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*Tritangent Planes to Toroidal Knots**

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ABSTRACT. A proof is given that, with the only exception of (3,2), all toroidal knots in $\mathfrak{R}^3$ obtained in the standard way by stereographic projection of knots in S^3 have tritangent planes.

Montesinos and Nuño [2] and Morton [3] have proved that two different presentations of the trefoil knot as a toroidal knot of type (3, 2) have no tritangent planes, answering thus in the negative a conjecture of Freedman [1].

Let p, q be relative prime integers and let a, $b>0$ be such that $a^2+b^2=1$; in the following, we shall put $a=\cos A$, $b=\sin A$, with $0<A<\frac{\pi}{2}$. We have a toroidal knot in $S^3\subset\mathfrak{R}^4$ given by the equations ($a\cos pt$, $a\sin pt$, $b\cos qt$, $b\sin qt$), $0\leq t\leq 2\pi$. Applying to it the stereographic projection given by $(x, y, z, w)\to\frac{1}{1-w}(x, y, z)$, we get the (p, q)–knot

$$N_{p,q}(t)=\frac{1}{1-b\sin qt}(a\cos pt,\ a\sin pt,\ b\cos qt),\quad 0\leq t\leq 2\pi, \tag{1}$$

on the torus in $\mathfrak{R}^3$ of equation

$$\left(\frac{1}{a}-\sqrt{x^2+y^2}\right)^2+z^2=\frac{b^2}{a^2}.$$

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We shall prove that all such knots, with the unique exception of the (3, 2) one, have tritangent planes.

As in [3], let $r_1 x + r_2 y + r_3 z = r$ be the equation of a plane P in $\mathbb{R}^3$. The points of $N_{p,q}$ that belong to P correspond to the values of t that satisfy

(2) $$r_1 a \cos pt + r_2 a \sin pt + r_3 b \cos qt + rb \sin qt = r.$$

Also, P is tangent to $N_{p,q}$ at $N_{p,q}(t)$ iff t satisfies (2) and its derivative with respect to t, that is

(3) $$-pr_1 a \sin pt + pr_2 a \cos pt - qr_3 b \sin qt + qrb \cos qt = 0.$$

We put $z = e^{it}$ in (2) and (3) and get the equations

(4) $$uz^p + \bar{u}\bar{z}^p + vz^q + \bar{v}\bar{z}^q - 2r = 0,$$

(5) $$p(uz^p - \bar{u}\bar{z}^p) + q(vz^q - \bar{v}\bar{z}^q) = 0,$$

where we have put

(6) $$u = a(r_1 - ir_2), \quad v = b(r_3 - ir).$$

Thus, the number of points of tangency of P with $N_{p,q}$ is the number of values of z in the unit circle of $\mathbb{C}$ that satisfy (4) and (5).

Let $r_1 = r_2 = 0$, $r_3 = a/b$, $r = 1$. Then $u = 0$ and $v = a - ib = e^{-iA}$. With these data, P is one of the two planes that are tangent to the torus along a whole circle. Then, (5) says that vz^q must be real, and consequently, (4) is equivalent to $vz^q = r = 1$, that is $z^q = e^{iA}$. Therefore, as expected, P has q points of tangency with $N_{p,q}$, given by the roots of the last equation. So, we can henceforth assume that $q = 2$, $p \geq 5$.

Lemma. *Let $p \geq 5$ be an odd integer; then, there exist a solution, τ, of the equation*

(7) $$p \sin t \cos \frac{p}{2} t - 2 \cos t \sin \frac{p}{2} t = 0,$$

such that $0 < \tau < \pi$.

Proof. Let $f(t) = p \sin t \cos \frac{p}{2} t - 2 \cos t \sin \frac{p}{2} t$. If $p \geq 7$, then

$$f\left(\frac{2\pi}{p}\right)=p\sin\frac{2\pi}{p}\cos\pi-2\cos\frac{2\pi}{p}\sin\pi=-p\sin\frac{2\pi}{p}<0$$

$$f\left(\frac{3\pi}{p}\right)=p\sin\frac{3\pi}{p}\cos\frac{3\pi}{2}-2\cos\frac{3\pi}{p}\sin\frac{3\pi}{2}=2\cos\frac{3\pi}{p}>0.$$

If $p=5$, we consider instead the interval $\left[\frac{3\pi}{5},\frac{4\pi}{5}\right]$:

$$f\left(\frac{3\pi}{5}\right)=5\sin\frac{3\pi}{5}\cos\frac{3\pi}{2}-2\cos\frac{3\pi}{5}\sin\frac{3\pi}{2}=2\cos\frac{3\pi}{5}<0$$

$$f\left(\frac{4\pi}{5}\right)=5\sin\frac{4\pi}{5}\cos 2\pi-2\cos\frac{4\pi}{5}\sin 2\pi=5\sin\frac{4\pi}{5}>0$$

In the first case, there is such a solution in $\left(\frac{2\pi}{p},\frac{3\pi}{p}\right)$; in the second, in $\left(\frac{3\pi}{5},\frac{4\pi}{5}\right)$, as claimed. ∎

In the following, τ will stand for a solution of (7) in $(0,\pi)$. We remark that from (7) it can be easily seen that $\sin 2\tau\neq 0$ and $\sin p\tau\neq 0$. Also, dividing (7) by $\cos\tau\cos\frac{p}{2}\tau$ we have

$$p\tan\tau=2\tan\frac{p}{2}\tau. \tag{8}$$

We define B by the conditions

$$\cot B=-\frac{2\sin 2\tau}{p\sin p\tau}\tan A,\quad 0<B<\pi. \tag{9}$$

We prove now that $0<\cos(A-B)/\sin B<1$. We have

$$\frac{\cos(A-B)}{\sin B}=\sin A+\cos A\cot B=\sin A\left(1-\frac{2\sin 2\tau}{p\sin p\tau}\right)$$

$$=\sin A\left(1-\frac{2\tan\tau\cos^2\tau}{p\tan\frac{p}{2}\tau\cos^2\frac{p}{2}\tau}\right)=(\textit{by}\,(8))\sin A\left(1-\frac{4\cos^2\tau}{p^2\cos^2\frac{p}{2}\tau}\right)$$

$$= \sin A\left(1-\frac{4(1+\tan^2\frac{p}{2}\tau)}{p^2(1+\tan^2\tau)}\right) = (by\ (8))\sin A\left(1-\frac{4(1+\frac{p^2}{4}\tan^2\tau)}{p^2(1+\tan^2\tau)}\right)$$

$$= \sin A\left(1-\frac{\frac{4}{p^2}+\tan^2\tau}{1+\tan^2\tau}\right).$$

Our claim follows from the fact that $\frac{4}{p^2}<1$. We define C by

$$\sin C = \frac{\cos(A-B)}{\sin B}, \quad 0<C<\frac{\pi}{2}. \tag{10}$$

We have now the ingredients for exhibiting a plane triply tangent to $N_{p,2}$. Let us put

$$r_1 = \cos B \cos\frac{p}{2}C, \quad r_2 = \cos B \sin\frac{p}{2}C, \quad r_3 = \sin B \cos C, \ r = \sin B \sin C.$$

Then

$$uz^p = \cos A \cos B e^{ip(t-\frac{1}{2}C)}, \quad vz^2 = \sin A \sin B e^{i2(t-\frac{1}{2}C)}.$$

Hence, one half of the left hand side of (4) is

$$\cos A \cos B \cos p(t-\tfrac{1}{2}C) + \sin A \sin B \cos 2(t-\tfrac{1}{2}C) - \cos(A-B)$$

$$= \cos A \cos B(\cos p(t-\tfrac{1}{2}C)-1) + \sin A \sin B(\cos 2(t-\tfrac{1}{2}C)-1)$$

So, it is obvious that $t=\frac{1}{2}C$ is a solution of (4). We show that $t=\tau+\frac{1}{2}C$ so is. In this case, by substitution in the last expression we obtain

$$\cos A \cos B(\cos p\tau - 1) + \sin A \sin B(\cos 2\tau - 1)$$

$$= -2(\cos A \cos B \sin^2\tfrac{p}{2}\tau + \sin A \sin B \sin^2\tau)$$

$$= -2\cos A \sin B(\cot B \sin^2\tfrac{p}{2}\tau + \tan A \sin^2\tau)$$

$$= -2\sin A \sin B\Big(-\frac{2\sin\tau\cos\tau}{p\sin\frac{p}{2}\tau\cos\frac{p}{2}\tau}\sin^2\frac{p}{2}\tau+\sin^2\tau\Big)$$

$$= -2\sin A \sin B\frac{\sin\tau}{p\cos\frac{p}{2}\tau}\Big(p\sin\tau\cos\frac{p}{2}\tau-2\cos\tau\sin\frac{p}{2}\tau\Big)$$

$$=0$$

by the Lemma. Therefore, $t=\frac{1}{2}C-\tau$ is clearly another solution. Let us show that the three values satisfy (5). The left hand side of this equation is now, up to a non-zero factor,

$$p\cos A\cos B\sin p\Big(t-\frac{1}{2}C\Big)+2\sin A\sin B\sin 2\Big(t-\frac{1}{2}C\Big).$$

For $t=\frac{1}{2}C$ this zero. For $t=\tau+\frac{1}{2}C$ this is

$$p\cos A\cos B\sin p\tau+2\sin A\sin B\sin 2\tau$$

$$=p\cos A\sin B\sin p\tau\left(\cot B+\frac{2\sin 2\tau}{p\sin p\tau}\tan A\right)$$

$$=0,$$

by (9). Clearly, the same occurs for $t=\frac{1}{2}C-\tau$. Since the three values modulo 2π are different, we have proved the announced result.

Theorem. *Let $p, q\geq 2$ be relative prime integers and $(p, q)\neq(3, 2)$; then, the (p, q) toroidal knot $N_{p,q}$ given by (1) has a tritangent plane.*

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