REVISTA MATEMÁTICA de la
Universidad Complutense de Madrid
Volumen 4, números 2 y 3; 1991.
http://dx.doi.org/10.5209/rev_REMA.1991.v4.n2.17957

Solidity in Sequence Spaces

I. J. MADDOX

ABSTRACT. Relations are established between several notions of solidity in vector-valued sequence spaces, and a generalized Köthe-Toeplitz dual space is introduced in the setting of a Banach algebra.

INTRODUCTION

The study of linear spaces of scalar sequences and their α-duals was initiated by Köthe and Toeplitz [4].

If s denotes the linear space of all infinite sequences $a=(a_k)$ of complex numbers a_k and if E is a linear subspace of s then, following [4]; see also [3] and [1], we define the α-dual of E as

$$E^\alpha=\{a\in s:\sum_{k=1}^{\infty}|a_k x_k|<\infty \text{ for all } x\in E\}.$$

Two related dual spaces are defined by

$$E^\beta=\{a\in s:\sum_{k=1}^{\infty} a_k x_k \text{ converges for all } x\in E\},$$

$$E^\gamma=\{a\in s:\sup_n|\sum_{k=1}^{n} a_k x_k|<\infty \text{ for all } x\in E\}.$$

Topologies on a sequence space, involving β and γ duality have been examined by Garling [2], who noted that $E^\alpha=E^\beta=E^\gamma$ when E is solid (or normal), i.e. when $x\in E$ and $|y_k|\leq|x_k|$ for all $k\in N$ together imply that $y\in E$.

1991 Mathematics Subject Classification: 46A45.
Editorial de la Universidad Complutense. Madrid, 1991.

Thus, for example, the space c_0 of null sequences is solid, but the space c of convergent sequences is not.

We shall be concerned with the more general situation of vector-valued sequences $x=(x_k)=(x_1, x_2, \ldots)$ with x_k in a complex linear space X. By $s(X)$ we denote the linear space of all sequences $x=(x_k)$ with $x_k \in X$ and the usual coordinatewise operations: $\alpha x=(\alpha x_k)$ and $x+y=(x_k+y_k)$, for each $\alpha \in C$.

If $\lambda=(\lambda_k)$ is a scalar sequence and $x \in s(X)$ then we shall write $\lambda x=(\lambda_k x_k)$.

In case X is also a normed space we denote by B the closed unit ball of X and by $B(X)$ the space of all bounded linear operators on X. As usual X^* denotes the continuous dual space of X. Two subspaces of $s(X)$ that we consider later are

$$l_\infty(X)=\{x \in s(X): \sup_k \|x_k\| < \infty\},$$

$$l_1(X)=\{x \in s(X): \sum_{k=1}^{\infty} \|x_k\| < \infty\}.$$

These spaces generalize the classical spaces l_∞ and l_1 which are subspaces of s.

If $A=(A_k)$ is a sequence in $B(X)$ we shall write $Ax=(A_k x_k)$ for each $x \in s(X)$.

Some information about types of generalized Köthe-Toeplitz duals involving sequences of linear operators may be found in Maddox [5].

We now consider the eight statements below, each of which expresses some notion of solidity for a linear subspace E of $s(X)$. It is statement (5) which generalizes the original idea of solid (or normal) as given by Köthe and Toeplitz [4].

The first three statements are meaningful in any complex linear space, but the last five statements require X to be normed. A statement such as $\|y_n\|=\|x_n\|$ is an abbreviation for $\|y_n\|=\|x_n\|$ for all $n \in N$. Also, in (6) to (8) the A_n are elements of $B(X)$.

(1) $x \in E$ and $\lambda \in l_\infty$ imply $\lambda x \in E$

(2) $x \in E$ and $|\lambda_n| \leq 1$ imply $\lambda x \in E$

(3) $x \in E$ and $|\lambda_n|=1$ imply $\lambda x \in E$

(4) $x \in E$ and $\|y_n\|=\|x_n\|$ imply $y \in E$

(5) $x \in E$ and $\|y_n\| \leq \|x_n\|$ imply $y \in E$

(6) $x \in E$ and $(\|A_n\|) \in l_\infty$ imply $Ax \in E$

(7) $x \in E$ and $\|A_n\| \leq 1$ imply $Ax \in E$

(8) $x \in E$ and $\|A_n\| = 1$ imply $Ax \in E$.

EQUIVALENCES

In Theorems 1, 2 and 3 we determine the relations between the statements (1) to (8).

Theorem 1. *In any complex linear space X the statements (1), (2) and (3) are equivalent.*

Proof. It is trivial that $(1) \to (2) \to (3)$. Let us show that $(3) \to (1)$. If (3) holds, $x \in E$ and $\lambda \in l_\infty$ then there exists $M > 0$ with $|\lambda_n| \leq M$ for all $n \in N$. Define

$$\mu_n = \lambda_n / M = \alpha_n + i\beta_n$$

where α_n and β_n are real. Then $|\alpha_n| \leq 1$ and $|\beta_n| \leq 1$, so we may choose γ_n and δ_n with

$$\alpha_n^2 + \gamma_n^2 = \beta_n^2 + \delta_n^2 = 1.$$

Define $z_n = \alpha_n + i\gamma_n$ and $w_n = \beta_n + i\delta_n$, whence

$$|z_n| = |\bar{z}_n| = |w_n| = |\overline{w}_n| = 1,$$

and so zx, $\bar{z}x$, wx, $\overline{w}x$ are all in E. Since E is a linear space it follows that αx, $\beta x \in E$ and so $\lambda x \in E$. Hence $(3) \to (1)$.

Theorem 2. *In any normed linear space X the statements (4), (5), (6), (7) and (8) are equivalent.*

Proof. Let (4) hold, $x \in E$ and $\|y_n\| \leq \|x_n\|$. If $\|x_n\| = 0$ we define $\lambda_n = 1$ and if $\|x_n\| > 0$ we define $\lambda_n = \|y_n\| / \|x_n\|$, so that in every case $0 \leq \lambda_n \leq 1$ and $\|y_n\| = \|\lambda_n x_n\|$. Now define μ_n such that $\lambda_n^2 + \mu_n^2 = 1$ and write $z_n = \overline{\lambda_n} + i\mu_n$. Then

$$\|z_n x_n\| = \|\bar{z}_n x_n\| = \|x_n\|$$

and so (4) implies that zx and $\bar{z}x$ are in E, whence $\lambda x \in E$. Since $\|y_n\| = \|\lambda_n x_n\|$ it follows from (4) that $y \in E$. Hence (4)→(5).

Now let (5) hold, $x \in E$ and $\|A_n\| < M$ for all $n \in N$. Then

$$\|A_n(M^{-1} x_n)\| \leq \|x_n\|.$$

Hence $M^{-1}(A_n x_n) \in E$, so $Ax \in E$, whence (5)→(6).

It is trivial that (6)→(7)→(8).

Finally, let (8) hold, $x \in E$ and $\|y_n\| = \|x_n\|$. By the Hahn-Banach theorem there exists $f_n \in X^*$ with $\|f_n\| = 1$ and $f_n(x_n) = \|x_n\|$. If $\|x_n\| = 0$ we define $A_n = I$, the identity operator of $B(X)$, and if $\|x_n\| > 0$ we define

$$A_n(w) = f_n(w) y_n / \|x_n\|$$

for each $w \in X$. Then, for all $n \in N$, it is clear that $\|A_n\| = 1$ and $y_n = A_n x_n$, so it follows from (8) that $y = Ax \in E$. Hence (8)→(4), which completes the proof.

Theorem 3. *In any normed linear space X any one of the statements (4) to (8) implies all of the statements (1) to (3). But (1) is equivalent to (4) if and only if X is one-dimensional.*

Proof. For the first part of the theorem it is sufficent to show that (8)→(3). Let (8) hold, $x \in E$ and $|\lambda_n| = 1$. Now define $A_n \in B(X)$ by $A_n(w) = \lambda_n w$ for each $w \in X$. Then $\|A_n\| = |\lambda_n| = 1$, whence (8) implies that $\lambda x = Ax \in E$, so (8)→(3).

It is straightforward to verify that if X is one-dimensional then (1)→(4).

Finally, suppose (1)→(4) but assume that the dimension of X exceeds 1. Let $\{b_1, b_2, \ldots\}$ be a Hamel base for X and let us define $E = s([b_1])$, so that $x \in E$ is of the form $x_n = \alpha_n b_1$ for all $n \in N$. It is clear that (1) holds, whence (4) holds. Now we define, for all $n \in N$,

$$x_n = \|b_2\| b_1 \quad \text{and} \quad y_n = \|b_1\| b_2.$$

Then $x \in E$ and $\|x_n\| = \|y_n\|$, whence $y \in E$, so $y_n = \alpha_n b_1$. Consequently we have $\alpha_1 b_1 = \|b_1\| b_2$, which is contrary to the fact that b_2 are elements of the base.

THE DELTA DUAL

Henceforth we assume that X is an abstract non-commutative normed algebra, not necessarily containing an identity element. As before $x=(x_k)$, $y=(y_k)$ denote elements of the space $s(X)$.

For any non-empty subset E of $s(X)$ we now introduce its delta dual E^δ defined as follows:

$$E^\delta=\{y\in s(X):\sum_{k=1}^{\infty}(\|x_k y_k\|+\|y_k x_k\|)<\infty \text{ for all } x\in E\}.$$

It is immediate that E^δ is a linear subspace of $s(X)$ even though E may not be a linear subspace. Also, it is clear that we have $E\subset E^{\delta\delta}$ for any non-empty E.

If it happens that $E=E^{\delta\delta}$ we shall define E to be *δ-perfect*.

Theorem 4. *If E is δ-perfect then E is solid in the sense of (1), (2) or (3).*

Proof. Let $x\in E$, $|\lambda_k|\leq 1$ for all $k\in N$ and $y\in E^\delta$. Then

$$\sum(\|\lambda_k x_k y_k\|+\|y_k \lambda_k x_k\|)\leq\sum(\|x_k y_k\|+\|y_k x_k\|)<\infty,$$

which implies $\lambda x\in E^{\delta\delta}=E$.

Of course there are solid spaces which are not δ-perfect; for example $E=c_0$ when X is the complex field C.

Next we examine the relation between the space $l_\infty(X)$ of bounded sequences and the δ-dual of $l_1(X)$. Since

$$\|y_k\|\leq\sup_k\|y_k\|:=\|y\|_\infty$$

for each $y\in l_\infty(X)$ we have

$$\sum_{k=1}^{\infty}(\|x_k y_k\|+\|y_k x_k\|)\leq 2\|y\|_\infty\sum_{k=1}^{\infty}\|x_k\|$$

for each $x\in l_1(X)$, whence

(9) $$l_\infty(X)\subset l_1^\delta(X)$$

for any normed algebra X. In case X contains a certain type of element, which we shall call an almost identity, we shall be able to prove in Theorem 5 below that there is equality in (9).

Given a normed algebra X we say that X has an *almost identity* $z\in X$ if for such a z there is a positive constant c such that

$$c\|x\|\leq\|xz\|+\|zx\|,\quad \text{for all } x\in X.$$

If X has identity e, in the usual sense that $xe=ex=x$ for all $x\in X$, then e is obviously an almost identity. We note also that the normed algebra c_0 with $xy:=(x_k y_k)$ and $\|x\|=\sup_k |x_k|$ has no almost identity. For if $z\in c_0$ were such an identity then for some positive c we have $c\|x\|\leq 2\|xz\|$ for all $x\in c_0$. Choose n with $|z_n|<c/2$ and let $x=e_n$, the n-th unit vector in c_0. Then $c\leq 2|z_n|<c$, a contradiction.

Theorem 5. *If X has an almost identity then $l_\infty(X)=l_1^S(X)$.*

Proof. In view of (9) we need only show that $l_1^S(X)\subset l_\infty(X)$. Let $y\in l_1^S(X)$, so that for all $x\in l_1(X)$,

$$\sum_{k=1}^{\infty}(\|x_k y_k\|+\|y_k x_k\|)<\infty.$$

Applying the Banach-Steinhaus theorem, there is a positive constant M such that for all $n\in N$ and all $x\in l_1(X)$,

$$\sum_{k=1}^{n}(\|x_k y_k\|+\|y_k x_k\|)\leq M\sum_{k=1}^{\infty}\|x_k\|.$$

Now take any $n\in N$ and define $x_n=z$ and $x_k=0$ for $k\neq n$, where z is an almost identity. Hence we have $c\|y_n\|\leq M\|z\|$, which implies that $y\in l_\infty(X)$, and the proof is complete.

It is interesting to note in the next theorem that there are normed algebras X without an almost identity such that equality holds in (9).

Theorem 6. $l_\infty(c_0)=l_1^S(c_0)$.

Proof. By the argument of Theorem 5 there is a number M such that for all $n\in N$ and all $x\in l_1(c_0)$,

$$2\|x_n y_n\|\leq M\sum_{k=1}^{\infty}\|x_k\|,$$

where we take $y \in l_1^\delta(c_0)$. Now take any $n \in N$, any $p \in N$ and define $x_k = 0$ $(k \neq n)$, $x_n = e_p$, the p-th unit vector in c_0. Then we have $2|y_{np}| \leq M$, where

$$y_n = (y_{n1}, y_{n2}, \ldots) \in c_0$$

for each $n \in N$. Since n and p are arbitrary it follows that $\|y_n\| \leq M/2$, so $y \in l_\infty(c_0)$, as required.

If E is any linear subspace of $s(X)$ then its delta dual generates a natural locally convex topology on E determined by the seminorms

$$p_y(x) = \sum_{k=1}^{\infty} (\|x_k y_k\| + \|y_k x_k\|),$$

for each $x \in E$ and each $y \in E^\delta$. We shall call this the E^δ topology on E.

In conclusion we give the following result:

Theorem 7. *If the normed algebra X has an almost identity z then the $l_1^\delta(X)$ topology on $l_1(X)$ coincides with the norm topology of $l_1(X)$.*

Proof. As usual, the norm topology of $l_1(X)$ is given by the norm

$$\|x\| = \sum_{k=1}^{\infty} \|x_k\|$$

for each $x = (x_k) \in l_1(X)$.

First we show that the $l_1^\delta(X)$ topology is weaker than the norm topology of $l_1(X)$ even when X has no almost identity. Let $\epsilon > 0$ and $y_1, y_2, \ldots, y_r \in l_1^\delta(X)$, where

$$y_i = (y_{i1}, y_{i2}, \ldots).$$

By the argument of Theorem 5 there are positive numbers $M_1, \ldots, M_r$ such that

$$p_{y_i}(x) \leq M_i \|x\|$$

for $i = 1, 2, \ldots r$ and for all $x \in l_1(X)$. Taking M to be the largest of the M_i it follows that if $\|x\| < \epsilon / M$ then

$$\sup\{p_{y_i}(x) : i = 1, 2, \ldots, r\} < \epsilon,$$

whence, in the usual notation, the sphere $S(0, \epsilon/M)$ is contained in the neighbourhood

$$U(0, p_{y_1}, p_{y_2}, \ldots, p_{y_r}, \epsilon).$$

Conversely, suppose that X has an almost identity z with corresponding constant c, and let $\epsilon > 0$ be given. If we define

$$y = (z, z, z, \ldots)$$

then $y \in l_1^\delta(X)$ by Theorem 5. Hence if $x \in U(0, p_y, c\epsilon)$ then

$$\sum_{k=1}^{\infty} (\|x_k z\| + \|z x_k\|) < c\epsilon,$$

and since $c\|x_k\| \leq \|x_k z\| + \|z x_k\|$ for all $k \in N$ it follows that $\|x\| < \epsilon$, so that $x \in S(0, \epsilon)$, and the proof is complete.

References

[1] COOKE, R. G., *Infinite matrices and sequence spaces*, Macmillan and Co., London, 1949.

[2] GARLING, D. J. H., *The β- and γ-duality of sequence spaces*, Proc. Camb. Phil. Soc., *63* (1967), 963-981.

[3] KÖTHE, G., *Topological Vector Spaces I* (English translation by D.J.H. Garling of Topologische Lineare Räume I, 1966), Springer-Verlag, 1969.

[4] KÖTHE, G., and TOEPLITZ, O., *Lineare Räume mit unendlichvielen Koordinaten und Ringe unendlicher Matrizen*, J.f. reine u. angew. Math., *171* (1934), 193-226.

[5] MADDOX, I. J., *Infinite matrices of operators*, Springer-Verlag, Berlin, 1980.

Department of Pure Mathematics
Queen's University of Belfast
Belfast BT7 1NN
United Kingdom

Recibido: 29 de mayo de 1990