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Cyclic Branched Coverings of Knots and Homology Spheres

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ABSTRACT. We study cyclic coverings of S^3 branched over a knot, and study conditions under which the covering is a homology sphere. We show that the sequence of orders of the first homology groups for a given knot is either periodic or tends to infinity with the order of the covering, a result recently obtained independently by Riley. From our computations it follows that, if surgery on a knot k with less than 10 crossings produces a manifold with cyclic fundamental group, then k is a torus knot.

0. INTRODUCTION

A knot k in a homology sphere M naturally gives rise to an infinite sequence of closed 3-manifolds, $B_n(k)$, the n-fold covers of M branched over k. The n-fold cyclic cover of $M - N(k)$ corresponds to the kernel of the homomorphism $\pi_1(M-k) \to \mathbf{Z}_n$ induced via the abelianisation map. The branched cover $B_n(k)$ is obtained by adding a solid torus whose core $\tilde{k}$ covers k, and whose meridian is a loop on the boundary which is an n—fold cover of the meridian of k (see for instance **[BZ]**, chapter 8).

We denote the order of the first homology group (all coefficients are in $\mathbf{Z}$) by $b_n = |H_1(B_n(k))|$. Fox showed (see **[Web]**) that this is the resultant of the (first) Alexander polynomial $\Delta_k(t)$ and the polynomial $t^n - 1$ (with the proviso that $b_n = \infty$ when this resultant is 0).

Conversely, Fried **[Fr]** has recently shown that the sequence $\{b_n\}$ determines $\Delta_k(t)$ if no term is ∞, i.e. when no root of $\Delta_k(t)$ is an n-th root of unity.

Gordon showed that the sequence $\{b_n\}$ is periodic if and only if all the roots of $\Delta_k(t)$ are roots of unity, and asked (**[Gor1]**, page 366) whether $\{b_n\}$ is

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either periodic, or tends to infinity with n. We answer in the affirmative. Let $\mu(p)$ denote the measure of the polynomial $p(t)$ i.e. $\mu(p)=$ $|a_0| \prod_i max(1, |\alpha_i|)$, the product taken over all the roots α_i of the polynomial p according to multiplicity, where a_0 is the leading coefficient of p. Note that $\mu(p)=1$ if and only if p is monic and all of its roots have unit modulus, and so by Kronecker's theorem (see e.g. [**P**, p.118]), are all roots of unity.

Theorem 1.

$$\lim_{\substack{n\to\infty \\ b_n\neq 0}} \sqrt[n]{b_n} = \mu(\Delta_k).$$

Corollary

The sequence $\{b_n\}$ is either periodic or tends to infinity.

Proof of theorem: By definition, $b_n = |a_0^n \prod(\alpha_i^n - 1)|$ so that $\sqrt[n]{b_n} =$ $= |a_0| \prod \sqrt[n]{|\alpha_i^n - 1|}$.

It is easy to see that if $|\alpha| > 1$ then $\lim_{n\to\infty} \sqrt[n]{|\alpha^n - 1|} = |\alpha|$ and that if $|\alpha| < 1$ then $\lim_{n\to\infty} \sqrt[n]{|\alpha^n - 1|} = 1$.

If α is an algebraic integer with $|\alpha| = 1$ and $|\alpha^n| \neq 1$ then (see [**Ge**] and [**B**, p. 2-3])

$$2 \geq |\alpha^n - 1| = 2\,sin\left(\frac{n}{2} \left| \arg\alpha - \frac{p}{n} 2\pi \right| \right) \geq \frac{2}{\pi} n \left| \log\alpha - \frac{2p}{n} \log(-1) \right|$$
$$> C \exp\{-(logn)^6\}$$

where p is an integer closest to $\left(\frac{n \arg\alpha}{2\pi}\right)$ and C is a positive constant depending only on Δ_k. Hence, if $\Delta(\alpha) = 0$ and $|\alpha| = 1$ then $\lim_{\substack{n\to\infty \\ \alpha^n \neq 1}} \sqrt[n]{|\alpha^n - 1|} = 1$. The theorem now follows. ∎

This result appears to be known in the context of dynamical systems; b_n is the number of points in the g-torus $S^1 \times S^1 \times \ldots \times S^1$ of period n under an autohomeomorphism with characteristic polynomial Δ (see [**L**]). Here Δ is monic.

In §2 we use recent work of Mignotte and Waldschmidt to obtain an explicit bound N (in terms of $\mu(\Delta)$ and the coefficients of Δ) such that for $n > N$ we have $b_n \neq 1$. This bound is of the order of 10^{16} for knots with up to 10 crossings. In §3 we show how to use continued fractions to reduce the number of values of n for which b_n may be 1 from N to approximately $log_{10} N$.

Note that some of these results have been obtained independently by R. Riley using similar methods **[R3]**. Part of this work was done at C.I.M.A.T (Guanajuato, México). The authors would like to thank Arturo Ramirez for very useful conversations and Wolfgang Lassner for help with high precision computation of roots of some knot polynomials. Also many thanks are due to Clifton Webb of I.B.M. and Greg Baran of M.S.R.I. for their help in calculations using I.B.M.'s Scratchpad II package.

1. PRELIMINARY RESULTS AND DEFINITIONS

Let k be a knot in S^3; we use (m, l) to denote a meridian longitude pair on $\partial N(k)$ in S^3, and $(k;p/q)$ to denote the manifold obtained by p/q surgery on k (adding a solid torus to $S^3 - N(k)$ killing the element $m^p l^q$). We refer to $p/1$ surgery as *integer* surgery, sometimes written p-surgery. Note that a knot longitude and the Alexander polynomial Δ are well defined in a homology sphere. We use $B_n(k)$ to denote the n-fold cover of S^3 branched over the knot k, and $\tilde{k}$ to denote the inverse image of k. We shall say that the closed 3-manifold M is a *meta-homology-sphere* if $H_1(M)$ is finite, and $\pi_1(M)$ has perfect commutator subgroup. We say that M is *lens-like* if $\pi_1(M)$ is cyclic non-trivial and the universal cover of M is S^3. Note that if $\pi_1(k;\, p/q) = \mathbf{Z}_p * G$ $(p \neq 0)$, then $(k;p/q)$ is a meta-homology-sphere.

Let $f(t) = a_n + a_{n-1}t + \dots a_0 t^n$, $g(t) = b_m + b_{m-1}t + \dots b_0 t^m$ be polynomials with integer coefficients, and with roots $\alpha_1, \alpha_2, \dots, \alpha_n$, and $\beta_1, \beta_2, \dots, \beta_m$. The *resultant* of f and g is

$$res(f, g) = (a_0^m \cdot b_0^n) \prod_{i,j} (\alpha_i - \beta_j) = \alpha_0^m \prod_i g(\alpha_i) = (-1)^{mn} b_0^n \prod_j f(\beta_j)$$

$$= \det \begin{pmatrix} a_0 & a_1 & \cdots & a_n & 0 & \cdots & 0 \\ 0 & a_0 & a_1 & \cdots & a_n & \cdots & 0 \\ \vdots & \vdots & \ddots & & \ddots & & \vdots \\ 0 & \cdots & 0 & a_0 & a_1 & \cdots & a_n \\ b_0 & b_1 & \cdots & b_m & 0 & \cdots & 0 \\ 0 & b_0 & b_1 & \cdots & b_m & 0 & 0 \\ \vdots & \vdots & \ddots & & \ddots & & \vdots \\ 0 & \cdots & 0 & b_0 & b_1 & \cdots & b_m \end{pmatrix}.$$

(This is an $(m+n)\times(m+n)$ matrix). Resultants have the following multiplicative and commutative properties:

$$Res(f_1 f_2, g) = Res(f_1, g)\, Res(f_2, g),\quad Res(f, g) = (-1)^{mn}\, Res(g, f).$$

Here we shall always have one of the polynomials equal to t^m-1, as $b_m(k) = |Res(\Delta_k(t), t^m-1)|$. In this case, $b_m(k)$ is the absolute value of the determinant of the circulant matrix $(a_{i,j})$, where $a_{i,j} = a_{i+j-2}$, and $\sum_{i=0}^{m-1} a_i t^i$ is $\Delta_k(t)$ reduced modulo t^m-1.

If p divides m, then t^p-1 divides t^m-1, so $b_p(k)$ divides $b_m(k)$, by multiplicativity of resultants. It can be shown that $b_m(k)=1$ if and only if the ideal generated by $\Delta_k(t)$ and t^m-1 is $\mathbf{Z}[t]$. Before obtaining access to the IBM Scratchpad II program at M.S.R.I., Berkeley, California, the authors implemented a Buchberger algorithm in (PASCAL) on a microcomputer which can be used for small values of m to obtain a canonical basis for the ideal, and thus see if it is trivial (in this case the basis consists of the element 1). For large values of m, the coefficients of the polynomials formed in the reduction process rapidly become too large for PASCAL to handle, and the algorithm becomes very slow and impractical.

Proposition 2.

Let p and q be coprime integers with $p>1$.

(1) *$(k; p/q)$ is a meta-homology-sphere if and only if $B_p(k)$ is a homology sphere.*

(2) *If $(k; p/q)$ is a lens-like space then $\pi_1(S^3-k)$ has a proper subgroup of finite index isomorphic to a knot group.*

Note: when k is not a cable about a torus knot the converse of (2) holds (see [**GW**]).

Proof:

(1) The (unbranched) p-fold cyclic cover of $M=(k; p/q)$ is a manifold which we denote $\widetilde{M}$. Then $S^3-N(k)=M-T$, where T is the solid torus added during surgery. Covering T in $\widetilde{M}$ is a solid torus $\widetilde{T}$, and $B_p(k)-N(\tilde{k}) = \widetilde{M}-\widetilde{T}$, where $\tilde{k}$ is the preimage of k under the branched covering map $\rho: B_p(k)\to S^3$. The surgery curve $m^p l^q$ on $\partial N(k)$ lifts to the curve $\tilde{m}\tilde{l}^q$ on $\partial N(\tilde{k})$ where $\tilde{m}$ is the preimage of m under ρ, and $\tilde{l}$ is a component of the

preimage of l under ρ. Notice that $\tilde{l}$ is homologically trivial in $B_p(k)-$ *int* $N(\tilde{k})$, and therefore $\widetilde{M}$ and $B_p(k)$ have the same homology.

(2) In this case $\widetilde{M}$ is in fact S^3, so that $\widetilde{M}-\tilde{T}=B_p(k)-N(\tilde{k})$ is a knot complement. ∎

In particular, if k is a knot with tunnel number ≤ 2 (for instance a 3-bridge knot) and no torus knot polynomial divides $\Delta_k(t)$, then, for $p>1$, $(k;p/q)$ is either irreducible, or a meta-homology-sphere (**[GS]** prop. 2.2.). By the above proposition, to show that all surgeries on k give prime manifolds, it suffices to show that $b_p \neq 1$ for all $p>1$.

By the cyclic surgery theorem (**[CGSL]** corollary 1), at most 2 non-trivial surgeries on a non-torus knot in S^3 may give manifolds with cyclic fundamental group, and these must be adjacent integer surgeries, n and $n+1$ say. By the above proposition, if there are two such surgeries, then $b_n=b_{n+1}=1$, and $\Delta_k(-1)=b_2=1$, by the division of resultants. This in fact occurs with the Fintushel-Stern example, where k is the $(-2, 3, 7)$ pretzel knot, and $(k;18)$ and $(k;19)$ are lens spaces **[FS]**;

$$\Delta_k(t)=1-t+t^3-t^4+t^5-t^6+t^7-t^9+t^{10}$$

(in fact, for $n<200$, we have that $b_n=1$ exactly when $n=2$, 3, 5, 6, 9, 10, 15, 17, 18, 19, 25, 34, 37, 43, 59, 74 — the last six of these values were computed using IBM's Scratchpad II program). The measure (see introduction) of this polynomial is 1.176280821.... Lehmer's conjecture (see **[Bo]**) states that, if the measure of an integral polynomial $P(t)$ is less than this number, then all the roots of $P(t)$ are roots of unity.

Thus we have

Proposition 3.

Let k be a knot in a homotopy sphere M such that $M-k$ is not a Seifert fibred space.

If two non-trivial surgeries on k give manifolds with finite cyclic fundamental group, then $|\Delta_k(-1)|=1$.

For example, for a 2-bridge knot, $|\Delta(-1)|\neq 1$, so that at most one non-trivial surgery on a non-torus 2-bridge knot yields a manifold with cyclic fundamental group. Takahashi has shown **[T]** that if a non-trivial surgery on a 2-bridge knot yields a manifold with finite fundamental group, then the knot is a torus knot.

Proposition 4.

A p/q surgery on a (r, s)-cable knot about the knot K in S^3 gives a meta-homology-sphere if and only if $(p, rs)=1$ and $b_p(K)=1$.

Proof: Let k denote the (r, s)-cable about K, and $T_{r,s}(t)$ the Alexander polynomial of the (r, s)-torus knot $C_{r,s}$. Then $\Delta_k(t)=T_{r,s}(t)\Delta_K(t^r)$. The p-fold covering of S^3 branched over $C_{r,s}$ is the Brieskorn manifold $M(p, r, s)$. This is a homology sphere if and only if the integers p, r, s are pairwise coprime (see for example **[GW, §5]**), which means in this case that $(p, rs)=1$ (r and s are already assumed to be coprime, in order that the cable be a knot). Also $b_p(k)=b_p(C_{r,s}).\prod_{i=1}^{p} \Delta_K(\xi^{ri})$; the first factor is 1 if and only if $(p, rs)=1$. It follows then that ξ^r is a primitive p-th root of unity if ξ is, so that the second term is $b_p(K)$. ∎

The following proposition is useful when dealing with non-monic Alexander polynomials.

Proposition 5.

Let $h(t)$ be a polynomial in $\mathbf{Z}[t]$ such that $h(1)=1$ and such that there are integers α, β, g satisfying $h(t)=\alpha f(t)+\beta t^g$. Let a_g be the coefficient of t^g in $f(t)$.

Then for all odd values of p greater than $\max\{g, deg(f)-g\}$, *we have that*

$$b_p=Res(h, t^p-1)\equiv(-1)^{p+1}(\beta^p+p.\alpha\beta^{p-1}a_g(\text{mod } \alpha^2).$$

If in addition $h(-1)>0$, then the result holds for all $p>\max\{g, deg(f)-g\}$.

Proof: Let ξ be a primitive p-th root of unity. First note that when p is odd,

$$\prod_{i=1}^{p} h(\xi^i)=h(1)\prod_{i=1}^{\left[\frac{p}{2}\right]} |h(\xi^i)|^2$$

as the roots occur in conjugate pairs. This is a positive number, so that

$$Res(h, t^p-1)=\prod_{i=1}^{p} h(\xi^i)$$

(no absolute value needs to be mentioned). When p is even, this product must be multiplied by $h(-1)$, so remains positive if this number too is positive.

By supposition

$$\prod_{i=1}^{p} h(\xi^i) = \prod_{i=1}^{p} (\alpha f(\xi^i) + \beta \xi^{gi}) = (-1)^{p+1}(\beta^p + \alpha\beta^{p-1} \sum_{i=1}^{p} f(\xi^i)\xi^{-gi}) + \alpha^2 A$$

where A is an algebraic integer (being an algebraic function of the ξ^i, which are all algebraic integers). But this is equal to

$$(-1)^{p+1}(\beta^p + \alpha\beta^{p-1} \sum_{i=1}^{p} \left(\sum_{j=0}^{deg(f)} a_j \xi^{i(j-g)} \right)) + \alpha^2 A.$$

When $j-g<p$, we have that $\xi^{i(j-g)}$ is a d-th root of unity, for some d dividing p. It follows that $\sum_{i=1}^{p} \xi^{i(j-g)} = 0$ except when $j=g$, whence

$$Res(h, t^p - 1) = \beta^p + a_g p\alpha\beta^{p-1} + \alpha^2 A.$$

But if two integers are conguent mod α^2 in the ring of algebraic integers, then they are congruent in the ring of integers, and the result follows. ∎

An application of proposition 5

If $\Delta(t)$ is an Alexander polynomial of degree 2 (normalized to have positive leading coefficient), then there is a positive integer α and a choice of $\epsilon = \pm 1$ such that

$$\Delta(t) = \alpha t^2 + (\epsilon - 2\alpha)t + \alpha = \alpha(t^2+1) + (\epsilon - 2\alpha)t.$$

If ϵ is negative then the roots are real, and the sequence $\{b_n\}$ is monotonic increasing.

We can apply the proposition to the remaining case, where $\epsilon = +1$, $g=1$, and $a_1 = 0$. Notice that now we have $\Delta(-1) = 4\alpha - \epsilon > 0$ and so the proposition applies to cases odd $p > 1$.

$$b_p \equiv (1-2\alpha)^p \equiv 1 - 2p\alpha \pmod{\alpha^2}.$$

But $2p\alpha \equiv 0 \pmod{\alpha^2}$ if and only if $\alpha | 2p$. For example, if α is odd and $b_\alpha > 1$, then $b_n > 1$ for all $n > 1$.

Hartley has in fact shown that $b_p = 1$ for a degree 2 knot polynomial for some p, only if the polynominal is $t^2 - t + 1$, using Plans' theorem (see [**H**, Prop. 1.7]).

For example, the knot 9_{49} has an Alexander polynomial which can be expressed as follows:

$$\Delta(t) = 3 - 6t + 7t^2 - 6t^3 + 3t^4 = 3(1 - 2t - 2t^3 + t^4) + 7t^2$$

so $b_p \equiv 7^p \bmod 9$. But 7 has order 3 in $\mathbf{Z}_9$, so that

$$\pm b_p \equiv 1 \mod 9 \text{ iff } p \equiv 0 \mod 3.$$

Also $b_3 = 100$ so that $b_{3p} > 1$ for all p. Thus no cyclic cover of S^3 branched along 9_{49} is a homology sphere.

The following proposition is useful when $\Delta(t)$ has no roots on the unit circle. For example, it can be used to show that if k is a 3-bridge knot with less than 10 crossings, and $\Delta_k(t)$ has no roots on the unit circle, then $b_n(k) > 1$ for all $n > 1$.

Proposition 6. *If $\Delta(t)$ has no roots in the annulus $1 \leq |z| \leq \sqrt[n_0]{\frac{3+\sqrt{5}}{2}}$ then $b_n > 1$ for $n \geq n_0$.*

Proof: If $n \geq n_0$ and α is a root of Δ with $|\alpha| > 1$ then $|\alpha^n| > \frac{3+\sqrt{5}}{2}$ and it follows that $|(\alpha^n - 1)(\alpha^{-n} - 1)| > 1$.

2. NUMERICAL ESTIMATES

Let $f(t) = a_d + a_{d-1}t + \ldots + a_0 t^d$ be a polynomial with integer coefficients such that $a_0 + a_d$, $f(1)$ and $f(-1)$ are not zero. Suppose that (counted according to multiplicity) there are d'' roots of modulus 1, and d' roots of modulus different from 1. Denote the roots of modulus 1 and argument between 0 and π by $\alpha_1, \ldots, \alpha_{d''/2}$ and the roots of modulus different from 1 by $\beta_1, \ldots, \beta_{d'}$.

We set $L_n = |a_0^n \prod_{j=1}^{d'} (1 - \beta_j^n)|$ and $S_n = \prod_{i=1}^{d''/2} |1 - \alpha_i^n|^2$; if d' (resp. d'') is 0 then set L_n (resp. S_n) equal to 1. We then have $|Res(f(t), t^n - 1)| = L_n S_n$, and we call L_n (resp. S_n) the *large* (resp. *small*) part of $|Res(f(t), t^n - 1)|$. Clearly $S_n < 2^{d''}$.

It is easy to find a lower bound for $\log L_n$ (Lemma 7). We shall use results from transcendental number theory to obtain a lower bound for S_n. We estimate the numbers involved for 9_{48}, to give an idea of the orders of magnitude involved, and we show that for this knot, $b_n > 1$ for $n > 10^{14}$. For knots with less than 11 crossings whose Alexander polynomial has a root that is not a root of unity, $b_n > 1$ for $n > 2.6 \times 10^{16}$.

Let M denote the measure of the polynomial f,

$$M = \mu(f) = |a_0| \prod_j^{d'} \max\{1, |\beta_j|\}.$$

Let

$$R = \max\{\{|\beta_j| : |\beta_j| < 1\} \cup \{|\beta_j|^{-1} : |\beta_j| > 1\}\}.$$

If $d' = 0$ then R is defined to be 0. Notice that $R < 1$. The following lemma is an improvement on proposition 6.

Lemma 7. An estimate for the Large Part.

i) $(1 - R^n)^{d'} M^n \leq L_n \leq (1 + R^n)^{d'} M^n$

ii) $\lim_{n \to \infty} \sqrt[n]{L_n} = M$.

Proof: Write $\hat{\beta}_j = \beta_j$ if $|\beta_j| < 1$, and $\hat{\beta}_j = \beta_j^{-1}$ if $|\beta_j| > 1$.
Then $R = \max\{|\hat{\beta}_j|\}$, and

$$L_n = |a_0^n \prod_{j=1}^{d'} (1 - \beta_j^n)| = M^n |\prod_{j=1}^{d'} (1 - \hat{\beta}_j^n)|.$$

Since $(1 - R^n)^{d'} \leq \prod_{j=1}^{d'} |1 - \hat{\beta}_j^n| \leq (1 + R^n)^{d'}$, the first part of the theorem follows.

Assertion ii) is a consequence of i). ∎

We now want to estimate S_n.

Lemma 8. ([W, Lemma 2.4])

Suppose that $|\lambda| = 1$ *and* p *is an integer closest to* $\dfrac{n \arg \lambda}{2\pi}$.

Then $|1 - \lambda^n| \geq \dfrac{2n}{\pi} |\dfrac{2p}{n} \log(-1) - \log(\lambda)|$.

Proof:

$$|1-\lambda^n| = 2\sin(|\arg\lambda^n|/2) = 2\sin(\tfrac{n}{2}|\tfrac{p}{n}2\pi - \arg\lambda|$$

$$\geq \frac{2n}{\pi}|\frac{2p}{n}\pi - \arg\lambda| = \frac{2n}{\pi}|\frac{2p}{n}\log(-1) - \log\lambda|.$$

∎

Assume now that $|\lambda| = 1, 0 < \arg\lambda < \pi$, λ is algebraic and $\lambda^n \neq 1$. Again let p be an integer closet to $\frac{n\arg\lambda}{2\pi}$. We obtain a lower bound for $|\frac{2p\pi}{n} - \log\lambda|$ by applying Mignotte and Waldschmidt's main theorem [MV] with the following substitutions (assuming $n \geq 3$):

$S_0 = 1 + \log n \qquad D_0 = 1$

$S_1 = 1 + \pi \qquad D_1 = 1$

$S_2 = D + A \qquad D_2 = D =$ degree of the minimal polynomial for λ

where $A \geq \max\{\log h(\lambda), \arg\lambda\}$, and $h(\lambda)$, the height of λ, is the maximum of the absolute values of the coefficients of the minimal polynomial for λ. Now we need that

$$T = 5 + \log(Dn(1+\pi)(D+A))$$

$$e < E \leq \min\left\{\frac{eD(1+\pi)}{\pi}, \frac{e(D+A)}{\arg\lambda}, e^{\frac{2\pi}{5}}\right\}.$$

The term $e^{\frac{2\pi}{5}}$ can be omitted if $n \geq \frac{1}{25}D^{1/2}$ and the term $\frac{e(D \pm A)}{\arg\lambda}$ can be omitted if $\arg\lambda \leq (1+\pi^{-1} - D^{-1})^{-1}$.

Making these substitutions, the main theorem of [MW, p. 242] gives:

Lemma 9.

If $n \geq 3$, *then*

$$|\frac{2p}{n}\log(-1) - \log\lambda| > exp\{-5.10^8(1+\pi)D^3(D+A)T^2(\log E)^{-3}\}$$

Because of the multiplicity property of resultants, to estimate b_n it suffices to estimate $|Res(f(t), t^n - 1)|$ when $f(t)$ is an irreducible factor of an Alexander polynomial. Moreover, if $\Phi_m(t)$ is the m-th cyclomotic polynomial, one can actually calculate $Res(\Phi_m(t), t^n-1)$. It depends only on m and δ, where $\delta=(m,n)$. One has (see [**A**, page 460]):

$$Res(\Phi_m(t), t^n-1)=\begin{cases} p^{\phi(m)/\phi(\frac{m}{\delta})}, & \text{if } \frac{m}{\delta}=p^\alpha,\ p \text{ prime}, \alpha>0 \\ 1, & \text{otherwise,} \end{cases}$$

where ϕ is Euler's function.

For these reasons the next theorem, which gives a lower bound for $|Res(f(t), t^n-1)|$, is only stated for primitive, irreducible, noncyclotomic polynomials.

Theorem 10.

Lef $f(t)=a_d+a_{d-1}t+\dots a_0t^d$ be a primitive, irreducible, noncyclotomic integral polynomial of degree $d>1$. Let

$$a_0\prod_{i=1}^{d''/2}(t-\alpha_i)\prod_{i=1}^{d''/2}(t-\alpha_i^{-1})\prod_{j=1}^{d''}(t-\beta_j)$$

be its factorization over **C**, *where $|\alpha_i|=1$, $0<\arg\alpha_i<\pi, (i=1,\dots,d'')$ and $|\beta_j|\neq 1$, $(j=1,\dots,d')$.*

Let

M be the measure of $f(t)$,

$R=\max\{|\beta_j|:|\beta_j|<1\}\cup\{|\beta_j|^{-1}:|\beta_j|>1\}$,

$a=\max\{\log|a_0|,\dots,\log|a_d|,\arg\alpha_1,\dots,\arg\alpha_{d''/2}\}$

$e_i=1+\log(\min\left\{\frac{d(1+\pi)}{\pi},\frac{d+a}{\arg\alpha_i}\right\})$

$\sigma=\sum_{i=1}^{d''/2}e_i^{-3}$

$c=10^9(1+\pi)d^3(d+a)\sigma(\log M)^{-1}$

$h=5+\log((1+\pi)d(d+a))$.

If $n\geq\max\{3,\frac{1}{25}d^{1/2}\}$ then

$$2^{d''}(1+R^n)^{d'}M^n \geq |Res(f(t), t^n-1)|$$
$$\geq (1-R^n)^{d'}(\frac{2n}{\pi})^{d''} M^{n-c(h+\log n)^2}.$$

Proof: $|Res(f(t), t^n-1)| = L_n S_n$ and $L_n \leq (1+R^n)^{d'} M^n$ (by Lemma 7), and $S_n \leq 2^{d''}$ from which the first inequality follows.

Let p_i be an integer closest to $\frac{n \arg \alpha_i}{2\pi}$. Using Lemmas 8 and 9 we have

$$S_n = \prod_{i=1}^{d''/2} |1-\alpha_i|^2$$
$$\geq (\frac{2n}{\pi})^{d''} \prod_{i=1}^{d''/2} |\frac{2pi}{n} \log(-1) - \log \alpha_i|^2$$
$$\geq (\frac{2n}{\pi})^{d''} \prod_{i=1}^{d''/2} \exp\{-10^9 (1+\pi) d^3 (d+a)$$
$$(5+\log((1+\pi) d(d+a)) + \log n)^2 e_i^{-3}\}$$
$$= (\frac{2n}{\pi})^{d''} M^{-c(h+\log n)^2}$$

Since, by Lemma 7, $L_n \geq (1-R^n)^{d'} M^n$ the second inequality follows. ∎

Corollary 11.

Suppose s is a positive number satisfying $e^s - 2s \geq h + \log c$.

If $n \geq ce^{2s}$ then $Res(f(t), t^n-1)| \geq (1-R^n)^{d'} (\frac{2n}{\pi})^{d''}$.

(As pointed out by Santiago López de Medrano, $e^s - 2s \geq h + \log c$ holds if $s \geq 1 + \sqrt{h - 1 + \log c}$.)

Proof: We may assume that $d'' > 0$. Then $n \geq ce^2 s > \max\{3, \frac{9}{25} d^{1/2}\}$ and

$$|Res(f(t), t^n-1)| \geq (1-R^n)^{d'} (\frac{2n}{\pi})^{d''} M^{n-c(h-\log n)^2}.$$

Now $n=ce^{2t}$ where $t\geq s\geq \ln 2$. Then $e^t-2t\geq e^s-2s\geq h+\log c$ and so $e^t\geq h+\log\ c+2t$ which implies that $ce^{2t}\geq c(h+\log c+2t)^2$, that is $n\geq c(h+\log n)^2$. Hence

$$M^{n-c(h-\log n)^2}\geq 1 \text{ and } |Res(f(t),t^n-1)|\geq(1-R^n)^{d'}(\frac{2n}{\pi})^{d''}.$$

∎

Example For the knot 9_{48} we have that $\Delta(t)=1-7t+11t^2-7t^3+t^4$.

There are two roots of unit modulus α, α^{-1}, where $\arg\alpha=\cos^{-1}(\frac{7-\sqrt{13}}{4})$ and there are two real roots, β,β^{-1} where

$$\beta=(7+\sqrt{13}+\sqrt{46+14\sqrt{13}})/4=5.109646....$$

We have $M=\beta$, $R=\beta^{-1}$, $a=\log(11)$, $e_1=1+\log 4(1+\pi^{-1})$, $\sigma=e_1^{-3}$, $c=55.092935...\times 10^9$ and $h=9.6633438....$

The inequality $e^s-2s\geq h+\log\ c$ holds if $s=3.73444$. Thus if $n\geq ce^{2s}=9.6558226...\times 10^{13}$ one has $b_n\geq(1-R^n)^2(\frac{2n}{\pi})^2>9.8\times 10^{26}$.

Using table 1 of **[BZ]** and table 1 of **[Bo]** one obtains the following bounds for noncyclotomic irreducible factors of Alexander polynomials of knots with less than 11 crossings: $d\leq 8$, $d+a\leq 8+\pi, \sigma\leq 3(1+\log(1+\frac{8}{\pi}))^{-3}$ and $M\geq 1.6355731$ ($M\geq 1.75$ for degree 8 polynomials). Hence $h+\log c\leq 40.929691$ and $e^s-2s\geq h+\log c$ holds if $s=3.88571$. One can conclude that, if $n\geq 2.6\times 10^{16}$ then $b_n>2.7\times 10^{32}$. Thus $n\geq 2.6\times 10^{16}$ and k is a knot with less than 11 crossings such that $\Delta_k(t)$ is not a product of cyclotomic polynomials, then the n-fold cyclic cover of S^3 branched over k is not a homology sphere.

3. USING CONTINUED FRACTIONS

When α is a complex number of unit modulus, the denominators of the convergents of $\frac{\arg\alpha}{2\pi}$ are the values q such that α is "near" a q-th root of unity. In order to discover for which values of n b_n may be 1, we study continued fraction expansions of the arguments, divided by 2π, of the roots of unit modulus of $\Delta_k(t)$. This shall enable us to reduce significantly the upper bound of $\{n\mid b_n(k)=1\}$ given in §2. Thus, for example, if k is a 3-

bridge knot with less than 10 crossings, and $\Delta_k(t)$ is not a product of cyclotomic polynomials, then $b_n(k)>1$ for $n>11$.

To an irrational number θ one associates a sequence of integers $c_0, c_1, c_2, \ldots$ (with $c_j>0$ if $j>0$) as follows:

$$n_0=\theta, c_i=\text{integer part of } n_i, \; n_{i+1}=\frac{1}{n_i-c_i}.$$

Then $\theta=[c_0, c_1, c_2, \ldots]=c_0+1/(c_1+1/(c_2+\ldots))$ is the continued fraction expansion of θ.

Also if $\theta=[c_0, c_1, c_2, \ldots]$ and j is a nonnegative integer we set

$$\frac{p_j}{q_j}=[c_0, c_1, c_2, \ldots, c_j]=c_0+1/(c_1+1/(c_2+\ldots+1/c_j)\ldots))$$

where p_j, q_j are relatively prime integers, and $q_j>0$.

The numbers $\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots$ (resp. $q_0, q_1, q_2, \ldots$) are called the *convergents* (resp. *denominators of the convergents*) of θ. One has $q_j=c_j q_{j-1}+q_{j-2}$ for $j>1$.

We denote by $\|x\|$ the distance from the real number x to the integers, i.e., $\|x\|=\inf_{p\in\mathbb{Z}}|x-p|$. The denominators of the convergents of θ have the following properties:

1) If $j\geq 1$, q_{j+1} is the smallest positive integer such that $\|q_{j+1}\theta\|<\|q_j\theta\|$.

2) If $j\geq 1$, then $\|q_j\theta\|=c_{j+2}\|q_{j+1}\theta\|+\|q_{j+1}\theta\|$.

Lemma 12.

Let θ be an irrational number in $(0, 1)$, let $[0, c_1, c_2 \ldots]$ be its continued fraction expansion, and let $q_0, q_1, \ldots$ be the denominators of its convergents. Suppose that $B>1$ and that j is an integer such that $\|q_j\theta\|>B^{-q_j}$. Then

i) $\|n\theta\|>B^{-n}$ if $q_j\leq n<q_{j+1}$.

ii) $\|q_{j+1}\theta\|>B^{-q_{j+1}}$ if $c_{j+2}+1\leq B^{q_{j+1}-q_j}$.

Proof: The first assertion is a consequence of the fact that $\|n\theta\| > \|q_j\theta\|$ if $q_j \leq n < q_{j+1}$. To prove ii) notice that

$$\|q_{j+1}\theta\| > \|q_{j+2}\theta\| = \|q_j\theta\| - c_{j+2}\, \|q_{j+1}\theta\|$$

and therefore

$$\|q_{j+1}\theta\| > (c_{j+2}+1)^{-1}\, \|q_j\theta\| > B^{q_j - q_{j+1}}\, B^{-q_j} = B^{-q_{j+1}}.$$

∎

Now let $f(t) = a_d + a_{d-1}\, t + \ldots + a_0\, t^d$ be a polynomial with integer coefficients such that $a_0 \neq 0$ and $a_d \neq 0$. We assume that no root of $f(t)$ is a root of unity. We follow the notation of §2.

Lemma 13.

If $\|n\, \frac{\arg \alpha_i}{2\pi}\| > M^{-n/d''}$, $(i = 1, \ldots, d''/2)$ *then* $|Res\,(f(t), t^n - 1)| > 4^{d''}(1 - R^n)^{d''}$.

Proof: Since $|\alpha_i^n - 1| = 2 \sin(\pi\, \|n\, \frac{\arg \alpha_i}{2\pi}\|)$, and by Lemma 7, $L_n \geq (1 - R^n)^{d'}\, M^n$, we have

$$L_n\, S_n \geq (1 - R^n)^{d'}\, M^n (2 \sin(\pi \min_{i \leq d''/2} \|n\, \frac{\arg \alpha_i}{2\pi}\|))^{d''}.$$

Using $\sin x \geq \frac{2}{\pi} x$ for $0 \leq x \leq \frac{\pi}{2}$, and $\|n \frac{\arg \alpha_i}{2\pi}\| > M^{-n/d''}$, one sees that the right hand side is greater than $4^{d''}\, (1 - R^n)^{d'}$. ∎

Theorem 14.

Let u and v be positive integers with the following property:

When $\theta \in \{\frac{\arg \alpha_1}{2\pi}, \ldots, \frac{\arg \alpha_{d''/2}}{2\pi}\}$, *and* $[0, c_1, c_2, \ldots]$ *is the continued fraction expansion of* θ, *and* $q_0, q_1, q_2, \ldots$ *are the denominators of the convergents of* θ,

i) $\|q_j\theta\| > M^{-q_j/d''}$ *if* $q_j \leq u < q_{j+1}$ *and*

ii) $c_{j+2} + 1 \leq M^{(q_{j+1} - q_j)/d''}$ *if* $u < q_{j+1}$ *and* $q_j < v$.

Then $|Res\,(f(t), t^n - 1)| > 4^{d''}(1 - R^n)^{d'}$ *for* $u \leq n < v$.

Proof: This follows from Lemmas 12 and 13, taking $B = M^{\frac{1}{d'}}$. ∎

Assuming that the Alexander polynomial of the knot k is not a product of cyclotomic polynomials, Corollary 11 provides a number v such that $b_n(k) > 1$ for $n \geq v$ (one may take $v = 2.6 \times 10^{16}$ if k has less than 11 crossings). Then Theorem 14 provides a number u such that $b_n(k) > 1$ if $u \leq n < v$ (for 3-bridge knots with less than 11 crossings one can take $u = 290$). To determine for which values of n (smaller than u) $b_n(k) = 1$ the following observations are useful:

Remarks

1) One has the inequalities

$$|Res(f(t), t^n - 1)| \geq (1 - R^n)^{d'} M^n 2^d \prod_{i-1}^{d''/2} \sin^2(\pi \| n \frac{\arg \alpha_i}{2\pi} \|)$$

(implicit in the proof of lemma 13) and $\| n \frac{\arg \alpha_i}{2\pi} \| \geq \| q_j \frac{\arg \alpha_i}{2\pi} \|$ where q_j is the largest denominator of a convergent of $\| \frac{\arg \alpha_i}{2\pi} \|$ smaller than n.

2) As noted before, if $b_n(k) = 1$ then $b_m(k) = 1$ for every m diving n. Thus if $\{n > 1 \mid b_n(k) = 1\}$ is not empty, the smallest element of the set is a prime number.

3) If $L_n S_n = 1$ and $L_m < L_n$ then $S_n < S_m$. Thus if $L_n S_n = 1$ and $L_n > \max_{1 \leq m < n} L_m$ then $S_n < \min_{1 \leq m < n} S_m$. One can show that, if $f(t)$ is reciprocal, then $L_n > \max_{1 \leq m < n} L_m$ holds for

$$n > \frac{\log(1 - R^{\frac{1}{2}}) - \log(1 + R^{-\frac{1}{2}})}{\log R}$$

and so, if in addition $L_n S_n = 1$, we must have $S_n < \min_{1 \leq m < n} S_m$. Notice that, if $d'' = 2$, the values of n satisfying $S_n < \min_{1 \leq m < n} S_m$ are precisely the denominators of the convergents of $\frac{\arg \alpha_i}{2\pi}$.

4) $b_n(k) = 1$ if and only if the image of $\Delta_k(t) \in \mathbf{Z}(\mathbf{Z})$ in the group ring $\mathbf{Z}(\mathbf{Z}_n)$ is a unit. If $n < 7$ and $n \neq 5$, then all units of $\mathbf{Z}(\mathbf{Z}_n)$ are of the form $\pm t^i$ where t generates $\mathbf{Z}_n$.

Applications

We consider the 34 3-bridge knots with less than 10 crossings. For 10 of these knots, $\Delta(t)$ has a quadratic, non-cyclotomic, reiprocal factor and so, by [**H**, Prop. 1.7], $b_n>1$ if $n>1$. For 9 of the remaining 24 knots, $\Delta(t)$ has a reciprocal factor with no roots of unit modulus, and it is not difficult to show, using Prop. 6 that $b_n>1$ if $n>1$. For three of the knots (8_{10}, 8_{19}, and 8_{20}) $\Delta(t)$ is a product of cyclotomic polynomials and $b_n=1$ if and only if $n\equiv\pm1$ mod 6. Handling the remaining 12 knots (the knots in Table 1) requires Corollary 11, Theorem 14, and high precision computation.

For purposes of calculation it is easier to approximate the roots of the Conway-like polynomial $P(x)$ where $P(t+t^{-1})\doteq\Delta(t)$; a real root of P between -2 and 2 corresponds to a modulus one root of Δ. This means that a real Newton's algorithm can be used for P. Denoting by α a modulus one root of the Alexander polynomial of one of the 12 knots of Table 1, and assuming that α is not a root of unity and $0<\arg\alpha<\pi$, a continued fraction expansion for $\frac{\arg\alpha_i}{2\pi}$ was obtained by finding values q such that $|1-\alpha^q|<\min\limits_{m<q}|1-\alpha^m|$. These calculations were checked by calculating the arcosine of half of the root of P (using Borchardt's algorithm to obtain the required number of digits), dividing by 2π, and then calculating the continued fraction of this by the standard procedure:

$$c_i=\text{integer part of } n_i,\ n_{i+1}=-\frac{1}{n_i-c_i}.$$

Calculations were done using a Mumath symbolic package, mainly for its ability to handle arbitrary precision arithmetic.

This work was subsequently checked at M.S.R.I. using the I.B.M. Scratchpad program on an I.B.M. RT, where all these calculations were much easier to implement.

For the 12 knots referred to above, Table 1 shows the polynomial $F(t)=\Delta(t)/(\text{cyclotomic factors})$ of the knot, the measure M of $F(t)$, the greatest modulus R of the roots of $F(t)$ inside the unit circle ($R=0$ if no such roots exist), the number d' (resp. d'') of roots of $F(t)$ off (resp on) the unit circle, the continued fraction expansion of $\frac{\arg\alpha_i}{2\pi}$ where α_i are the modulus one roots of $F(t)$ with argument between 0 and π, a few denominators of convergents of $\frac{\arg\alpha_i}{2\pi}$, and values of u and v such that the hypothesis of Theorem 14 is satisfied..

Table 1

8_5 and 9_{42} $f(t)=t^4-2t^3+t^2-2t+1$

$M=1.883203\ldots$, $R=0.531011\ldots$, $d'=2$, $d''=2$

$$\frac{\arg\alpha_1}{2\pi}=[0, 3, 1, 1, 7, 1, 1, 4, 4, 1, 7, 1, 1, 82, 1, 3, 5, 5, 1, 3, 14, 7, 2, 3, 1, 1, 3, 2, \ldots]$$

$q_1=3$, $q_2=4$, $q_3=7$, $q_4=53$, $q_5=60, \ldots, q_{26}\approx 3.1\times 10^{16}$

$u=53$, $v=2.6\times 10^{16}$

8_{16} $f(t)=t^6-4t^5+8t^4-9t^3+8t^2-4t+1$

$M=3.219439\ldots$, $R=0.557326\ldots$, $d'=4$, $d''=2$

$$\frac{\arg\alpha_1}{2\pi}=[0, 4, 2, 1, 17, 1, 1, 6, 2, 4, 5, 3, 3, 1, 2, 2, 1, 2, 1, 2, 3, 5, 1, 5, 1, 1, 1, 2, 1, 129, 2, 3, 3, 231, 1, \ldots]$$

$q_1=4$, $q_2=9$, $q_3=13$, $q_4=230$, $q_5=243, \ldots, q_{33}\approx 2.1\times 10^{17}$

$u=13$, $v=2.6\times 10^{16}$

9_{16} $f(t)=2t^4-3t^3+3t^2-3t+2$

$M=2$, $R=0$, $d'=0$, $d''=4$

$$\frac{\arg\alpha_1}{2\pi}=[0, 3, 1, 2, 26, 3, 6, 1, 10, 5, 2, 1, 6, 12, 8, 3, 1, 2, 1, 5, 1, 2, 401, 17, 10, 18, 3, \ldots]$$

$q_1=3$, $q_2=4$, $q_3=11$, $q_4=290$, $q_5=881, \ldots, q_{25}\approx 2.8\times 10^{17}$

$$\frac{\arg\alpha_2}{2\pi}=[0, 13, 3, 2, 1, 1, 3, 1, 1, 11, 6, 1, 7, 1, 54, 3, 1, 1, 2, 3, 6, 1, 3, 1, 26, 3, 30, 1, 1, 1, 5, 1, 8, 1, \ldots]$$

$q_1=13$, $q_2=40$, $q_3=93$, $q_4=133$, $q_5=226, \ldots, q_{32}\approx 7.1\times 10^{16}$

$u=290$, $v=2.6\times 10^{16}$

9_{22} $f(t)=t^6-5t^5+10t^4-11t^3+10t^2-5t+1$

$M=4.53532\ldots$, $R=0.469565\ldots$, $d'=4$, $d''=2$

$$\frac{\arg\alpha_1}{2\pi}=[0, 4, 4, 1, 1, 1, 2, 1, 1, 25, 1, 1, 1, 4, 1, 2, 1, 1, 3, 4, 1, 1, 1, 3, 4, 4, 1, 3, 1, 1, 2, 3, 2, 2, 15, 1, 2, 32, 2, 1, 30, 1, \ldots]$$

$q_1=4,\ q_2=17,\ q_3=21,\ q_4=38,\ q_5=59, \ldots, q_{40}\approx 2.9\times 10^{17}$

$u=4,\ v=2.6\times 10^{16}$

9_{25} $f(t)=3t^4-12t^3+7t^2-12t+3$

$M=6.304498\ldots,\ R=0.475851\ldots,\ d'=2,\ d''=2$

$$\frac{\arg \alpha_1}{2\pi}=[0,\ 8,\ 16,\ 3,\ 1,\ 1,\ 8,\ 5,\ 7,\ 1,\ 1,\ 8,\ 1,\ 3,\ 6,\ 3,\ 9,\ 1,\ 7,\ 2,\ 10,\ 1,\ 4,\ 4,\ 2,\ 1,\ 1,\ 40,\ 2,\ 3,\ 2,\ldots]$$

$q_1=8,\ q_2=129,\ q_3=395,\ q_4=524,\ q_5=919,\ \ldots,\ q_{29}\approx 2.9\times 10^{16}$

$u=8,\ v=2.6\times 10^{16}$

9_{32} $f(t)=t^6-6t^5+14t^4-17t^3+14t^2-6t+1$

$M=5.562469\ldots,\ R=0.424001\ldots,\ d'=4,\ d''=2$

$$\frac{\arg \alpha_1}{2\pi}=[0,\ 5,\ 8,\ 9,\ 4,\ 4,\ 1,\ 32,\ 1,\ 2,\ 22,\ 2,\ 1,\ 1,\ 5,\ 6,\ 299,\ 1,\ 2,\ 1,\ 4,\ 10,\ 6,\ 11,\ 1,\ 7,\ 65,\ldots]$$

$q_1=5,\ q_2=41,\ q_3=374,\ q_4=1537,\ q_5=6522, \ldots,\ q_{25}\approx 1.1\times 10^{17}$

$u=5,\ v=2.6\times 10^{16}$

9_{36} $f(t)=t^6-5t^5+8t^4-9t^3+8t^2-5t+1$

$M=3.165265\ldots,\ R=0.315928\ldots,\ d'=2,\ d''=4$

$$\frac{\arg \alpha_1}{2\pi}=[0,\ 3,\ 1,\ 3,\ 1,\ 6,\ 4,\ 183,\ 9,\ 1,\ 1,\ 8,\ 1,\ 5,\ 1,\ 2,\ 9,\ 1,\ 6,\ 1,\ 5,\ 1,\ 12,\ 8,\ 11,\ 1,\ 1,\ 3,\ 2,\ 5,\ 3,\ \ldots]$$

$q_1=3,\ q_2=4,\ q_3=15,\ q_4=19,\ q_5=129, \ldots,\ q_{30}\approx 5\times 10^{16}$

$$\frac{\arg \alpha_1}{2\pi}=[0,\ 11,\ 8,\ 1,\ 1,\ 1,\ 23,\ 5,\ 1,\ 2,\ 22,\ 1,\ 1,\ 1,\ 3,\ 1,\ 1,\ 11,\ 6,\ 2,\ 1,\ 2,\ 10,\ 7,\ 1,\ 42,\ 1,\ 14,\ 1,\ 1,\ 1,\ 1,\ 7,\ 3,\ldots]$$

$q_1=11,\ q_2=89,\ q_3=100,\ q_4=189,\ q_5=289,\ \ldots,\ q_{32}\approx 8.8\cdot 10^{16}$

$u=89,\ v=2.6\times 10^{16}$

9_{43} $f(t)=t^6-3t^5+2t^4-t^3+2t^2-3t+1$

$M=2.225868\ldots,\ R=0.449263\ldots,\ d'=2,\ d''=4$

$$\frac{\arg\alpha_1}{2\pi}=[0, 2, 1, 4, 1, 3, 447, 1, 116, 1, 3, 1, 10, 3, 3, 3, 1, 1, 1, 2, 3, 3, 1, 2, 1, 4, 5, 5, 6, 1, 1, 1, 3, 1, \ldots]$$

$q_1=2$, $q_2=3$, $q_3=14$, $q_4=17$, $q_5=65$, ..., $q_{32}\approx 3.1\times 10^{16}$

$$\frac{\arg\alpha_2}{2\pi}=[0, 4, 13, 1, 7, 11, 6, 3, 1, 2, 2, 1, 2, 42, 3, 1, 1, 7, 17, 1, 13, 1, 3, 2, 1, 27, 1, 1, 10, 1, \ldots]$$

$q_1=9$, $q_2=118$, $q_3=127$, $q_4=1007$, $q_5=1134$, ..., $q_{29}\approx 4\times 10^{16}$

$u=127$, $v=2.6\times 10^{16}$

9_{45} $f(t)=t^4-6t^3+9t^2-6t+1$

$M=4.174674\ldots$, $R=0.239539\ldots$, $d'=2$, $d''=2$

$$\frac{\arg\alpha_1}{2\pi}=[0, 9, 1, 1, 2, 3, 4, 1, 2, 116, 1, 3, 1, 1, 2, 2, 4, 1, 1, 3, 2, 2, 2, 7, 3, 2, 1, 1, 1, 1, 1, 1, 2, 1, 62, 1, 3, 1, 1, 1, 6, 1, 6, 3, \ldots]$$

$q_1=9$, $q_2=10$, $q_3=19$, $q_4=48$, $q_5=163$, ..., $q_{42}\approx 1.6\times 10^{17}$

$u=9$, $v=2.6\times 10^{16}$

9_{47} $f(t)=t^6-4t^5+6t^4-5t^3+6t^2-4t+1$

$M=3.832725\ldots$, $R=0.510795\ldots$, $d'=4$, $d''=2$

$$\frac{\arg\alpha_1}{2\pi}=[0, 3, 2, 2, 81, 1, 29, 4, 1, 5, 14, 1, 1, 1, 1, 2, 2, 1, 11, 2, 2, 2, 2, 1, 5, 8, 297, 1, 4, 2, \ldots]$$

$q_1=3$, $q_2=7$, $q_3=17$, $q_4=1384$, $q_5=1401$, ..., $q_{28}\approx 2.8\times 10^{16}$

$u=7$, $v=2.6\times 10^{16}$

9_{48} $f(t)=t^4-7t^3+11t^2-7t+1$

$M=5.109646\ldots$, $R=0.195708\ldots$, $d'=2$, $d''=2$

$$\frac{\arg\alpha_1}{2\pi}=[0, 11, 3, 1, 2, 6, 2, 1, 1, 13, 244, 3, 1, 2, 4, 4, 1, 1, 2, 26, 1, 5, 52, 3, 4, 1, 2, 1, 17, 1, \ldots]$$

$q_1=11$, $q_2=34$, $q_3=45$, $q_4=124$, $q_5=789$, ..., $q_{29}\approx 2.4\times 10^{17}$

$u=11$, $v=2.6\times 10^{16}$

Using Corollary 11, Theorem 14 and the remarks above, it was found that the only values of (k, n) such that $b_n(k)=1$, $n>1$ and k is one of the knots appearing in Table 1 are $(8_5, 7)$, $(9_{16}, 11)$ and $(9_{42}, 7)$.

Using this, we show now that p/q-surgery $(q\neq 0)$ on a nontrivial nontorus knot with less than 10 crossings produces a manifold with noncyclic fundamental group. Since such knots are known to have property P by [CGLS, Corollary 7] and [AM], we may assume that $|p|>1$. By the cyclic surgery theorem ([CGLS, Corollary 1]) $q=1$. Also nontrivial surgery on nontorus 2-bridge knots produces manifolds with infinite fundamental group by [T]. Thus, by proposition 2(1), the only pairs (k, p) that need to be considered, where k is the knot and p is the surgery, are $(8_5, \pm 7)$, $(9_{16}, \pm 11)$, $(9_{42}, \pm 7)$, $(8_{10}, p)$ and $(8_{20}, p)$ with $p\equiv\pm 1 \bmod 6$ (notice that 8_{19} is a torus knot).

By [R2, p. 282] if k is 8_{10} or 8_{20} there is an epimorphism from the group of k onto PSL (2, Z) sending the longitude to the trivial element and the meridian to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and therefore $\pi_1(k;p)$, with $p\equiv\pm 1 \bmod 6$, projects onto PSL $(2, Z_{p_1})$, p_1 a prime divisor of p.

Recall that if M^3 is a homology 3-sphere and M bounds a smooth oriented 4-manifold Y such that $H_1(Y)$ has no 2-torsion and the quadratic form of Y is even then $\mu(M)=\frac{\sigma(Y)}{8} \bmod 2$. This is well defined by Rohlin's theorem ([Ro]). If k is a knot in S^3 then $\mu(B_p(k))$ can be computed using [C, Theorem 5] or [K, Theorem 12.6].

Also if k is a knot in a homology 3-sphere Σ^3 then the Arf invariant $\chi(k)$ of k is $\frac{b^2-1}{8} \bmod 2$ where b is the order of the first homology group of the 2-fold cyclic cover of Σ^3 branched over k ([*Le*], [*M*]).

Proposition. *If $\pm p/1$-surgery on a nontorus knot k produces a manifold with cyclic fundamental group then $\mu(B_p(k))=\chi(k)$.*

Remark. One can prove the stronger assertion that the Casson invariant ([AM]) of $B_p(k)$ equals $\pm\tilde{\Delta}''(1)$ where $\tilde{\Delta}(t)$ is obtained by symmetrizing the polynomial $\prod_{i=0}^{p-1} \Delta(w^i t^{1/p})$, $\Delta(t)$ is the Alexander polynomial of k and w is a primitive p-th root of unity.

Proof. We use the notation of the proof of Prop. 2(1). The manifold $\widetilde{M}$ is a homotopy 3-sphere and so, by Casson's theorem ([AM]), $\mu(\widetilde{M})=0$. Now $\widetilde{M}$ (resp. $B_p(k)$) is obtained from $B_p(k)-int\,N(\tilde{k})$ by adding a solid torus $\widetilde{T}$ (resp. $N(\tilde{k})$) in such a way that $\tilde{m}\tilde{l}^{\pm1}$ (resp. $\tilde{m}$) bounds a 2-disk in the solid torus. Hence, by [Gor2, Theorem 2], $\mu(B_p(k))=\chi(\tilde{r})$, where $\tilde{r}$ is the core of $\widetilde{T}$. But $\chi(\tilde{r})$ (resp. $\chi(k)$) is $\dfrac{b_{2p}^2-1}{8}$ mod 2 (resp. $\dfrac{b_2^2-1}{8}$ mod 2) where $b_n=|H_1(B_n(k))|$. The numbers b_2 and b_{2p} are odd and, by [H, Lemma 2.1] $\dfrac{b_{2p}}{b_2}$ is the square of an integer. It follows that $\chi(\tilde{r})=\chi(k)$ and therefore $\mu(B_p(k))=\chi(k)$. ∎

One has $\mu(B_7(8_5))=1$, $\chi(8_5)=0$, $\mu(B_7(9_{42}))=0$, $\chi(9_{42})=1$ and therefore, $\pi_1(8_5;\pm7)$ and $\pi_1(9_{42};\pm7)$ are not cyclic.

Finally if k is 9_{16} then $\pi_1(S^3-k)$ has a Wirtinger presentation with generators $x_1,\ldots,x_9$ and relations

$$x_2x_1=x_4x_2=x_5x_4=x_6x_5,$$

$$x_3x_6=x_8x_3=x_7x_8=x_9x_7,$$

$$x_3x_1=x_1x_9=x_9x_5.$$

An epimorphism from $\pi_1(k;-11)$ onto SL(2,23) (essentially due to Riley [R1, p. 609]) can be defined by sending x_1 and x_3 to $\begin{pmatrix}1&1\\0&1\end{pmatrix}$ and x_2 to $\begin{pmatrix}1&0\\-1&1\end{pmatrix}$ and there is an epimorphism from $\pi_1(k;11)$ onto the alternating group A_7 sending x_1 to (23576), x_2 to (36574) and x_3 to (12654). The last epimorphism was found using the program CAYLEY (We thank Gerardo Raggi for help with CAYLEY). Hence $\pi_1(9_{16};\pm11)$ is not cyclic.

Therefore, if $\pi_1(k;p/q)$ is cyclic, $q\neq0$ and k is a nontrivial knot with less than 10 crossings then k is a torus knot.

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