,a \neq 0$$

$$a + q_{0,0} + \textstyle\sum_{i=1}^{d} q_{i,0}\, \omega^{ki} \neq 0 \quad k = 1, \ldots d+1 \tag{1.5}$$

Sufficient conditions for p_0 being singular of multiplicity $1 + (d+1)^2$ may be obtained as follows: from (1.1), define Y_3 by $X_3 = z_3 Y_3$; one observes that

$$\begin{aligned} \mu\langle X_2, z_3 Y_3; p_0\rangle &= \mu\langle X_2, z_3; p_0\rangle + \mu\langle X_2, Y_3; p_0\rangle \\ &= \mu\langle X_2(z_2, 0); 0\rangle + \mu\langle X_2, Y_3; p_0\rangle \\ &= 1 + \mu\langle X_2, Y_3; p_0\rangle \end{aligned}$$

so that $\mu\langle X; p_0\rangle \geq 2$ if $\mu\langle X_2, Y_3; (0,0)\rangle \geq 1$, which is the case, from (1.1), if

$$q_{00} = 0 \tag{1.6}$$

since q_{00} is the independent term of Y_3. Assuming (1.6), we may now increase $\mu\langle X_2, Y_3; p_0\rangle$ by forcing the curves $(X_2 = 0)$ and $(Y_3 = 0)$ to be very tangent at a common chosen line: The linear terms of X_2 and $Y_3 = z_3^{-1} X_3$ in (1.1) define, respectively, the tangent line at p_0 to the curves $(X_2 = 0)$ and $(Y_3 = 0)$. The line $(z_3 = 0)$ is then tangent to both curves if these linear terms satisfy the conditions

$$p_{0,0} = q_{0,1} = 0 \tag{1.7}$$

$$a \neq 0, q_{1,0} \neq 0 \tag{1.8}$$

Assuming (1.6), (1.7) and (1.8) we rewrite the vector field (1.1) as

$$X(z_2, z_3) = [-az_2 + z_2(az_2^{d+1} + z_3\,\mathbf{L}_d) + z_3\,\mathbf{P}_1^d]\frac{\partial}{\partial z_2}$$

$$+ z_3[q_{1,0}z_2 + az_2^{d+1} + z_3\,\mathbf{L}_d + \mathbf{Q}_2^d]\frac{\partial}{\partial z_3}$$

$$= X_2\frac{\partial}{\partial z_2} + z_3\,Y_3\frac{\partial}{\partial z_3} \tag{1.9}$$

From the last expression, consider $X_{23} = (\frac{a}{q_{1,0}} - z_2)\,Y_3 + X_2$. The ideals (X_2, Y_3) and (X_{23}, Y_3) coincide.

Now, if $X_{23} \in \mathbb{C} \cdot z_2^{d+1}$ and $Y_3(0, z_3) \in (z_3^{d+1})$ then

$$\mu\langle X_2, Y_3; p_0\rangle = \mu\langle X_{23}, Y_3; p_0\rangle = \mu\langle kz_2^{d+1}, Y_3; p_0\rangle$$

$$= (d+1)\cdot\mu\langle Y_3(0, z_3); 0\rangle = (d+1)^2$$

so that $\mu\langle X; p_0\rangle = 1 + (d+1)^2$, as desired.

We proceed now to write equations for the previous conditions: From (1.9) define $\mathbf{Q}'$ and $\mathbf{Q}''$ by

$$\mathbf{Q}_2^d(z_2, z_3) = \sum_{i=2}^{d} q_{i,0}\,z_2^i + z_3\Big(\sum_{i+j=2,\,j\geq 1}^{d} q_{ij}\,z_2^i z_3^{j-1}\Big)$$

$$= \mathbf{Q}'(z_2) + z_3\,\mathbf{Q}''(z_2, z_3)$$

Then X_{23} is a scalar multiple of z_2^{d+1} if the following conditions are verified

$$\frac{a}{q_{1,0}}(\mathbf{L}_d+\mathbf{Q}'')-z_2\,\mathbf{Q}''+\mathbf{P}_1^d=0 \tag{1.10}$$

$$\frac{a}{q_{1,0}}(\mathbf{Q}'+az_2^{d+1})-z_2\,\mathbf{Q}'-q_{1,0}\,z_2^2\in(z_2^{d+1}) \tag{1.11}$$

We may solve equation (1.10) for the coefficients of $\mathbf{P}_1^d$:

$$\mathbf{P}_1^d=z_2\,\mathbf{Q}''-\frac{a}{q_{1,0}}(\mathbf{L}_d+\mathbf{Q}'') \tag{1.12}$$

(1.12) is a system of $d(d+3)/2$ equations and hence the p_{ij}'s are not free variables anymore.

Factorizing the powers of z_2 in (1.11), one finds the following sufficient conditions for (1.11) to be satisfied

$$q_{1,0}=\lambda a, \qquad q_{j,0}=\lambda^j a \qquad \text{for } j=2,\dots,d \tag{1.13}$$

$$a^2+q_{d,0}\,q_{1,0}=(\lambda^{d+1}-1)\,a^2\neq 0 \qquad \text{for some } \lambda\in\mathbb{C}^* \tag{1.14}$$

Finally, the conditions for $Y_3(0,z_3)\in(z_3^{d+1})$ are

$$q_{0,j}=0 \qquad \text{for } j=1,\dots,d \tag{1.15}$$

$$l_{0,d}\neq 0 \tag{1.16}$$

The total number of equations in (1.6), (1.7), (1.12), (1.13), and (1.15) is $(d^2+7d+6)/2$, in $(d+2)^2$ variables. One may find solutions to this system which are consistent with the open conditions (1.5), (1.8), (1.14) and (1.16) since, up to now, the parameter λ introduced in (1.13) and (1.15) only has to satisfy that it is not zero and that it is not a $(d+1)$-root of 1. Summarizing, we have proved:

Lemma 1.1 *The space of polynomial $(d+1)$–homogeneous vector fields $\{W\}$ in $\mathbb{C}^3$ whose blown-up foliation $\mathcal{F}_{\tilde{W}}$ satisfies (i), (ii) and (iii), contains a quasi-projective subvariety $\mathcal{V}_1$ formed by vector fields whose blown-up foliation $\mathcal{F}_{\tilde{W}}$ has simple singularities at p_k, $k=1,\dots d+1$, and a singularity at p_0 of multiplicity $1+(d+1)^2$. It is defined by the solutions of (1.6), (1.7), (1.12), (1.13) and (1.15) that satisfy the open conditions (1.5), (1.8), (1.14) and (1.16). Each non-empty irreducible component has codimension at most $(d^2+11d+14)/2$.*

1.2 Repeated Eigenvalues

Now we begin to work on the whole $\tilde{\mathbb{C}}^3$, so it is just now when the variables $h_{i,j,k}$ introduced in (1.3) appear. From (1.4), it follows that the condition for a repeated non-zero eigenvalue at p_0 is

$$h_{d,0,0}=-a \tag{1.17}$$

which we will assume in what follows. As X is tangent to a line $\mathscr{L}\simeq\mathbb{CP}^1$, those eigenvalues which have $\mathscr{L}$ as generalized eigenspace do satisfy a relation (see [2]). To avoid it, we are forced to make a clever choice of those pairs of eigenvalues that might be repeated. To say, we impose the condition that the eigenvalue which is normal to the exceptional divisor E, coincides with the one *tangent* to $\mathscr{L}$ only at the d points p_k, $k=1,\ldots,d$ and, at the remaining one $p_{d+1}=(0,1,0)$, to coincide with the one not tangent to $\mathscr{L}$ on E. From (1.4), these conditions are

$$\begin{aligned} &-2a+\sum_{i=0}^{d-1} h_{i,d-i,0}\,\omega^{k(d-i)}-a(d+1)=0 \qquad k=1,\ldots,d \\ &-3a+\sum_{i=0}^{d-1} h_{i,d-i,0}-\sum_{i=1}^{d} q_{i,0}=0 \end{aligned} \tag{1.18}$$

We now proceed to find a solution to equations (1.18) satisfying the conditions in Lemma 1.1.

For simplicity, let's replace $h_{j,d-j,0}$ by h_j. Then substituing (1.13) on (1.18) we arrive to the system of equations

$$\begin{aligned} &\textstyle\sum_{j=0}^{d-1} h_j\omega^{kj}= \quad a(3+d) \qquad k=1,\ldots,d \\ &\textstyle\sum_{j=0}^{d-1} h_j= \quad a(3+\sum_{j=1}^{d}\lambda^j) \end{aligned} \tag{1.19}$$

which in matrix form is

$$\mathbf{A}_{d+1}\begin{pmatrix} h_{d-1} \\ h_{d-2} \\ \vdots \\ h_0 \\ 0 \end{pmatrix}=\begin{pmatrix} a(3+\sum_{j=1}^{d}\lambda^j) \\ a(3+d) \\ \vdots \\ a(3+d) \\ a(3+d) \end{pmatrix}$$

where $\mathbf{A}_{d+1}$ denotes the transpose of the $(d+1)\times(d+1)$ Vandermonde matrix with values $\omega, \omega^2, \ldots, \omega^{d+1}=1$

$$\begin{pmatrix} 1 & 1 & \ldots & 1 & 1 \\ \omega & \omega^2 & \ldots & \omega^d & 1 \\ \omega^2 & \omega^4 & \ldots & \omega^{2d} & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \omega^d & \omega^{2d} & \ldots & \omega^{d^2} & 1 \end{pmatrix}$$

Since $\mathbf{A}_{d+1}^{-1}$ equals $\frac{1}{d+1}$ times the transpose conjugate of $\mathbf{A}_{d+1}$, the solution of equations (1.19) is then

$$\begin{pmatrix} h_{d-1} \\ h_{d-2} \\ \vdots \\ h_0 \\ 0 \end{pmatrix} = \frac{1}{d+1} \cdot \overline{(\mathbf{A}_{d+1})^{\mathrm{T}}} \begin{pmatrix} a(3+\sum_{j=1}^{d} \lambda^j) \\ a(3+d) \\ \vdots \\ a(3+d) \\ a(3+d) \end{pmatrix}$$

The last equation imposes a condition on the parameter λ obtained in equations (1.13) which is equivalent to

$$\sum_{j=1}^{d} \lambda^j = -(d^2+3d+3) \qquad (1.20)$$

To say, that λ is a root of

$$\mathbf{P}(x) = (d+1)(d+2) + \sum_{j=0}^{d} x^j \qquad (1.21)$$

In particular, if λ is real, it most be negative.

Taking absolute value on both sides of (1.20) one verifies that no root λ of (1.21) has modulus ≤ 1. Each one gives a compatible election of the parameter introduced in (1.13) and so, the solutions h_j of the system of linear equations (1.19)) are readily seen to be

$$h_j = -a(d+3) \qquad j=0, 1, \ldots, d-1 \qquad (1.22)$$

Remark: Define $\mathbf{Q}(z_2)=\sum_{j=0}^{d} z_2^j$. Then for $k=1,\ldots,d+1$, the lower right eigenvalue of (1.4) at p_k is non-zero and its value may be written as

$$a\cdot\mathbf{Q}(\lambda\omega^k)=a(1+\sum_{j=1}^{d}\lambda^j\omega^{kj})=a\frac{\lambda^{d+1}-1}{\lambda\omega^k-1}$$

$$=-a(d+1)(d+2)\cdot\frac{\lambda-1}{\lambda\omega^k-1} \tag{1.23}$$

The last equality follows from the fact that λ is a root of (1.21). Thus, the points p_k are indeed simple singularities.

Lemma 1.2 *The space of polynomial $(d+1)$-homogeneous vector fields in $\mathbb{C}^3$, contains a non-empty quasi-projective subvariety $\mathscr{V}$ defined by the equations (1.6), (1.7), (1.12), (1.13), (1.15), (1.17), (1.18), (1.20), and by the open conditions (1.5), (1.8), (1.14), (1.16). Its elements are vector fields $\{W\}$ in $\mathscr{V}_1$ (of Lemma 1.1) such that, at each singular point, the linear part of $\mathscr{F}_{\tilde{w}}$, has a double non-zero eigenvalue. Each non-empty irreducible component has codimension at most $(d^2+13d+20)/2$.*

Proof. The assertion will follow if we exhibit an element in $\mathscr{V}$. Let λ satisfy (1.20), $\omega=e^{2\pi i/(d+1)}$, $a\neq 0$ and $l_{0,d}\neq 0$.

Let $\mathbf{H}(x_1,x_2,x_3)=-a(d+3)\sum_{j=0}^{d-1}x_1^jx_2^{d-j}-ax_1^d$. Then, the vector field

$$\begin{aligned}W^H(x_1,x_2,x_3)=&[x_1\mathbf{H}-(ax_2^{d+1}+l_{0,d}x_3^{d+1})]\frac{\partial}{\partial x_1}\\&+[x_2\mathbf{H}-ax_1^dx_2-\lambda^{-1}l_{0,d}x_3^{d+1}]\frac{\partial}{\partial x_2}\\&+x_3[\mathbf{H}+a\textstyle\sum_{j=1}^{d}\lambda^jx_2^jx_1^{d-j}]\frac{\partial}{\partial x_3}\end{aligned} \tag{1.24}$$

belongs to $\mathscr{V}$.

1.3 Non-Semisimplicity

Now consider a polynomial $(d+2)$-homogeneous vector field in $\mathbb{C}^3$

$$\tilde{W}=\sum_{m=1}^{3}\tilde{W}_m\frac{\partial}{\partial x_m}$$

$$
\begin{aligned}
&= \textstyle\sum_{i+j+k=d+2} a_{i,j,k}\, x_1^i\, x_2^j x_3^k \frac{\partial}{\partial x_1} \\
&+ \textstyle\sum_{i+j+k=d+2} b_{i,j,k}\, x_1^i\, x_2^j x_3^k \frac{\partial}{\partial x_2} \qquad (1.25)\\
&+ \textstyle\sum_{i+j+k=d+2} c_{i,j,k}\, x_1^i\, x_2^j x_3^k \frac{\partial}{\partial x_3}
\end{aligned}
$$

the sum $V = W^H + \hat{W}$ and the blown-up foliation $\mathscr{F}_{\tilde{V}}$ whose generator in the coordinates (1.2) is

$$\tilde{V}(z_1, z_2, z_3) = \frac{1}{z_1^d}(D\sigma^{-1})\, V(z_1, z_1 z_2, z_1 z_3) \qquad (1.26)$$

The linear part of $\mathscr{F}_{\tilde{V}}$ restricted to $\mathscr{L}$ leaves (1.4) unchanged, except for the first column

$$\begin{pmatrix} W_1^H(1, z_2, 0) & 0 & 0 \\ \hat{W}_2 - z_2 \hat{W}_1(1, z_2, 0) & (d+2)\, a z_2^{d+1} - a & \frac{\partial X_2}{\partial z_3}(z_2, 0) \\ \hat{W}_3(1, z_2, 0) & 0 & a z_2^{d+1} + q_{0,0} + \sum_{i=1}^{d} q_{i,0} z_2^i \end{pmatrix}$$

For simplicity in what follows, let

$$
\begin{aligned}
\hat{W}_{21}(z_2) &= \hat{W}_2 - z_2 \hat{W}_1(1, z_2, 0) \\
\hat{W}_3\ (z_2) &= \hat{W}_3(1, z_2, 0)
\end{aligned}
$$

Assume now that W^H lies in the subvariety $\mathscr{V}$ defined in Lemma 1.2, then the linear parts $D\tilde{V}$ at each p_k are, respectively,

$$\begin{pmatrix} -a & 0 & 0 \\ \hat{W}_{21}(0) & -a & 0 \\ \hat{W}_3(0) & 0 & 0 \end{pmatrix} \qquad \text{for } p_0 = (0,0,0) \qquad (1.27)$$

$$\begin{pmatrix} -a(d^2+3d+2) & 0 & 0 \\ \hat{W}_{21}(1) & a(1+d) & \frac{\partial X_2}{\partial z_3}(1, 0) \\ \hat{W}_3(1) & 0 & -a(d^2+3d+2) \end{pmatrix} \text{for } p_{d+1} = (0, 1, 0) \qquad (1.28)$$

and

$$\begin{pmatrix} a(1+d) & 0 & 0 \\ \hat{W}_{21}(\omega^k) & a(1+d) & \dfrac{\partial X_2}{\partial z_3}(\omega^k,0) \\ \hat{W}_3(\omega^k) & 0 & a(1+\sum_{j=1}^{d} \lambda^j \omega^{kj}) \end{pmatrix} \quad \text{for } p_k=(0,\omega^k,0),\ k=1,\dots,d \tag{1.29}$$

So, necesary and sufficient conditions for $D\tilde{V}$ to be non-semisimple at each p_k are, respectively,

$$\hat{W}_{21}(0)\neq 0 \qquad \text{for } p_0=(0,0,0), \tag{1.30}$$

$$\frac{\partial X_2}{\partial z_3}(1,0)\cdot\hat{W}_3(1)\neq 0 \quad \text{or} \quad [-a(d^2+3d+2)+a(d+1)]\,\hat{W}_3(1)\neq 0$$

$$\text{for } p_{d+1}=(0,1,0), \tag{1.31}$$

and

$$\frac{\partial X_2}{\partial z_3}(\omega^k,0)\cdot\hat{W}_3(\omega^k)+\hat{W}_{21}\cdot[a(1+d)-a(1+\sum_{j=1}^{d}\lambda^j\omega^{kj})]\neq 0$$

$$\text{for } p_k=(0,\omega^k,0), \qquad k=1,\dots,d \tag{1.32}$$

Remark: It is clear that we have two different eigenvalues in (1.27) and in (1.28) but this is not *a priori* true for (1.29). They are different if and only if

$$\mathbf{Q}(\lambda\omega^j)\neq d+1.$$

holds for every $j\in\{1,\dots,d\}$.

Anyhow, it is clear that, once the $(d+1)-$ homogeneous part has been choosen in $\mathscr{V}$, one can always find polynomials (1.25) such that equations (1.30), (1.31) and (1.32) are verified.

We now define the subvariety $\mathscr{W}$ mentioned at the begining of the paper.

Definition 1.3 *Let $\mathscr{W}$ be the complement, in $\mathscr{V}$, of the subvariety defined by (1.30), (1.31) and (1.32) when the inequality is replaced by equality.*

Summarizing, we have shown

Lemma 1.4 *$\mathscr{W}$ is a non-empty quasi-projective subvariety in the projectivized space of polynomial vector fields $\{V\}$ in $\mathbb{C}^3$, which have homogeneous non-zero terms only in degrees $d+1$ and $d+2$. For $V\in\mathscr{W}$, the blown-up foliation $\mathscr{F}_{\tilde{V}}$ satisfies conditions (i) to (v), with singular set $\mathscr{K}=\{p_0=(0,0,0),\ p_k=(0,\omega^k,\ 0);\ k=1,\dots,d+1\}$ and fixed line $\mathscr{L}$: $(z_1=z_3=0)$ in the coordinates (1.2). Each non-empty irreducible component has codimension at most $(d^2+13d+20)/2$.*

To say, the $(d+1)$-degree part of $\{V\}$ lies in $\mathscr{V}$ (of Lemma 1.2) and the $(d+2)$-terms satisfy the open conditions (1.30), (1.31) and (1.32).

Remark: If one could drop the $2(d+2)$ conditions required to choose an invariant line and the singularities on it, the codimension of the family thus founded would be d^2+3d+3.

2. NON EXISTENCE OF SEPARATRICES

Let $\mathscr{F}_0$ be the holomorphic foliation described by a holomorphic vector field X in a neighbourhood of 0 in $\mathbb{C}^3$. Let $\sigma_j\colon \mathbb{C}^3_j\to\tilde{\mathbb{C}}^3_{j-1}$, $j=1,\dots,r$, be a sequence of quadratic transformations based on p_{j-1}, $p_0=0$, E_0 a plane through 0, $E_j=\sigma_j^{-1}(p_{j-1})$, $p_j\in E_j$. For $k>j$ let E_j^k be the strict transform of E_j under $\sigma_k\circ\dots\circ\sigma_{j+1}$ and $E_k^k=E_k$. Let $\mathscr{F}_j$ be the (adapted) foliation obtained by pulling back under σ_j the foliation $\mathscr{F}_{j-1}$.

Definition 2.1. A singular point p of the foliation $\mathscr{F}_j$ is called a *simple corner* (see [3], p. 163) if we may find a coordinate chart (z_1,z_2,z_3) around p such that:

1) $p\in E^j_{j_0}\cap E^j_{j_1}$, with $j_0<j_1\leq j$ and $z_1z_2=0$ is a local equation for $E^j_{j_0}\cup E^j_{j_1}$ at p.

2) We may describe the foliation $\mathscr{F}_j$ in a neighbourhood of p by a means of a vector field of the form

$$Y=z_1(1+g_1)\frac{\partial}{\partial z_1}+z_2(\beta+g_2)\frac{\partial}{\partial z_2}+g_3\frac{\partial}{\partial z_3}\tag{2.1}$$

with β a complex number which is not a strictly positive rational number and $g_i(0)=0$, for $i=1,2,3$.

We note that in (2.1) g_1 may be taken to be 0, by dividing Y by $(1+g_1)$ and that β may be 0.

Theorem 2.2 *Assume that the vector field V lies in $\mathscr{W}$, as in Definition 1.3. Then every separatrix of $\mathscr{F}_{\tilde{V}}$ is contained in E.*

Proof: We shall consider three main cases, according to the different choices of pairs of repeated eigenvalues and subject to their corresponding non-semisimplicity conditions. Namely, the multiple point p_0 corresponds to the first case, the simple point p_{d+1} to the second, and the remaining simple points $\{p_1, \ldots, p_d\}$ to the third.

Case 1. We begin with p_0. Subject to condition (1.30), a linear change of coordinates puts (1.27) in Jordan canonical form. Then the typical element in the family looks like

$$Y(\zeta_1, \zeta_2, \zeta_3) = (\zeta_1 \cdot (-a + h_1),\ \zeta_1 - a\zeta_2 + h_2,\ \zeta_3 h_3)$$

where $deg\, h_1 \geq 1$, $deg\, h_3 \geq 1$, $deg\, h_2 \geq 2$.

The $-a$ and 0 eigenspaces are respectively $r_0 = (0:0:1)$ and $r_1 = (0:1:0)$ so they are isolated singularities of the blown-up foliation $\mathscr{F}_{\tilde{Y}}$ restricted to the new exceptional divisor E_1.

Let (y_1, y_2, y_3) be coordinates around r_0 such that

$$\sigma(y_1, y_2, y_3) = (y_1 y_3,\ y_2 y_3,\ y_3) = (\zeta_1, \zeta_2, \zeta_3) \tag{2.2}$$

The foliation $\mathscr{F}_{\tilde{Y}}$ is tangent to the divisor $D_0 (= E_0^1 \cup E_1$, where $E_0^1 = \sigma^{-1}(E))$ which in this coordinate chart is given by $y_1 y_3 = 0$, and is generated by

$$(D\sigma^{-1})\, Y = (y_1 \cdot (-a + y_3 H_1 - H_3),\ y_1 - a y_2 + y_3 H_2 - y_2 H_3,\ y_3 H_3)$$

where $H_1 = y_3^{-1} h_1$, $H_2 = y_3^{-2} h_3$, and $H_3 = y_3^{-1} h_3$. After dividing by $-a$ we see from this expression that r_0 is a simple corner with $\beta = 0$

Now let (x_1, x_2, x_3) be coordinates around r_1 such that

$$\sigma(x_1, x_2, x_3) = (x_1 x_2,\ x_2,\ x_3 x_2) = (\zeta_1, \zeta_2, \zeta_3) \tag{2.3}$$

In this coordinate chart D_0 is given by $x_1 x_2 = 0$ and the foliation is generated by

$(D\sigma^{-1})\, Y =$
$(x_1(-x_1 + x_2(G_1 - G_2)),\ x_2(-a + x_1 + x_2 G_2),\ x_3(a - x_1 + x_2 x_3)(G_3 - G_2))$

where now $G_1 = x_2^{-1} h_1$, $G_2 = x_2^{-2} h_2$, and $G_3 = x_2^{-1} h_3$. Again, after dividing by $-a$, we see from this expression that r_1 is a simple corner with $\beta = 0$.

Case 2. Consider the point p_d Subject to the second condition of (1.31), we put (1.28) in Jordan canonical form. With the conventions on the h_j's adopted in Case 1, we now consider the vector field

$$Y(\zeta_1, \zeta_2, \zeta_3) = (\zeta_1 \cdot (e_2 + h_1),\ e_3 \zeta_2 + h_2,\ \zeta_1 + e_2 \zeta_3 + \zeta_3 h_3)$$

$$\text{where } e_2 = -a(d^2 + 3d + 2),\ e_3 = a(d+1)$$

By an abuse of notation we have used the ζ_j's and Y to denote the coordinates and the vector field, as before.

Once again, the e_2 and e_3 eigenspaces are respectively $r_0 = (0:0:1)$ and $r_1 = (0:1:0)$. The blown-up foliation $\mathscr{F}_{\tilde{Y}}$ is tangent to the divisor $D_d = E_0^1 \cup E_d$, where $E_0^1 = \sigma^{-1}(E)$ and E_d stands for the new exceptional divisor, phrasing Case 1:

Around r_0 we use the coordinates given by (2.2). D_d is given by $y_1 y_3 = 0$ and $\mathscr{F}_{\tilde{Y}}$ is generated by

$$(D\sigma^{-1})\, Y =$$
$$(y_1(-y_1 + y_3(H_1 - H_3)), (e_3 - e_2) y_2 - y_1 y_2 + y_3(H_2 - y_2 H_3), y_3(e_2 + y_1 + y_3 H_3))$$

with the H_j's defined on Case 1. After dividing by e_2 we see from this expression that r_0 is a simple corner with $\beta = 0$.

For r_1 we use the coordinates given by (2.3). D_d is given by $x_1 x_2 = 0$ and the generator is

$$(D\sigma^{-1})\, Y =$$
$$(x_1[(e_2 - e_3) + x_2(G_1 - G_2)],\ x_2(e_3 + x_2 G_2),\ x_1 + (e_2 - e_3) x_3 + x_2 x_3 (H_3 - H_2))$$

with the G_j's defined on Case 1. Dividing by $e_2 - e_3$ we see that r_1 is a simple corner with $\beta = \dfrac{-1}{d+1}$.

Case 3. Consider the points $\{p_1, \ldots, p_d\}$. Since we are not able to evaluate the lower right eigenvalues $\mathbf{Q}(\lambda\omega^k)$, $k = 1, \ldots, d$, we must consider the various relevant posibilities on the relative position of them in the complex plane. Call them $e_3 = a(d+1)$ and $c_k = a \cdot \mathbf{Q}(\lambda\omega^k) = a(1 + \sum_{j=1}^{d} \lambda^j \omega^{kj})$.

Case 3.1. If $e_3 = c_k$ for some $k \in \{1, \ldots, d\}$, then the condition (1.32) of non-semisimplicity turns into one of

$$\hat{W}_{21}(\omega^k),\ \hat{W}_3(\omega^k),\ \text{or}\ \frac{\partial X_2}{\partial z_3}(\omega^k, 0) \neq 0 \tag{2.4}$$

since condition (*iv*) might be satisfied, we are forced to choose the first or the second one (say the first). Then, after a linear change of coordinates around p_k, $\mathscr{F}_Y$ is generated by

$$Y(\zeta_1, \zeta_2, \zeta_3) = (\zeta_1 \cdot (e_3 + h_1),\ \zeta_1 + e_3 \zeta_2 + h_2,\ \zeta_3 \cdot (e_3 + h_3)) \tag{2.5}$$

Y is *linearizable* keeping E invariant (see the Remarks after Lemma A3 and Proposition A, in the Appendix). Explicit integration of the linear vector field obtained shows that, by the condition chosen on (2.4), every separatrix of Y is contained in E.

Case 3.2. When e_3 and c_k are different, the situation is similar to Case 2: with the notation of the preceeding cases, a linear change of coordinates allows us to consider the vector field

$$Y(\zeta_1, \zeta_2, \zeta_3) = (\zeta_1 \cdot (e_3 + h_1),\ \zeta_1 + e_3 \zeta_2 + h_2,\ \zeta_3 \cdot (c_k + h_3)) \tag{2.6}$$

The e_3 and c_k eigenspaces in this coordinates are, respectively, $r_1 = (0:1:0)$ and $r_0 = (0:0:1)$. $\mathscr{F}_{\tilde{Y}}$ is tangent to the divisor D_k, analogously defined.

With the notation of Case 2, around r_1 and r_0, D_k is, respectively, locally given by $x_1 x_2 = 0$ and $y_1 y_3 = 0$.

Around $r_1 = (0:1:0)$, the foliation is generated by

$$(D\sigma^{-1})\, Y =$$
$$(x_1[-x_1 + x_2(G_1 - G_2)],\ x_2[e_3 + x_1 + x_2 G_2],\ x_3[(c_k - e_3) - x_1 + x_2(G_3 - G_2)])$$

Dividing by e_3 we see that r_1 is a simple corner with $\beta = 0$.

The generator near $r_0 = (0:0:1)$ is

$$(D\sigma^{-1})\, Y = \tag{2.7}$$
$$(y_1[(e_3 - c_k) + y_3(H_1 - H_3)],\ y_1 + (e_3 - c_k) y_2 + y_3(H_2 - y_2 H_3),\ y_3[c_k + y_3 H_3])$$

The following discussion contains concepts and results which are detailed at the Appendix.

Let $\Lambda=\{e_3, c_k\}\subset\mathbb{C}$.

If Λ is in the Siegel Domain ($\frac{e_3}{c_k}\in\mathbb{R}^-$), one divides by c_k in (2.7). The expression obtained shows that r_0 is a simple corner with $\beta=\frac{e_3-c_k}{c_k}\in\mathbb{R}^-$

If Λ is in the Poincaré Domain, two main cases arise:

The non-resonant case. The vector field (2.6) is linearizable keeping E invariant and one can integrate the linear vector field thus obtained. Its separatrices are contained in E.

In the **resonant case,** there are two possibilities:

If $c_k=ne_3$ for some $n\in\mathbb{N}\backslash\{1\}$, then r_0 is a simple corner with the same β of the Siegel Case, since now it equals $\frac{1}{n}-1\in\mathbb{R}^-$.

If $e_3=rc_k$ for some $r\in\mathbb{N}\backslash\{1\}$, the preceeding arguments must be modified since some separatrices may appear outside the exceptional divisor E. Indeed, in this case we may rewrite (2.6) as

$$Y=\begin{pmatrix} r & 0 & 0\\ 1 & r & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\zeta_1\\ \zeta_2\\ \zeta_3\end{pmatrix}+\begin{pmatrix}\zeta_1\cdot h_1\\ h_2\\ \zeta_3\cdot h_3\end{pmatrix}\quad (r\in\mathbb{N}\backslash\{1\} \tag{2.8}$$

Let $A=A\zeta$ denote the linear part of Y. The unique **P**-resonant monomial associated to the homological equation (A.5) is $\zeta_3^r\,\frac{\partial}{\partial\zeta_2}$ (Lemma A3). From Proposition A, Y is analytically conjugated, keeping the plane $E\colon(\zeta_1=0)$ invariant, with the vector field

$$Y_*=\begin{pmatrix} r & 0 & 0\\ 1 & r & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\zeta_1\\ \zeta_2\\ \zeta_3\end{pmatrix}+\begin{pmatrix}0\\ \zeta_3^r\frac{\partial}{\partial\zeta_2}\\ 0\end{pmatrix} \tag{2.9}$$

The holomorphic change of coordinates

$$F(\zeta_1,\zeta_2,\zeta_3)=(\zeta_1+\zeta_3^r,\zeta_2,\zeta_3)=(w_1,w_2,w_3)$$

conjugates Y_* with the linear vector field A but takes the invariant plane $E:(\zeta_1=0)$ onto the invariant surface $S:(w_1-w_3^{\zeta}=0)$.

The linear vector field Aw has separatrices on the (invariant) plane $w_1=0$ which are not contained in S.

We do not know if this type of resonances appear or not in our examples since we are not able to compute the roots of (1.21). However, we can ensure that they will not appear if we make a particular choice of the root λ of (1.21). First we show that for each choice of λ there is at most one resonance.

Lemma 2.3. *If $e_3=rc_k$ for some $r\in\mathfrak{N}\setminus\{1\}$, and some $k\in\{1,\dots,d\}$, then k is unique.*

Proof: Supose that there exists $j\in\{1,\dots,d\}$ and $s\in\mathfrak{N}\setminus\{1\}$, such that $s(\lambda^{d+1}-1)/(\lambda w^j-1)=d+1$, then $(\lambda w^k-1)/(\lambda w^j-1)=r/s\in\mathbf{Q}^+$ and this implies that (λw^k-1) and (λw^j-1) have the same argument (mod 2π). On the other hand, they lie in the circle centered at -1 of radius $|\lambda|>1$ and $0\in\Delta(-1,|\lambda|)$. This implies that both points are the same and $j=k$

Now, $r\cdot\mathbf{Q}(\lambda w^k)=d+1$ for some $r\in\mathfrak{N}\setminus\{1\}$ and some $k\in\{1,\dots,d\}$, if and only if $\lambda=(1+r(d+2))/(w^k+r(d+2))$. The real part of this complex number is positive for every choice of r, k and d and this computation shows that if a resonance occurs, then the chosen root necesarily has positive real part. If this number fails to be a root of (1.20) then we are done and the resonance relation cannot appear, but if it is, all we have to do is to choose λ with negative real part. This root always exists, since the sum of the roots of (1.21) equals -1 (because the coefficient of x^{d-1} of (1.21) is 1). In particular, if the degree d is odd, then the (negative) real root of (1.20) is a good choice.

Let $\mathscr{F}$ be a holomorphic foliation with singular point p. If V a germ of a holomorphic vector field at p defining $\mathscr{F}$, recall that the *algebraic multiplicity* of $\mathscr{F}$ at p is the degree of the smallest non-zero coefficient in the power series expansion of V.

Adding up the former condition on λ to the definition of Ψ, we have

Theorem 2.4 *Let V be a germ at 0 of a holomorphic vector field on $\mathbb{C}^3$, with algebraic multiplicity $d+1$ such that its $(d+2)$-jet belongs to $\mathscr{W}$, then V does not have a separatrix through 0.*

Remark. Although $\mathcal{W}$ has big dimension, the freedom in the above calculation is constrained just to one discrete variable ($\lambda \in \mathbb{C}$ which satisfies (1.20)). There are many variables in $\mathcal{W}$ which never apeared (for example, $h_{ij,k}$, $k>0$).

A APPENDIX

Here we discuss the normal forms for the vector fields obtained in Chapter 2. A complete exposition on the general theory can be found in [1].

Consider (the germ at 0 of) an autonomous differential equation defined by (the germ at 0 of) a holomorphic vector field

$$\dot{x} = Ax + \sum_{m \geq 2} v_m(x) \qquad x \in \mathbb{C}^n \tag{A.1}$$

Let $(x_1, \ldots, x_n)$ denote coordinates with respect to the basis $(e_1, \ldots, e_n)$ and $x^m = x_1^{m_1} \ldots x_n^{m_n}$. denote by

$$L_A : \Xi^m_{(\mathbb{C}^n,0)} \to \Xi^m_{(\mathbb{C}^n,0)} \tag{A.2}$$

the linear operator which transforms each homogeneous polynomial vector field of degree (or weight) m, into the Poisson Bracket of the linear vector field Ax, with it.

Let $(\mathbf{m}, \phi) = \sum_{j=1}^n m_j \phi_j$. If $A = \mathrm{diag}\{\phi_j\}$ then L_A is diagonal too

$$L_A(x^m e_j) = [\phi_j - (\mathbf{m}, \phi)]\, x^m e_j \tag{A.3}$$

If A has Jordan blocks, then L_A also has Jordan blocks, but even in this case, its eigenvalues are given by the formula above.

A n-tuple $\phi = (\phi_1, \ldots, \phi_n) \in \mathbb{C}^n$ is called *resonant* if there exists a relation $\phi_j - (\mathbf{m}, \phi) = 0$, for some $j \in \{1, \ldots, n\}$; otherwise is called *nonrsonant*. A vector field (A.1) is called resonant (nonresonant) if the eigenvalues of A are (are not) resonant. In the presence of resonances, the corresponding eigenvector of (A.3) is called a *resonant monomial* or simply a *resonance*.

We say that the n-tuple $\phi = (\phi_1, \ldots, \phi_n)$ of eigenvalues of A is in the Poincaré Domain if $0 \in \mathbb{C}$ is not contained in the convex hull of ϕ.

Poincaré-Dulac Theorem *If the eigenvalues ϕ of the linear part A of a holomorphic vector field at a singular point belong to the Poincaré Domain, then, in a neighbourhood of the singular point:*

1. *If ϕ is nonresonant, the vector field is biholomorphically equivalent to the linear vector field A.*

2. *If ϕ is resonant, it is biholomorphically equivalent to a polynomial normal form of the type $A+\{$resonant monomials$\}$.*

The proof is divided in two parts, the first of which is the *formal* construction of the desired change of coordinates: For each $m \geq 2$, the solution h_m of the homological equation associated to (A.1)

$$L_A h_m = v_m \tag{A.4}$$

gives a (polynomial, tangent to the identity) change of coordinates which annihilates all nonresonant monomials on that fixed m. Repeated aplication of this procedure gives rise to a sequence of changes of coordinates whose product, in the limit, is the one we are looking for.

In the second step, the convergence of that power series is proved, using the fact that the eigenvalues belong to the Poincaré Domain.

To say, *at most* the resonances will not be annihilated by this change of coordinates.

Now we come back to the vector field (2.8), in $\mathbb{C}^3$. We will obtain a Poincaré-Dulac-type normal form for it.

Let $\Xi^s_{(\mathbf{P},0)}$ denote the space of homogenoeus polynomial vector fields of degree s tangent to the plane $E:(\zeta_1=0)$, *i.e.*, those for which the $\frac{\partial}{\partial \zeta_1}$-component lies in the ideal (ζ_1). Consider the restriction $\widehat{L_A}$ of (A.2) to them.

Lemma A1 *$\widehat{L_A}$ leaves $\Xi^s_{(\mathbf{P},0)}$ invariant, for every degree s:*

$$\widehat{L_A}: \Xi^s_{(\mathbf{P},0)} \to \Xi^s_{(\mathbf{P},0)} \tag{A.5}$$

Proof: Make the calculation using (A.6) below.

A monomial in $\Xi^s_{(\mathbf{P},0)}$ will be called **P**-*resonant* if it is not in the range of $\widehat{L_A}$.

Lemma A2 *Let A be the linear part of (2.8), $s>1$, $r>1$. The resonant monomials asociated to $\widehat{L_A}: \Xi^s_{(\mathbb{C}^3,0)} \to \Xi^s_{(\mathbb{C}^3,0)}$ are $\zeta_3^r \frac{\partial}{\partial \zeta_1}$ and $\zeta_3^r \frac{\partial}{\partial \zeta_2}$.*

Proof: Let $\mathbf{e}^k_{lmn}=\zeta_1^l\zeta_2^m\zeta_3^n\frac{\partial}{\partial\zeta_k}$, $k=1, 2, 3$ and $l+m+n=s$. Then

$$L_A(\mathbf{e}^k_{lmn})=\begin{cases}(lr+mr+n-r)\,\mathbf{e}^1_{lmn}+m\,\mathbf{e}^1_{l+1,m-1,n}-\mathbf{e}^2_{lmn} & \text{if } k=1\\ (lr+mr+n-r)\,\mathbf{e}^2_{lmn}+m\,\mathbf{e}^2_{l+1,m-1,n} & \text{if } k=2\\ (lr+mr+n-1)\,\mathbf{e}^3_{lmn}+m\,\mathbf{e}^3_{l+1,m-1,n} & \text{if } k=3\end{cases} \tag{A.6}$$

Since s, $r>1$, then $lr+rm+n-1$ is allways non-zero. $lr+rm+n-r=0$ implies that $l+m=0$ and $s=n=r$. This shows that $\mathbf{e}^1_{00r}=\zeta_3^r\frac{\partial}{\partial\zeta_1}$ and $\mathbf{e}^2_{00r}=\zeta_3^r\frac{\partial}{\partial\zeta_2}$ are the only resonances.

Lemma A3 Let *A be the linear part of* (2.8). *The unique* **P**-*resonant monomial asociated to* (A.5) *is* $\zeta_3^r\frac{\partial}{\partial\zeta_2}$.

Proof: By Lemma A2, $\zeta_3^r\frac{\partial}{\partial\zeta_2}$ is **P**-resonant. (but $\zeta_3^r\frac{\partial}{\partial\zeta_1}$ is not). A **P**-resonant monomial which is not resonant necesarily lies in $L_A(\Xi^s_{(\mathbb{C}^n,0)}\setminus\Xi^s_{(\mathbf{P},0)})$.

Let $\mathbf{e}^k_{lmn}\in\Xi^s_{(\mathbf{P},0)}$ be **P**-resonant. For each degree $s\geq 2$, $\Xi^s_{(\mathbb{C}^n,0)}\setminus\Xi^s_{(\mathbf{P},0)}$ is generated by the monomial vector fields of the form

$$\mathbf{e}^1_{0mn}=\{\zeta_2^m\zeta_3^n e_1 : m+n=s;\ m,\ n=0, 1, \dots\}$$

From (A.6):

$$L_A(\mathbf{e}^1_{0mn})=(mr+n-r)\,\mathbf{e}^1_{0mn}+m\,\mathbf{e}^1_{1,m-1,n}-\mathbf{e}^2_{0mn}$$

The last two summands on the right belong to $\Xi^s_{(\mathbf{P},0)}$. Hence $L_A(\mathbf{e}^1_{0mn})\in\Xi_{(\mathbf{P},M)}$ if and only if $0=n+r(m-1)$, if and only if $n=0$ and $m=1$ or $n=r$ and $m=0$. Since the first one is linear, the only resonance is $\mathbf{e}^2_{00r}=\zeta_3^r\frac{\partial}{\partial\zeta_2}$

Remark. Case 3.1 admits a similar treatment: one may assume that the linear part of (2.5) is that of (2.8) with $r=1$. From the calculations above, the homological equation associated to this operator has no **P**-resonances.

Proposition A. *Let Y denote the vector field* (2.8). *There exists a holomorphic change of coordinates* $x=\varphi(\zeta)$ *such that* $(\zeta_1=0)_*=(x_1=0)$ *and* $\varphi_* Y = Ax + \tau \cdot x_3^r \frac{\partial}{\partial x_2}$, *for some* $\tau \in \mathbb{C}$.

Sketch of Proof: The proof follows by phrasing that one from the usual Poincaré-Dulac Theorem: By Lemma A1, one is allowed to consider the homological equation associated with the operator (A.5). It is solvable for every degree $s \neq r$ by Lemma A3, and the solutions produce a sequence of changes of coordinates which leave invariant the plane E. The product of them, in the limit, in principle gives rise to a formal conjugation with the normal form in question. However, since the eigenvalues belong to the Poincaré Domain, the usual majorant norm estimates show that this formal conjugation is indeed convergent.

Corolary-Remark. *Case 3.1 is linearizable keeping E invariant, as well as* (2.6) *in the non-resonant case.*

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