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A Normability Condition on Locally Convex Spaces

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ABSTRACT. In a previous work [11] we introduced a certain property (y) on locally convex spaces and used it to remove the assumption of separability from the theorem of Bellenot and Dubinsky on the existence of nuclear Köthe quotients of Fréchet spaces. Our purpose is to examine condition (y) further and relate it to some other normability conditions. Some of our results were already announced in [10].

1. PRELIMINAIRES

Our terminology and notation for locally convex spaces is quite standard (cf. e.g. [5]). By $\mathcal{U}(E)$ we always denote a base of neighborhoods of a locally convex space (lcs) E which consists of absolutely convex and closed neighborhoods. Consequently the topology of a Fréchet space E is defined by a basic sequence of seminorms, i.e. an increasing sequence of seminorms $(\| \ \|_k)$ such that the corresponding unit balls $U_k = \{x \in E: \|x\|_k \leq 1\}$ form a base of neighborhoods. A linear operator $T: E \to F$ is *bounded* if $T(U)$ is a bounded subset of F for some neighborhood U. In case every continuous linear operator from a lcs E into a lcs F is bounded, we write $(E, F) \in B$. A complete characterization of those pairs of Fréchet spaces satisfying $(E, F) \in B$ was given by Vogt [16].

Following Nachbin, we say that a lcs E satisfies the *openness condition* [12] if for every $U \in \mathcal{U}(E)$ there is a $V \in \mathcal{U}(E)$ such that for each $W \in \mathcal{U}(E)$ there is a $\rho > 0$ with

$$V \subset p_u^{-1}(0) + \rho W$$

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where p_u denotes the gauge of U. A Fréchet space satisfying this condition is called a *quojection* [2]. A Fréchet space is a quojection if and only if it can be represented as the projective limit of a sequence of Banach spaces with surjective linking maps.

A lcs E is said to satisfy the *boundedness condition (b)* if for each $U\in\mathcal{U}(E)$ there is a $V\in\mathcal{U}(E)$ such that for each $W\in\mathcal{U}(E)$ we have

$$W^0\cap E'[U^0]\subset \rho V^0$$

for some $\rho>0$ ([12], [8]). Here and throughout $E'[U^0]$ stands for the span of U^0 in E'. Clearly a lcs E satisfies (*b*) if and only if for each $U\in\mathcal{U}(E)$, $V\subset U$, such that for each $W\in\mathcal{U}(E)$, $W\subset V$, $E'[V^0]$ and $E'[W^0]$ induce the same topology on $E'[U^0]$. In this case the closure of $E'[U^0]$ in $E'[V^0]$ and in $E'[W^0]$ is the same. Vogt has proved in [18] that a Fréchet space satisfies *(b)* if and only if its bidual is a quojection. Hence we follow Moscatelli and call a Fréchet space which satisfies *(b)* a *prequojection*. For a recent survey on quojection and prequojections we refer to [7].

A lcs E has property *(y)* if there is a neighborhood $U_1\in\mathcal{U}(E)$ such that

$$E'=\bigcup_{U\in\mathcal{U}(E)}\overline{E'[U_1^0]\cap U^0}$$

where the closure is taken with respect to any topology compatible with the duality $\langle E, E'\rangle$ [11]. Condition *(y)* implies that $E'[U_1^0]$ is dense in E' and therefore p_{u_1} is a continuous norm on E.

Bellenot and Dubinsky have proved in [2] that a separable Fréchet space which is not a prequojection, has a quotient space which is nuclear, admits a continuous norm and has a basis, i.e. it has a *nuclear Köthe quotient*. In [11] we have proved that if F is a Fréchet space which has *(y)* and if there is an unbounded continuous linear operator $T: E\to F$, then there is a nuclear Köthe space $\lambda(A)$, a surjection $Q: F\to\lambda(A)$ such that $QT: E\to\lambda(A)$ is also a surjection. On the other hand, it is not difficult to show that $(E, F)\in B$ if E satisfies *(b)* and F satisfies *(y)* [11]. In fact a Fréchet space is a prequojection if and only if $(E,\lambda(A))\in B$ for any nuclear Köthe space $\lambda(A)$ ([12]). Hence the assumption that E is not a prequojection is also necessary in the theorem of Bellenot and Dubinsky. However as a corollary of the main result in [11], the assumption of separability in the theorem of Bellenot and Dubinsky can be removed.

2. CONDITION *(y)*

We have already noted that condition *(y)* implies the existence of a continuous norm. In this section we shall relate this condition to some other normability conditions. We recall first a condition introduced in [14] for Fréchet spaces: A lcs E is *asymptotically normable* if there is a neighborhood $U_1 \in \mathcal{U}(E)$ such that for each $V \in \mathcal{U}(E)$ there is a $W \in \mathcal{U}(E)$ so that for every $\epsilon > 0$ one can find $M > 0$ with

$$p_v(x) \leq M p_{u_1}(x) + \epsilon p_w(x).$$

Proposition 1. *Every asymptotically normable lcs satisfies (y).*

Proof: For $V \in \mathcal{U}(E)$, we choose $W \subset V$ by asymptotic normability and polarizing obtain for each $\epsilon > 0$

$$V^0 \subset E'[U_1^0] + \epsilon W^0.$$

This implies however $V^0 \subset \overline{E'[U_1^0] \cap (2W^0)}$.

We already know that a lcs which has the bounded approximation property and a continuous norm also satisfies *(y)*[11]. In [14] a Köthe-Montel space $\lambda(A)$ is constructed which is not asymptotically normable. So although this Köthe space has *(y)*, it is not asymptotically normable and therefore the converse of Prop. 1. is false.

In case of Fréchet spaces condition *(y)* can be strengthened.

Lemma 1. *A Fréchet space E satisfies (y) if and only it has a base of neighborhoods* (U_n) *such that for each k there is an m with* $U_k^0 \subset \overline{E'[U_1^0] \cap U_m^0}$.

Proof: We construct the base (U_n) so that the first one U_1 is as in condition *(y)* and let $A_k = \overline{E'[U_1^0] \cap U_k^0}$. Since $U_k \subset A_k^0$, $\{A_k^0 : k = 1, 2, \ldots\}$ defines a metrisable topology τ on E which is weaker than the given one. However condition *(y)* says $E' = \cup_{k=1}^{\infty} A_k$ and so by the weak homorphism theorem ([5]), τ coincides with the original topology of E.

Behrends, Dierolf and Harmand [1] constructed a proper prequojection which admits a continuous norm. We recall that a Fréchet space is *countably normed* if it can be expressed as the intersection of a sequence of Banach spaces. Moscatelli [9] (cf. also [7]) has devised a method for constructing proper prequojections which are even countably normed (cf. also [4]). Such a countably normed space E cannot have *(y)*, because it would then satisfy $(E, E) \in B$[11]. However the converse is true.

Proposition 2. *A Fréchet space which satisfies (y) is countably normed.*

Proof: Let (U_n) be a base of neighborhoods as in Lemma 1 and set $I = E'[U_1^0] \setminus \{0\}$. For each k, let $\| \ \|_k^*$ denote the dual norm defined on $E'[U_k^0]$ and $a_u^k = (1/\|u\|_k^*)$, $u \in I$. We let Λ be the space of all scalar-valued functions f on I with

$$|f|_k = \sup_{u \in I} |f(u)| \, a_u^k < +\infty$$

for each $k = 1, 2, \ldots$. With these norms, Λ becomes a countably normed Fréchet space. Define now $T: E \to \Lambda$ by $Tx = (u(x))_{u \in I}$. T is continuous and by (y), it is one to one. Since for each k there is an m with $U_k^0 \subset (\overline{I \cap U_m^0})$ by Lemma 1, we have that T is an isomorphism of E onto a subspace of Λ. Hence E is countably normed.

Remark: Using a somewhat different approach, Vogt also proved that one can imbed a Fréchet space with (y) into a weighted sup-norm space [19].

Following Komatsu [7], we call a lcs E a *Komura space* if for every $U \in \mathcal{U}(E)$ there is a $V \in \mathcal{U}(E)$ so that the linking map $\tilde{\rho}_{v,u}$ is a weakly compact map of the associated Banach spaces $\tilde{E}_v$ and $\tilde{E}_u$. We now give a partial converse of our last result.

Proposition 3. *Let E be a Komura space. If there is a neighborhood $U_1 \in \mathcal{U}(E)$ with the property that for each $U \in \mathcal{U}(E)$ there is a $V \subset U$, $V \in \mathcal{U}(E)$ with $\tilde{\rho}_{vu_1}$ one to one, then F satisfies (y).*

Proof: Let $v \in E'$. We may assume $v \in V^0$ where $\tilde{\rho}_{vu_1}$ is one to one and choose $W \in \mathcal{U}(E)$ such that $\tilde{\rho}_{wv}$ is weakly compact. The adjoint of $\tilde{\rho}_{vu_1}$ which imbeds $E'[U_1^0]$ into $E'[V^0]$, has a dense range. That is $cl(E'[U_1^0]) = E'[V^0]$ where cl denotes the closure with respect to the duality $< E'[V^0], \tilde{E}_v >$. The adjoint of $\tilde{\rho}_{wv}$ imbeds $E'[V^0]$ into $E'[W^0]$ and it is continuous if we equip $E'[V^0]$ with the Mackey topology $\mu(E'[V^0], \tilde{E}_v)$ and $E'[W^0]$ with its norm topology. So we obtain

$$E'[V^0] = cl(E'[U_1^0]) \subset \overline{E'[U_1^0]}$$

where bar denotes the closure with respect to the norm of $E'[W^0]$. Hence for each $\epsilon > 0$ we have

$$V^0 \subset E'[U_1^0] + \epsilon W^0$$

and this yields (y) as in the proof of Prop. 1.

A lcs E is called totally reflexive if every quotient space of E is reflexive. Valdivia [15] has proved that a Fréchet space is totally reflexive if and only if it a Komura space. Hence we get the following result as a direct consequence of Propositions 2 and 3.

Corollary 1. *A totally reflexive Fréchet space satisfies (y) if and only if it is countably normed.*

In case E is a Fréchet-Schwartz space, we know that E is countably normed if and only if it is asymptotically normable ([17]; 5.7. Lemma). So both of these conditions and condition *(y)* coincide in this case [11].

3. BOUNDEDNESS OF OPERATORS

In this section we give some results relating conditions *(b)* and *(y)* to the boundedness of all continuous operators. For related results in the restricted context of Fréchet spaces we refer to [3] and [12]. Vogt has proved that a prequojection is always quasinormable [18]. Whether this is true in general seems to be an open question.

Lemma 2. *Let E be a lcs which satisfies (b). The following conditions are equivalent:*

(1) *E is quasinormable.*

(2) *For each $U \in \mathcal{U}(E)$ there is $V \in \mathcal{U}(E)$, $V \subset U$, such that $E'[V^0]$ and $(E', \beta(E', E))$ induce the same topology on $E'[U^0]$.*

(3) *For each $U \in \mathcal{U}(E)$ there is $V \in \mathcal{U}(E)$ and a bounded subset B of E such that $B^0 \cap E'[U^0] \subset V^0$.*

Moreover in the situation of condition (2) the closure of $E'[U^0]$ in $E'[V^0]$ and in $(E', \beta(E', E))$ is the same.

Proof: It is easy to see that (2) and (3) are equivalent and that condition (3) implies that E is quasinormable.

Assume that E is quasinormable and fix $U \in \mathcal{U}(E)$. According to condition *(b)* we find $V \in \mathcal{U}(E)$, $V \subset U$, such that for each $W \in \mathcal{U}(E)$, $W \subset V$, $E'[V^0]$ and $E'[W^0]$ induce the same topology on $E'[U^0]$. Now given V we can find $W \in \mathcal{U}(E)$, $W \subset V$, such that $E'[W^0]$ and $(E', \beta(E', E))$ induce the same topology on V^0. Let τ denote the topology induced by $E'[V^0]$ on $E'[U^0]$. We have that τ is finer than $\beta(E', E)$ on $E'[U^0]$ but both coincide on $V^0 \cap E'[U^0]$, which is a 0-neighbourhood in $(E'[U^0], \tau)$. Consequently $\tau = \beta(E', E)$ on $E'[U^0]$ and hence $(E', \beta(E', E))$ and $E'[V^0]$ induce the same topology on $E'[U^0]$.

We present now our result on the boundedness of all continuous operators: We note that (i) is already proved in [11] and included here only for the sake of completeness.

Proposition 4. *Each one of the following implies $(E, F) \in B$.*

(i) *E has (b) and F has (y).*

(ii) *E is a quasinormable lcs which satisfies (b) and F'' with its natural topology admits a continuous norm.*

(iii) *E has (b) and F is a B_r-complete Komura space which admits a continuous norm.*

Proof: Let $T: E \to F$ be continuous, E and F as in (ii). If $W \in \mathcal{U}(F)$ is such that $F'[W^0]$ is $\beta(F', F)$-dense in F', we find $U \in \mathcal{U}(E)$ with $T(U) \subset W$. By Lemma 2 there is $V \in \mathcal{U}(E)$ such that the closure of $E'[U^0]$ in $E'[V^0]$ *and in* $(E', \beta(E', E))$ is the same. We have

$$T'(F') \subset T'\overline{(F'[W^0])} \subset \overline{T'(F'[W^0])} \subset \overline{E'[U^0]} \subset E'[V^0]$$

where the first closure is taken in $(F', \beta(F', F))$ and the second and the third in $(E', \beta(E', E))$. Therefore T is bounded ([11], 1.2. Lemma).

Let $T: E \to F$ be continuous, E and F as in (iii), $\tilde{W} \in \mathcal{U}(F)$ be such that its gauge is a norm on F and by continuity find $U \in \mathcal{U}(E)$ such that $T(U) \subset \tilde{W}$. For this neighborhood we choose $V \in \mathcal{U}(E)$ as in condition *(b)*. Let

$$M = T'^{-1}\Big(\bigcap_{\epsilon > 0} (E'[U^0] + \epsilon V^0)\Big)$$

Since $F'[\tilde{W}^0] \subset M$, this is $\sigma(E', F)$-dense in F'. If we show $M = F'$, we then get $T'(F') \subset E'[V^0]$. This will imply that T is bounded (cf. [11]).

Now to show $M = F'$, it is enough to prove that $W^0 \cap M$ is $\sigma(F', F)$-closed for every $W \in \mathcal{U}(F)$: Let (v_α) be a net in $W^0 \cap M$ with limit v. Since F is a Komura space, we can find $W_1 \in \mathcal{U}(F)$ so that the corresponding linking map $\tilde{\rho}_{W_1', W}$ is weakly compact. Hence W^0 is a $\sigma(F'[W_1^0], \tilde{F}''_{W_1})$-compact subset of the

Banach space $F'[W_1^0]$. Therefore the $\sigma(F'[W_1^0], \tilde{F}''_{w_1})$-topology coincides with $\sigma(F', F)$ on W^0 and so (v_α) converges to v in the $\sigma(F'[W_1^0], \tilde{F}''_{w_1})$-topology. This means that v is in the closure of $W^0 \cap M$ with respect to the norm topology coming from $F'[W_1^0]$. We now chose $U_1 \in \mathcal{U}(E)$ with $T(U_1) \subset W_1$. Since the topologies defined by the balls V^0 and U_1^0 on $E'[U^0]$ coincide by *(b)*, this means that $T'v \in \overline{E'[U^0]}$ where the closure is taken with respect to the topology defined by the ball V^0. So $v \in M$ and the proof is finished.

One can easily derive several interesting results from Prop. 4. immediately. For example from (i) one gets that a lcs satisfying *(b)* which has the bounded approximation property and admits a continuous norm must necessarily be a normed space (see also [8]). In fact using their methods for constructing prequojections with specified properties, Metafune and Moscatelli have shown that to have the bounded approximation property is not a three space property within the class of Fréchet spaces (cf. [7], [9]).

To study the relationship between the conditions *(b)*, *(y)* and the bounded approximation property, we need a somewhat technical result.

Lemma 3. *Let E be a lcs with the bounded approximation property. Let (T_α) be a equicontinuous net of finite rank operators as in the definition of bounded approximation property. Then for each $U \in \mathcal{U}(E)$ there is $V \in \mathcal{U}(E)$ such that the inclusion*

$$T'_\alpha(\overline{E'[U^0]}) \subset E'[V^0]$$

holds for each α, where the closure is taken with respect to $\sigma(E', E)$.

Proof: For $U \in \mathcal{U}(E)$ we find $V \in \mathcal{U}(E)$ so that $T_\alpha(V) \subset U$ holds for each α. We consider $E/p_u^{-1}(0)$ with the quotient topology. Let $Q: E \to E/p_u^{-1}(0)$ be the canonical quotient map. The dual of $E/p_u^{-1}(0)$ is $p_u^{-1}(0)^\perp = \overline{E'[U^0]}$. If $u \in \overline{E'[U^0]} \cap W^0$ for some $W \in \mathcal{U}(E)$, $W \subset U$, then we have

$$|<T'_\alpha Q'(u), x>| = |<u, QT_\alpha x>| \leq p_{QW}(QT_\alpha x).$$

p_{QW} and p_{QU} are equivalent norms on the finite dimensional subspace $QT_\alpha(E)$ and so there is some $\rho_\alpha > 0$ with

$$|<T'_\alpha Q'(u), x>| \leq \rho_\alpha\, p_{QU}(QT_\alpha x).$$

If $x \in V$, since $T_\alpha(V) \subset U$ we get

$$|<T'_\alpha u, x>| = |<T'_\alpha Q'u, x>| \leq \rho_\alpha$$

and therefore $T'_\alpha u \in \rho_\alpha V^0$ for each α.

One consequence of the above results is the equivalence of condition *(b)* and the openness condition when the space has the bounded approximation property. This result was obtained independently by A. Galbis in his thesis (University of Valencia, 1988).

Corollary 2. *If E has the bounded approximation property and satisfies (b), then it also satisfies the openness condition.*

Proof: We will show that for each $U \in \mathcal{U}(E)$ there is $V \in \mathcal{U}(E)$ so that $\overline{E'[U^0]} \subset E'[V^0]$ where the closure is taken respect to $\sigma(E', E)$. This is equivalent to the openness property. By Lemma 3 we choose $V_1 \in \mathcal{U}(E)$ so that $T'_\alpha(\overline{E'[U^0]}) \subset E'[V_1^0]$ holds for each α. For V_1 we find $V \in \mathcal{U}(E)$, as in condition *(b)*; i.e. for each $W \in \mathcal{U}(E)$ we have

$$W^0 \cap E'[V_1^0] \subset \rho\, V^0$$

for some $\rho > 0$. Let $u \in \overline{E'[U^0]} \cap W_1^0$ for some $W_1 \in \mathcal{U}(E)$. By equicontinuity we find $W \in \mathcal{U}(E)$ with $T_\alpha(W) \subset W_1$ for each α. So,

$$T'_\alpha u \in W^0 \cap E'[V_1^0] \subset \rho\, V^0.$$

Since $u = \lim T'_\alpha u$ in $\sigma(E' E)$-topology, we get $u \in E'[V^0]$.

Another corollary of Proposition 5 is the following result which was already proved in [11].

Corollary 3. *If E admits a continuous norm and has the bounded approximation property, then E satisfies (y).*

Proof: If U is the unit ball of a continuous norm on E, we have $E' = \overline{E'[U^0]}$. By Lemma 3 we find $U_1 \in \mathcal{U}(E)$ such that $T'_\alpha(\overline{E'[U^0]}) \subset E'[U_1^0]$ for every α. So $T'_\alpha(E') \subset E'[U_1^0]$. For $u \in E'$, by equicontinuity we find $W \in \mathcal{U}(E)$ so that $T'_\alpha u \in W^0$ for each α. Since $u = \lim T'_\alpha u$, and $T'_\alpha u \in E'[U_1^0] \cap W^0$, we have shown that E satisfies *(y)*.

We finish by giving by a generalization of the main result of [13].

Proposition 5. *Let $T: E \to F$ be a continuous linear operator which is unbounded, where E is a Fréchet space and F a Fréchet space which satisfies (y). Then there is a subspace M of E which is isomorphic to a nuclear Köthe space such that $T: M \to T(M)$ is an isomorphism.*

Proof: By the theorem in [11], there is a nuclear Köthe space $\lambda(A)$, a quotient map $Q: F \to \lambda(A)$ such that $QT(E) = \lambda(A)$ also. Hence QT is unbounded. So we can apply the theorem of [13] to QT and find a nuclear Köthe subspace M of E such that the restriction of QT to M is an imbedding. It is easily seen that $T(M)$ is a closed subspace of F.

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