REVISTA MATEMATICA de la
Universidad Complutense de Madrid
Volumen 4, números 2 y 3; 1991.
http://dx.doi.org/10.5209/rev_REMA.1991.v4.n2.17959

On Exact Sequences of Quojections

G. METAFUNE and V. B. MOSCATELLI

ABSTRACT. We give some general exact sequences for quojections from which many interesting representation results for standard twisted quojections can be deduced. Then the methods are also generalized to the case of nuclear Fréchet spaces.

INTRODUCTION

A Fréchet space E is a *quojection* if it is the projective limit of a sequence (E_n, R_n) of Banach spaces E_n and surjective maps $R_n: E_{n+1} \to E_n$. In this case we write $E = \text{quoj}_n(E_n, R_n)$. A quojection is called *twisted* if it is not isomorphic to a countable product of Banach spaces.

It is well known that a quotient of a quojection is again a quojection (possibly Banach). Here we want to investigate quotients of products with respect to quojection subspaces with regard to the property of being twisted or not. This has an intrinsic interest as well as implications to the existence or not of unconditional bases. It is clear that the problem is equivalent to the study of short exact sequences of quojections of the form

$$(*) \qquad 0 \to F \to G \to E \to 0,$$

where, in our case, G is a product. Thus, in §1 we characterize the vanishing of the first derived functor $\text{Ext}^1(E, F)$, which is equivalent, for fixed E and F, to the splitting of any exact sequence as (*). To introduce the argument we show that, for any given quojection E, there always exist a quojection F and a cardinal d for which we have an exact sequence as (*) with $G = (l_d^1)^{\mathbb{N}}$. This

Research supported by the Italian Ministry of Education (MPI).
1991 Mathematics Subject Classification: 46A06, 46A12; 46A35.
Editorial de la Universidad Complutense. Madrid, 1991.

could also be deduced by carrying on further the proof of the result in [1], but, for the convenience of the reader (and because the effort is the same), we prefer to give here the full proof. In §2 we then give a general exact sequence for standard quojections. Based on the fact, proved in [16], that if E or F are standard twisted quojections, then no sequence (*) can split, in §3 we are able to obtain from our general sequence many interesting representation results involving standard twisted quojection and, consequently, unconditional bases. Last, but not least, applications are made in §4 to the case of nuclear spaces.

The notation we follow is standard, but, for brevity's sake, we use the term «surjection» to mean a linear and continuous surjective map.

1. ON THE VANISHING OF THE FUNCTOR Ext^1 FOR QUOJECTIONS

Theorem 1.1. *If E is a quojection, then there is a cardinal d and a surjection $T:(l^1_d)^{\mathbb{N}} \to E$ such that ker T is a quojection.*

Proof. By definition we have $E = \mathrm{quoj}_n(E_n, R_n)$, where the E_n are Banach spaces and the maps $R_n: E_{n+1} \to E_n$ are surjective. Let $d \geq \mathrm{dens}\,(E)$ and let $T_1: l^1_d \to E_1$ be a surjection. Supposing that $T_n: (l^1_d)^n \to E_n$ has been defined, let $\tilde{T}_n$ be a lifting of T_n into E_{n+1} (i.e., $R_n \tilde{T}_n = T_n$), let $S_n: l^1_d \to \ker R_n$ be a surjection and define $T_{n+1}: (l^1_d)^n \times l^1_d \to E_{n+1}$ by $T_{n+1} = \tilde{T}_n + S_n$. The required map is then obtained by setting $T(x_n) = (T_n(x_1, \ldots, x_n))_n$ for $(x_n) \in (l^1_d)^{\mathbb{N}}$. For the surjectivity, take any $(y_n) \in E$; since $y_1 \in E_1$ there is $x_1 \in l^1_d$ with $y_1 = T_1 x_1 = R_1 \tilde{T}_1 x_1$. But $y_1 = R_1 y_2$ with $y_2 \in E_2$, hence $y_2 - \tilde{T}_1 x_1 = S_1 x_2$ for some $x_2 \in l^1_d$, i.e. $y_2 = T_2(x_1, x_2)$. Inductively we obtain $y_n = T_n(x_1, \ldots, x_n)$ for all n, hence $(y_n) = T(x_n)$ with $(x_n) \in (l^1_d)^{\mathbb{N}}$. It remains to show that ker T is a quojection.

For each n let P_n: $(l^1_d)^{n+1} \to (l^1_d)^n$ be the canonical projection; then $\ker T = \mathrm{proj}_n(\ker T_n, P_n)$. Let $z = (x_1, \ldots, x_n) \in \ker T_n$; then $0 = T_n z = R_n \tilde{T}_n z$ implies $\tilde{T}_n z \in \ker R_n$ and we may choose $x_{n+1} \in l^1_d$ with $\tilde{T}_n z + S_n x_{n+1} = 0$. This means $(z, x_{n+1}) \in \ker T_{n+1}$ hence each $P_n: \ker T_{n+1} \to \ker T_n$ is surjective and ker T is a quojection, as claimed.

Theorem 1.1 is the analogue for quojections of the classical result for Banach spaces. However, the analogy stops here. In fact, while for a separable Banach space X not isomorphic to l^1 there is a unique, up to an automorphism of l^1, surjection $T: l^1 \to X$ (cf. [10, I, Theorem 2.f.8., p. 108]), so that the kernels of all such maps T are isomorphic, for quojections the situation is completely different, as the following example shows.

Example 1.2. Let $T: (l^1)^{\mathbb{N}} \to (l^2)^{\mathbb{N}}$ be the canonical surjection induced by the quotient map $q: l^1 \to l^2$. Clearly, ker $T = (\ker q)^{\mathbb{N}}$ and so ker T is a quojection. Now let N be a nuclear Fréchet space not isomorphic to ω: by [11, Proposition 1.14] $(l^2)^{\mathbb{N}}/N$ is isomorphic to $(l^2)^{\mathbb{N}}$ and we denote by J one such isomorphism. It follows that, if $Q: (l^2)^{\mathbb{N}} \to (l^2)^{\mathbb{N}}/N$ is the quotient map, then the map $S = JQT$ is a continuous surjection of $(l^1)^{\mathbb{N}}$ onto $(l^2)^{\mathbb{N}}$ for which ker S $(= T^{-1}(N))$ is not a quojection, hence not isomorphic to ker T.

Remark 1.3. At this stage one might think that the kernels of two maps from $(l^1)^{\mathbb{N}}$ onto the same separable quojection (not isomorphic to a complemented subspace of a product of the form $X \times (l^1)^{\mathbb{N}}$ with X Banach) would have to be isomorphic if they were both quojections. However, not even this is true, as will be shown in Example 3.13 below.

In the rest of this section we shall discuss some consequences of Theorem 1.1 and related results. We recall that the vanishing of the first derived functor $\mathrm{Ext}^1(E, F)$ for Fréchet spaces E and F means that every exact sequence

$$0 \to F \to G \to E \to 0$$

splits and, consequently, $G \simeq E \oplus F$. For a discussion of this topic see [20].

Proposition 1.4. *Let E be a quojection. The following are equivalent:*

(i) $\mathrm{Ext}^1(E, F) = 0$ *for every quojection F;*

(ii) *E is isomorphic to a complemented subspace of $(l^1_d)^{\mathbb{N}}$ for some cardinal d (and hence either $E \simeq l^1_{d'}, \omega, l^1_{d'} \times \omega$ $(d' \leq d)$ or E contains a complemented copy of $(l^1)^{\mathbb{N}}$).*

Proof. (i) $\to$ (ii): By Theorem 1.1 there is an exact sequence

$$0 \to \ker T \to (l^1_d)^{\mathbb{N}} \xrightarrow{T} E \to 0$$

and by assumption this sequence splits. Then the assertion in brackets follows essentially from the proof of Theorem 1.2 in [15].

(ii) $\to$ (i): This is a consequence of [20, Remark b), pag. 173].

Remark 1.5. In the separable case, assertion (ii) above may be replaced by $E \simeq l^1, \omega, l^1 \times \omega$ or $(l^1)^{\mathbb{N}}$ (cf. [15, Theorem 1.2]).

Proposition 1.6. *Let F be a Fréchet space. The following are equivalent:*

(i) $\mathrm{Ext}^1(E, F)=0$ *for every quojection E;*

(ii) *F is isomorphic to a complemented subspace of* $(l_d^\infty)^{\mathbb{N}}$ for some cardinal d (and hence either $F \simeq X, \omega, X \times \omega$ *(X an injective Banach space) or F contains a complemented copy of* $(l^\infty)^{\mathbb{N}}$).

Proof. (i) → (ii): Let d be such that F is a subspace of $(l_d^\infty)^{\mathbb{N}}$. By assumption the exact sequence

$$0 \to F \to (l_d^\infty)^{\mathbb{N}} \to (l_d^\infty)^{\mathbb{N}}/F \to 0$$

splits and (ii) follows from [15, Proposition 3.8 or Propisition 3.14].

(ii) → (i): This is immediate, since F is injective.

Proposition 1.7. *Let F be a Fréchet space. The following are equivalent:*

(i) $\mathrm{Ext}^1((l_d^1)^{\mathbb{N}}, F)=0$ *for every cardinal d;*

(ii) $\mathrm{Ext}^1((l^1)^{\mathbb{N}}, F)=0$;

(iii) $\mathrm{Ext}^1(\omega, F)=0$;

(iv) F is a quojection.

Proof. (i) → (ii) → (iii): Clear.

(iii) → (iv): By [19, Theorem 5.2].

(iv) → (i): By Proposition 1.4.

Proposition 1.8. *Let F be a separable Fréchet space. The following are equivalent:*

(i) $\mathrm{Ext}^1(l^1, F)=0$;

(ii) *F is quasinormable.*

Proof. First of all, observe that $\mathrm{Ext}^1(l^1, F)=0$ is quickly seen to be equivalent to the following: for any exact sequence

$$(1) \qquad 0 \to F \to G \xrightarrow{q} E \to 0$$

with G separable and Fréchet and $F=\ker q$, every map $T: l^1 \to E$ can be lifted to a map $\tilde{T}: l^1 \to G$ such that $q\tilde{T}=T$. In turn, this happens if and only if q lifts the bounded subsets of E to bounded subsets of G. Thus it suffices to show that F is quasinormable if and only if for any sequence (1) the map q lifts the bounded subsets. Now if F is quasinormable, then q lifts the bounded subsets by [4, Proposition 2]. Conversely, if this is the case, represent F as a reduced projective limit $F=\mathrm{proj}_n F$ of Banach spaces F_n. By [20, Lemma 1.1] we have an exact sequence

$$0 \to F \to \prod_n F_n \xrightarrow{q} \prod_n F_n \to 0$$

Since q lifts the bounded subsets by assumption and $\prod_n F_n$ is quasinormable by [3, Theorem, Pag. 159] (cfr. also [5], Pag. 33) F is quasinormable. Note that in general F is quasinormable if and only if $\mathrm{Ext}^1(l^1_d, F)=0$ with $d=\mathrm{dens}\,(F)$.

We conclude this section by showing that Proposition 1.4 cannot be improved by relaxing the condition that F be a quojection. In other words, from $(l^1)^{\mathbb{N}}$ being a quotient it does not follow, in general, that it is also a complemented subspace. At the same time, this will show that Proposition 3.3 and Corollary 3.4 of [15] do not hold for quojections. Precisely, we now exhibit examples of separable quojections having $(l^1)^{\mathbb{N}}$ as a quotient but not even as a subspace.

Example 1.9. Let X be a separable Banach space such that no power X^n contains an isomorphic copy of l^1 (e.g., $X=c_0$ or l^p, $1<p<\infty$) and put $E=l^1\times X^{\mathbb{N}}$. Next, choose a biorthogonal system $\{(y_n, y'_n): y_n\in X, y'_n\in X'\}$ with (y'_n) total over X and $\|y'_n\|=1$.

Then, define a map $i: X\to l^1$ by $i(x)=(2^{-n}\langle x, y'_n\rangle)$; clearly i is continuous and has a dense range. Now write l^1 in E as $l^1(l^1)$ and define $T: E\to (l^1)^{\mathbb{N}}$ by $T((a_n), (x_n))=(a_n+i(x_n))$ for $(a_n)\in l^1(l^1)$ and $(x_n)\in X^{\mathbb{N}}$. T is continuous and surjective: in fact, given $(b_n)\in (l^1)^{\mathbb{N}}$, choose $(x_n)\in X^{\mathbb{N}}$ so that $(b_n-i(x_n))\in l^1(l^1)$; if $a_n=b_n-i(x_n)$, then $T((a_n), (x_n))=(b_n)$. Thus E has a quotient isomorphic to $(l^1)^{\mathbb{N}}$. We show that no subspace of E is isomorphic to $(l^1)^{\mathbb{N}}$.

Arguing by contradiction, suppose that there exists an isomorphism J of $(l^1)^{\mathbb{N}}$ into E. For $(a, (x_n))\in E$, with $a\in l^1$ and $(x_n)\in X^{\mathbb{N}}$, put

$$p_1(a, (x_n))=\|a\|_1$$

and

$$p_{k+1}(a,(x_n)) = \|a\|_1 + \sum_{n=1}^{k} \|x_n\|_X.$$

Also, for $(a_n) \in (l^1)^{\mathbb{N}}$ put

$$q_k(a_n) = \sum_{n=1}^{k} \|a_n\|_1.$$

Then (p_k) and (q_k) are fundamental sequences of seminorms for E and $(l^1)^{\mathbb{N}}$ respectively and for any $b \in (l^1)^{\mathbb{N}}$ we must have

$$(2) \qquad c_1 p_1(Jb) \leq q_k(b) \leq q_{k+1}(b) \leq c_2 p_{j+1}(Jb) \leq c_3 q_m(b)$$

for suitable positive constants c_1, c_2, c_3 and integers $k \leq j < m$. If J_1 and J_2 are the restrictions of J to $(l^1)^k$ and $(l^1)^m$ respectively, from (2) we obtain the following diagram

$$\begin{array}{ccc} (l^1)^m & \xrightarrow{J_2} & l^1 \times X^j \\ Q\downarrow & & \downarrow P \\ (l^1)^k & \xrightarrow[J_1]{} & l^1 \end{array}$$

where P, resp. Q, is the projection onto the first component, resp. the first k components, and $PJ_2 b = J_1 Qb$ for all $b \in (l_1)^m$. Now consider the $(k+1)-$st copy of l^1 in $(l^1)^m$ and denote it by l^1_{k+1}. It follows from (2) that J_2 is an isomorphism on l^1_{k+1}. But if $b \in l^1_{k+1}$, then $Qb = 0$, hence $PJ_2\, b = 0$ and, therefore, $J_2 b \in X^j$. However, this contradicts our assumption on X.

Note that, with reference to the above example, it follows from Proposition 1.4 that for any surjection $S: E \to (l^1)^{\mathbb{N}}$, ker$S$ is *not* a quojection. It is also follows from Theorem 1.1 that every separable quojection is a quotient of E.

Finally, we observe that Theorem 1.1 has as a consequence also the following:

Proposition 1.10. *Every strict (LB)-space G is a subspace of $(l^\infty_d)^{\mathbb{N}}$ for some cardinal d.*

Proof. G' is a quojection and hence a quotient of $(l^1_d)^{\mathbb{N}}$ with respect to a quojection subspace, by Theorem 1.1. Since the quotient map lifts the bounded subsets, G'' is a subspace of $(l^\infty_d)^{\mathbb{N}}$, whence so is G.

2. AN EXACT SEQUENCE FOR STANDARD QUOJECTIONS

Theorem 1.1 of the previous section tells us that every quojection E can be realized as the quotient of $(l_d^1)^{\mathbb{N}}$ by a quojection subspace F. Now, in general, even if E is given, there is nothing we can say about F or about the possibility of replacing $(l_d^1)^{\mathbb{N}}$ by some other product (depending on E, of course). This is due to our lack of knowledge about general quojections. There is, however, an important class of quojections for which we can say a great deal and which we shall examine in this section.

We recall that at present there are only two methods for constructing twisted quojections, plus an exceptional case discussed in [16, §2]. The first was originally devised in [17]: because of its semplicity and versatility, it is still the major source of examples and counterexamples. The second, treated in [12], is much less concrete and, so far, it has been of very little use. Therefore, it is not surprising that we shall appeal once again to the former method. In doing so, we find it convenient to introduce a specific notation, for which we recall the construction in [17] in the following form.

Let (X_n) and (Z_n) be two sequences of Banach spaces for which there are surjections $s_n : X_n \to Z_n$. If L is a Banach sequence space, we form the steps

$$F_1 = (\bigoplus_n Z_n)_L = L(Z_n)$$

and, for all k,

$$F_{k+1} = [(\bigoplus_{n \leq k} X_n) \oplus (\bigoplus_{n>k} Z_n)] = L[(X_n)_{n \leq k}, (Z_n)_{n>k}].$$

Clearly the maps s_n induce, together with the identies of X_n and Z_n, surjections $S_k : F_{k+1} \to F_k$. The projective limit of the sequence (F_k, S_k) is, therefore, a quojection, which we shall denote by $Q[(X_n), (Z_n), (s_n); L]$. Such quojections form a subclass of the class of Fréchet spaces defined in [2] and we shall refer to quojections isomorphic to them as *standard* quojections. Note that $Q[(X_n), (\{0\}), (0); L] = \prod_n X_n$ so that every countable product of Banach spaces is a standard quojection.

The above construction has the added advantage that it may be performed also when the X_n, Z_n and L are Fréchet spaces (as implicitly done in [17]) and we shall give an interesting application of this in the last section.

If the spaces Z_n above are quotients of the X_n with respect to closed subspaces Y_n and if the s_n are the quotient maps, the corresponding standard quojection will de denoted by $Q[(X_n), (X_n/Y_n); L]$ if no confusion is likely to arise. Clearly, we always have

$$Q[(X_n), (Z_n), (s_n); L] \simeq Q[(X_n), (X_n/\ker s_n); L],$$

To end our preparations, we find it useful to state the following lemma, which is essentially Theorem 1.1 of [16] (the «only if» part of the lemma being obvious).

Lemma 2.1. *$Q[(X_n), (Z_n), (S_n); L]$ is twisted if and only if ker s_n is not complemented in X_n for infinitely many n.*

Now the proof of our main result of this section begins with the following lemma, which is at the heart of the matter.

Lemma 2.2. *Let $Q[(X_n), (X_n/Y_n); L]$ be any standard quojection and let (Z_n) be a sequence of Banach spaces admitting a sequence (f_n) of continuous linear maps $f_n: Z_n \to Y_n$ with dense ranges. Then there is a surjection $S: L(X_n) \times \prod_n Z_n \to Q[(X_n), (X_n/Y_n); L]$ such that ker S is a Fréchet space of Moscatelli type* (cfr. [2, Definition 1.3]).

Proof. First of all, put $E = Q[(X_n), (X_n/Y_{nn}); L]$ and observe that, by [2, Proposition 1.4], algebraically we have

$$E = \{(x_n) \in \prod_n X_n : (q_n(x_n)) \in L(X_n/Y_n)\}, \tag{3}$$

where $q_n: X_n \to X_n/Y_n$ is the quotient map for every n. Moreover, the topology of E may be defined by the system (p_k) of seminorms given by

$$\begin{aligned} p_1(x_n) &= \|(\|q_n(x_n)\|_{X_n/Y_n})\|_L, \\ p_{k+1}(x_n) &= p_1(x_n) + \sum_{n=1}^{k} \|x_n\|_{X_n}. \end{aligned} \tag{4}$$

Now for $((x_n), (z_n)) \in L(X_n) \times \prod_n Z_n$ the equation

$$S((x_n), (z_n)) = (x_n + f_n(z_n)) \tag{5}$$

defines a linear map $S: L(X_n) \times \prod_n Z_n \to \prod_n X_n$. Using (3) and (4) it is easy to see that S takes its values in E and is continuous as a map into E; hence, it remains to prove that S is surjective and ker S is of the required type.

Let $(x_n) \in E$; by (3) there is $(v_n) \in L(X_n)$ with $q(v_n) = q_n(x_n)$ for all n. Put $y_n = x_n - v_n$; since $y_n \in Y_n$ and f_n has a dense range, we can find $z_n \in Z_n$ such that $(y_n - f_n(z_n)) \in L(Y_n)$. Thus, if $u_n = v_n + y_n - f(z_n)$ for all n, then $(u_n) \in L(X_n)$ and $S((u_n), (z_n)) = (x_n)$.

Finally, by (5) we have

$$\ker S = \{((-f_n(z_n)), (z_n)) : (z_n) \in \coprod_n Z_n, (f_n(z_n)) \in L(Y_n)\} \tag{6}$$

from which the assertion about ker S follows, again by [2, Proposition 1.4].

Theorem 2.3. *Under the assumptions of Lemma 2.2 suppose that the maps f_n can be chosen to be surjections and denote them by s_n. Then*

$$\ker S \simeq Q[(Z_n), (Y_n), (s_n); L]. \tag{7}$$

Consequently:

(a) *ker S is Banach if and only if ker $s_n = 0$ for all but finitely many n;*

(b) *ker $S \simeq \omega \times$ Banach if and only if $0 < \dim$ (ker $s_n) < \infty$ for all but finitely many n;*

(c) *ker S is twisted if and only if ker s_n is not complemented in Z_n for infinitely many n.*

Moreover, we have established the existence of an exact sequence

$$O \to Q[(Z_n), (Y_n), (s_n); L] \to L(X_n) \times \coprod_n Z_n \xrightarrow{S} Q[(X_n), (X_n/Y_n); L] \to 0. \tag{8}$$

Proof. (7) follows from (6) and Proposition 1.4 of [2], while (8) follows from (7) and Lemma 2.2. It is then an easy matter to verify that (a), (b) and (c) hold (for (c) use Lemma 2.1).

The above theorem has many interesting corollaries, which will be given in the next section. Here we conclude with the following consequence of Lemma 2.2.

Proposition 2.4. *Let E be any standard quojection of the form $Q[(X_n), (X_n/Y_n); l^p]$, with Y_n separable for all n. Then for every sequence (Z_n) of Banach spaces there is a surjection $S : l^p(X_n) \times \coprod_n Z_n \to E$ such that ker S is not distinguished (hence not quasinormable). In particular, this holds for $E = \coprod_n Y_n$*

Proof. For each n take a compact map $f_n : Z_n \to Y_n$ with a dense range (f_n may be constructed more or less as i was in Example 1.9). Then the surjection

S constructed in the proof of Lemma 2.2 meets the requirement, by [2, Corollary 2.5].

Remark 2.5. It is interesting to note that, if in the above proposition we take $X_n = Y_n = Z_n = l^1$ and $f_n = f$ for all n, with $f(a_k) = (k^{-1} a_k)$ for $(a_k) \in l^1$, then for the corresponding surjection S: $(l^1)^N \to (l^1)^N$ we find that ker S is the non-distinguished Fréchet space of Köthe-Grothendieck (cfr. [9, §31, 7, pag. 435]).

3. CONSEQUENCES

This section is devoted to the investigation of various consequences of Theorem 2.3. Here «product» means a countable (and not finite) product of Banach spaces. Moreover, whenever all the Banach spaces X_n (resp., Z_n) and maps s_n are taken equal to one fixed space X (resp., Z) and map s, we simply write $Q(X, Z, s; L)$ for the corresponding standard quojection.

The first consequence of Theorem 2.3 is the following quite strange

Corollary 3.1. *Every standard twisted quojection is isomorphic to the quotient of a product by a Banach space.*

Proof. Taking in Theorem 2.3 $Z_n = Y_n$ and s_n = the identity map of Y_n for all n, we obtain from (8) an exact sequence

$$0 \to L(Y_n) \to L(X_n) \times \prod_n Y_n \to Q[(X_n), (X_n/Y_n); L] \to 0. \tag{9}$$

It is clear that we can also represent a standard twisted quojection as the quotient of a product by a product. Consequently, since a standard twisted quojection is never complemented in a product [16, Theorem 1.5 (b)], we have

Corollary 3.2. *Let E be any standard twisted quojection. Then there is Banach space X and a product F such that*

$$\mathrm{Ext}^1(E, X) \neq 0 \text{ and } \mathrm{Ext}^1(E, F) \neq 0.$$

Next, we observe that Corollary 3.1 has the following counterpart.

Corollary 3.3. *Every standard twisted quojection can be embedded as subspace F of a product G such that G/F is again a product.*

Proof. Represent the given quojection as $Q[(Z_n), (Y_n), (s_n); L]$, put $X_n = Y_n$ for all n and apply (8) to obtain an exact sequence

$$0 \to Q[(Z_n), (Y_n), (s_n); L] \to L(Y_n) \times \prod_n Z_n \to \prod_n Y_n \to 0. \tag{10}$$

Corollary 3.4. *Let F be any standard twisted quojection. Then there exists a product E such that* $\mathrm{Ext}^1(E, F) \neq 0$.

Here we pause to note that it is not possible to obtain an exact counterpart of Corollary 3.1, namely, the conclusion of Corollary 3.3 cannot be improved to «G/F is Banach» on account of the following

Proposition 3.5. *Let E be a quojection and let F be a subspace of E such that E/F is Banach. Then F is a quojection. Moreover, if E is a product, then also F is a product.*

Proof. Let $E = \mathrm{quoj}_n(E_n, R_n)$ and write F as the reduced projective limit $F = \mathrm{proj}_n(F_n, R_n)$, where each F_n is a closed subspace of E_n. Then $E/F = \mathrm{quoj}_n(E_n/F_n, \overline{R}_n)$, where the maps $\overline{R}_n: E_{n+1}/F_{n+1} \to E_n/F_n$ are induced by the maps $R_n: E_{n+1} \to E_n$. If E/F is Banach, we may assume that each $\overline{R}_n$ is an isomorphism. In particular, for all n,

$$0 = \ker \overline{R}_n = (R_n^{-1}(F_n) + F_{n+1})/F_{n+1},$$

which implies that in E_{n+1} we must have $R_n^{-1}(F_n) \subset F_{n+1}$. Hence $F = \mathrm{quoj}_n(F_n, R_n)$.

Suppose now that E is a product $\prod_k X_k$ of Banach spaces X_k and, for each n, let $R_n: \prod_{k=1}^{n+1} X_k \to \prod_{k=1}^{n} X_k$ be the canonical projection. With $F_1 \subset X_1$, from $R_1^{-1}(F_1) = F_2$ we obtain $F_2 = F_1 \times X_2$ and, in general, $F_n = F_1 \times \prod_{k=2}^{n} X_k$ $(n > 1)$, so that F is a product, as claimed.

Remark 3.6. Note that F may be a product even if E is twisted, as shown in [11, Corollary 2.3].

Going back to the consequences of Theorem 2.3, now we have

Corollary 3.7. *Every quojection $Q[(X_n), (X_n/Y_n); L]$, with $Y_n \not\approx l_d^1$ for infinitely many n and all cardinal numbers d, is isomorphic to the quotient of* *a product by a standard twisted quojection. In particular, the conclusion holds for every product of Banach spaces Y_n as above (cfr. (10)).*

Proof. Take $d \geq \text{dens}(Y_n)$ for all n, then choose surjections $s_n: l_d^1 \to Y_n$ and apply Theorem 2.3 (c) with $Z_n = l_d^1$.

Corollary 3.8. *Let (Y_n) be a sequence of Banach spaces and suppose that there are two infinite subsets N_1, N_2 of* **N** *such that, if $n \in N_1$, Y_n is not injective and, if $n \in N_2$, Y_n is not projective. Put*

$$E = Q[(X_n), (X_n/Y_n); L] \text{ and } F = Q[(Z_n), (Y_n), (s_n); L].$$

Then, given any twisted E (resp., F) there exist a twisted F (resp., E) and a product G for which we have an exact sequence

$$0 \to F \to G \to E \to 0$$

and, consequently, $\text{Ext}^1(E, F) \neq 0$.

We note the following special case of Corollary 3.8 (or 3.7), which should be compared with Theorem 1.1.

Corollary 3.9. *Let Y be a subspace of l^1 with $Y \not\simeq l^1$ and let $s: l^1 \to Y$ be a surjection. Then there is an exact sequence*

$$0 \to Q(l^1, Y, s; l^1) \to (l^1)^{\mathbf{N}} \to Q(l^1, l^1/Y; l^1) \to 0,$$

where, of course, both quojections are twisted.

Remark 3.10. We do not know if the above result holds with l^1 replaced by l^p ($1 < p < \infty, p \neq 2$). It is easy to see that it holds for l^∞ and c_0 and also for $C([0, 1])$ and $L^1(0, 1)$, with appropriate choices of Y, but again we know nothing about the case of $L^p(0, 1)$ ($1 < p < \infty$, $p \neq 2$). However, it «almost» holds for $L^t(0, 1)$ in the sense that we have, putting $L^t(0, 1) = L^t$ for all $1 < t < \infty$,

Corollary 3.11. *Let p, q, r, s be positive real numbers satisfying*

$$1 < p < q < 2 < r < s < \infty$$

and let L be reflexive. Then the reflexive product

$$G = L(L^p \oplus L^r) \times (L^q \oplus L^s)^{\mathbb{N}}$$

contains a standard twisted quojection F such that also G/F is a standard twisted quojection.

Proof. By [10, II, Corollary 2.f.5, p. 212] L^q is (isometrically) a subspace of L^p, whence L^r is a quotient of L^s. Since l^r is a subspace of L^r, we see from [8, Corollary 3, p. 168] that L^r cannot be isomorphic to a subspace of L^s. This implies also that L^q cannot be isomorphic to a complemented subspace of L^p. Now observe that $L^p \oplus L^r / L^q \oplus L^r = L^p / L^q$ and form the standard quojection $E = Q(L^p \oplus L^r, L^p / L^q; L)$. Then, by Theorem 2.3 (or Corollary 3.8), $E \simeq G/F$, where $F \simeq Q(L^q \oplus L^s, L^q \oplus L^r, s; L)$ (here $s: L^q \oplus L^s \to L^q \oplus L^r$ is the map which is the identity on L^q and the quotient map on L^s).

Clearly the above result holds also with L^q and L^r replaced by l^q and l^r respectively. Moreover, an application of (9) and (10) establishes the following particular case of Corollaries 3.1 and 3.3 which we find worth mentioning because all the spaces involved are reflexive.

Corollary 3.12. *There are exact sequences*

$$0 \to L^q \to l^q(L^p) \times (L^q)^{\mathbb{N}} \to Q(L^p, L^p/L^q; l^q) \to 0,$$

$$0 \to Q(L^s, L^r; l^r) \to L^r \times (L^s)^{\mathbb{N}} \to (L^r)^{\mathbb{N}} \to 0,$$

where $1 < p < q < 2 < r < s < \infty$.

We terminate our list of direct consequences of Theorem 2.3 by giving the example promised in Remark 1.3.

Example 3.13. Let $X = C([0, 1])$ and $E = Q(X, X/l^1; l^1)$. From Theorem 2.3 (a) we obtain a surjection $S_1 : l^1(X) \times (l^1)^{\mathbb{N}} \to E$ with $\ker S_1 \simeq l^1$. If $S_2 : l^1 \times (l^1)^{\mathbb{N}} \to l^1(X) \times (l^1)^{\mathbb{N}}$ is the surjection which on $(l^1)^{\mathbb{N}}$ is the identity and on l^1 is the quotient map onto $l^1(X)$, then $S = S_1 S_2$ is a surjection of $(l^1)^{\mathbb{N}}$ onto E such that $\ker S = S_2^{-1}(l^1)$ is Banach. Now let T: $(l^1)^{\mathbb{N}} \times (l^1)^{\mathbb{N}} \to E$ be the surjection which is S on the first copy of $(l^1)^{\mathbb{N}}$ and is the zero map on the second copy. Then $\ker T = \ker S \times (l^1)^{\mathbb{N}} \not\simeq \ker S$.

Now we shall give some less direct consequences of Theorem 2.3.

Corollary 3.14. *There are reflexive quojections E_1, E_2, E_3 having unconditional bases and containing subspaces F_1, F_2, F_3 respectively, such that:*

(a) *F_1, but not E_1/F_1, has an unconditional basis;*

(b) *E_2/F_2, but not F_2, has an unconditional basis;*

(c) *F_3 and E_3/F_3 have no unconditional bases.*

Proof. Follows from Corollaries 3.11, 3.12 and the result in [7]. Part (c) of the above corollary shows that the property of not having an unconditional basis is not a three-space property for the class of separable Fréchet spaces, thus providing the counterpart to Corollary 2.4 in [11].

At this point we cannot drop the word «unconditional» in the above statement, since twisted quojections may indeed have (conditional) bases, as shown in [13]. This, however, will be done in §4.

Now, as in [11], denote by SUM the class of countable direct sums of Banach spaces and call s-LB the class of strict (LB)-spaces. Evidently, s-LB\SUM is the class of *twisted*, strict (LB)-spaces (cf. [12, §1]). Then, dualizing Corollaries 3.11 and 3.12 we obtain (for the «moreover part» use [6, Proposition 3.1]).

Corollary 3.15. *For $i=1, 2, 3$ there are reflexive $G_i \in$ SUM containing respectively subspaces $H_i \in$ s-LB such that $G_i/H_i \in$ s-LB and*

(a) $H_1 \in$ SUM, $G_1/H_1 \notin$ SUM;

(b) $H_2 \notin$ SUM, $G_2/H_2 \in$ SUM;

(c) H_3, $G_3/H_3 \notin$ SUM.

Moreover, the analogue of Corollary 3.14 holds.

Finally, it is perhaps superfluous to note that, by duality, (8), (9) and (10) yield corresponding exact sequences for reflexive spaces in s-LB.

4. APPLICATIONS TO NUCLEAR SPACES

First of all we observe that, with reference to the construction of standard quojections $Q[(X_n), (Z_n), (s_n); L]$ outlined at the beginning of §2, if the spaces

X_n, Z_n and L are Fréchet spaces with continuous norms (each space having a fundamental system of norms fixed once and for all), then the same construction yields a Fréchet space without continuous norm which is twisted if and only if $\ker s_n$ is not complement in X_n for infinitely many n (cfr. Lemma 2.1). The details of all this are left to the reader; but we point out that in the nuclear case the above construction was implicitly performed in [17, §2], where it was explicitly carried out in the dual space. Here too we shall confine ourselves to the nuclear case and indicate how some of the results of §§2 and 3 go over to this case. (Note that §1 has no nuclear analogue, since there is no quotient-universal nuclear space).

The first result is the following analogue of Theorem 2.3.

Theorem 4.1. *Let (X_n), (Y_n) be two sequences of nuclear Fréchet spaces with continuous norms for which there exist surjections $r_n: X_n \to Y_n$. Further, let (Z_n) be a sequence of nuclear Fréchet spaces with continuous norms for which there are surjections $s_n: Z_n \to \ker r_n$. Then we have an exact sequence*

$$0 \to Q[(Z_n), (\ker r_n), (s_n); L] \to L(X_n) \times \prod_n Z_n \to Q[(X_n), (Y_n), (r_n); L] \to 0, \tag{11}$$

from which the analogues of (9) and (10) may also be derived.

Proof. It suffices to note that the proofs of Lemma 2.2 and Theorem 2.3 go over to this case.

Of course, Theorem 4.1 is not as far reaching as Theorem 2.3 so that, although Corollaries 3.1 and 3.3 have nuclear analogues, there is no hope of getting something like Corollaries 3.7 and 3.8. However, we wish to draw attention to the following analogue of Corollaries 3.11 and 3.12.

Let $\alpha = (\alpha_n)$ be an increasing sequence of positive real numbers such that

$$\lim_n \frac{\log n}{\alpha_n} = 0 \text{ and } \sup_n \frac{\alpha_{n^2}}{\alpha_n} < \infty \tag{12}$$

and let $\Lambda_1 = \Lambda_1(\alpha)$ be the associated nuclear power series space of finite type. Then we have

Corollary 4.2. *There is a surjection $r: \Lambda_1 \to \Lambda_1$ such that the space $F = Q(\Lambda_1, \Lambda_1, r; \Lambda_1)$ is twisted. Moreover, we have the following three exact sequences, none of which splits:*

$$(13) \qquad 0 \to F \to \Lambda_1^{\mathbb{N}} \to F \to 0,$$

$$(14) \qquad 0 \to \Lambda_1 \to \Lambda_1^{\mathbb{N}} \to F \to 0,$$

$$(15) \qquad 0 \to F \to \Lambda_1^{\mathbb{N}} \to \Lambda_1^{\mathbb{N}} \to 0.$$

Proof. Since $\Lambda_1 \oplus \Lambda_1 \simeq \Lambda_1$, by [18, Satz 5.4] there is an exact sequence

$$0 \to \Lambda_1 \to \Lambda_1 \xrightarrow{r} \Lambda_1 \to 0$$

with ker r not complemented in Λ_1. Since ker $r \simeq \Lambda_1$, we have a surjection s: $\Lambda_1 \to$ ker r with ker s not complemented in Λ_1. Also, $\Lambda_1(\Lambda_1) \simeq \Lambda_1 \hat{\otimes} \Lambda_1 \simeq \Lambda_1$ by $(12)_2$ and (13) follows from (11) by appropriately choosing everything. Similarly for (14) and (15). By [16, Theorem 2.3] none of the sequences splits.

Remark 4.3. We note the curious fact that all the steps in $Q(\Lambda_1, \Lambda_1, r; \Lambda_1)$ are isomorphic to Λ_1.

From Corollary 4.2 (13) we immediately derive, via [17, §2] or [6, Proposition 4.1], the announced improvement of Corollary 3.14 (c).

Corollary 4.4. *The property of not having a basis is not a three-space property for the class of separable (even nuclear) Fréchet spaces. The same is true also for the classes of separable (LB)— or (DF)— spaces.*

Remark 4.5. The first assertion in the above corollary should be compared with the result in [14].

Finally, we observe that (14) also holds for nuclear power series spaces of infinite type. In fact, suppose that α satisfies $(12)_2$ and $\sup_n \log n/\alpha_n < \infty$ and put $\Lambda_\infty = \Lambda_\infty(\alpha)$. Then we have

Corollary 4.6. *There is a nuclear Fréchet space Y (depending on α) with a continuous norm and surjection* r *: $\Lambda_\infty \to$ Y such that $Q(\Lambda_\infty, Y, r; \Lambda_\infty)$ is twisted and we have the following exact sequence (which does not split):*

$$0 \to \Lambda_\infty \to \Lambda_\infty^{\mathbb{N}} \to Q(\Lambda_\infty, Y, r; \Lambda_\infty) \to 0.$$

Proof. Let $\beta=(\beta_n)$ be an increasing sequence of positive real numbers satisfying $(12)_1$ and $\lim_n \alpha_n/\beta_n=0$. By [21, Lemma 3.3] there is a subspace $\tilde{E}$ of Λ_∞ and an exact sequence

$$0 \to \Lambda_\infty \to \Lambda_\infty \xrightarrow{r} \Lambda_1(\beta) \oplus \tilde{E} \to 0.$$

Since $\Lambda_1(\beta)$ is not isomorphic to a complemented subspace of Λ_∞, ker r is not complemented in Λ_∞ and the result follows by taking $Y=\Lambda_1(\beta)\oplus\tilde{E}$.

References

1. J. BONET, M. MAESTRE, G. METAFUNE, V. B. MOSCATELLI and D. VOGT: «Every quojection is the quotient of a countable product of Banach spaces», in *Advances in the Theory of Fréchet Spaces* (T. Terzioglu ed.). Kluwer Academic Publishers, 1989, 355-356.
2. J. BONET and S. DIEROLF: «Fréchet spaces of Moscatelli type», *Rev. Matem. Univ. Complutense Madrid* 2 (suppl.) (1989), 77-92.
3. V. E. CHOLODOVSKII: «On quasinormability of semimetrizable topological vector spaces», *Funkc. Anal.* 7 (1976), 157-160.
4. M. DE WILDE: «Sur le relèvement des parties bornées d'un quotient d'espaces vectoriels topologiques», *Bull. Soc. Roy. Sci. Liège* 43 (1971), 299-301.
5. S. DIEROLF and W. ROELCKE: «On the three-space-problem for topological vector spaces», *Collect. Math.* 32 (1981), 13-35.
6. K. FLORET and V. B. MOSCATELLI: «On bases in strict inductive and projective limits of locally convex spaces», *Pacific J. Math.* 119 (1985), 103-113.
7. — : «Unconditional bases in Fréchet spaces», *Arch. Math.* 47 (1986), 129-130.
8. M. I. KADEC and A. PEŁCZYŃSKI: «Bases, lacunary sequences and complemented subspaces in the spaces L_p», *Studia Math.* 21 (1962), 161-176.
9. G. KÖTHE: «Topological Vector Spaces», I, Springer, Berlin, 1969.
10. J. LINDENSTRUSS and L. TZAFRIRI: «Classical Banach Spaces», I, II, Springer, Berlin, 1977, 1979.
11. G. METAFUNE and V. B. MOSCATELLI: «On the three-space property for locally convex spaces», *Collect. Math.* 37 (1986), 287-296.
12. — : «Another construction of twisted spaces», *Proc. Roy. Irish Acad. Sect. A* 87 (1987), 163-168.
13. : «A twisted Fréchet space with basis», *Monatsh. Math.* 105 (1988), 127-129.
14. : «Nuclear Fréchet spaces with basis do not have the three-space property», *Arch. Math.* 50 (1988), 369-370.
15. : «Complemented subspaces of sums and products of Banach spaces», *Ann. Mat. Pura Appl.* 153 (1988), 175-190.
16. : «On twisted Fréchet and (LB)-spaces», *Proc. Amer. Math. Soc.*, 108 (1990), 145-150.
17. V. B. MOSCATELLI: «Fréchet spaces without continuous norms and without bases», *Bull. London Math. Soc.* 12 (1980), 63-66.
18. D. VOGT: «Eine Charakterisierung der Potenzreihenräume von endlichen Typ und ihre Folgerungen», *Manuscripta Math.* 37 (1982), 269-301.

19. — : «Some results on continuous linear maps between Fréchet spaces», in *Functional Analysis: Surveys and Recent Results III* (eds.: K. D. Bierstedt and B. Fuchssteiner), *North-Holland Math. Studies* 90, 1984, 349-381.
20. — : «On the functors $Ext^1 (E, F)$ for Fréchet spaces», *Studia Math.* 85 (1987), 163-197.
21. D. VOGT and M. J. WAGNER: «Charakterisierung der Unterräume und Quotientenräume der nuklearen stabilen Potenzreihenräume von unendlichen Typ», *Studia Math.* 70 (1981), 63-80.

Dipartimento di Matematica, Universitá
C.P. 193, I-73100, Lecce,
Italy.

Recibido: 6 de julio de 1990