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Coverings of S³ *Branched over Iterated Torus Links*

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ABSTRACT. Coverings of S^3 branched over iterated torus links appear naturally and very often in Algebraic Geometry. The natural graph-manifold structure of the exterior of an iterated torus link induces a graph-structure in the branched covers. In this paper we give an algorithm to compute valued graphs representing a branched cover given the monodromy representation associated to the covering. The algorithm is completely mechanized in order to be programmed, and can also be used for finding representations of groups of iterated torus links.

1. INTRODUCTION

Iterated torus links constitute a class of links whose more important examples are algebraic links, that is, links (in the sense of Algebraic Geometry) of isolated singularities of algebraic complex curves in $\mathbf{C}^2$.

Coverings of S^3 branched over algebraic links appear naturally in the study of singularities of complex surfaces. Given any isolated normal singularity of an algebraic complex surface (V, p), the singularity link (i.e., the boundary of suficiently small regular neighborhood of p in V) is a well defined three-manifold which can be represented as a covering of S^3 branched over an algebraic link whenever V is endowed with a finite map into $\mathbf{C}^2$. Through this paper, branched coverings are understood in the sense of Fox (see [F1]). The (normal) 2-dimensional singularity has a unique minimal resolution. Associated to it there is the so called dual graph of the resolution. This graph can be also thought as a plumbing graph, and the plumbed three-manifold represented by it is precisely the singularity link.

Iterated torus links are obtained by a sequence of iterations, or satellizations. So, a given iterated torus link is (roughly) the last of a sequence of iterated torus links, each obtained from the preceding one by an iteration. The exterior of a link obtained by a sequence of iterations in endowed with

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a natural structure of graph-manifold. A graph-manifold is defined (in [W]) to be a connected compact orientable 3-manifold with a finite system of tori that decomposes the manifold into S^1-bundles. Any graph-manifold can be described by a valued graph, which determines the manifold up to homeomorphism. A finite covering of S^3 branched over an iterated torus link induces a graph-manifold structure in the cover. Thus, the topological type of the cover is described by a valued graph. In this paper we work out an algorithm to find graphs of the cover for any given finite covering of S^3 branched over an iterated torus link, starting from the monodromy representation of the link group into a symmetric group. Our algorithm provides at the end Waldhausen graphs of the cover. When the cover is a singularity link, one can go further and obtain plumbing graphs, and obtain explicitely the graph of the singularity resolution. The algorithm for this last step is contained in [Ne].

In fact there is a theoretical procedure for constructing coverings of a triangulable manifold using a «splitting complex» of the manifold, due to Neuwirth ([Neu], [Mon]). But most times this procedure cannot be carried out practically; it depends on how complicated the manifold obtained from cutting open along the splitting complex is. Instead of using a splitting complex, we use for the exterior of an iterated torus link its decomposition into S^1-bundles, which can be determined algorithmically.

In brief, our procedure is the following. We completely codify the given iterated torus link by means of a sequence (§2). Then we uniquely associate to this sequence a graph-structure of the link exterior. In §3 we derive an inductive presentation of the link group which allows reading the generators of the fundamental group of each torus of the graph-structure-and computing the permutations associated to them by the monodromy representation of the link group- in terms of a set of meridians of the link. This is applied in §6 to compute, for a given S^1-bundle of the graph-structure of S^3, the characteristic numbers of the S^1-bundles which cover it, and also the matrices which describe the gluing of the family of S^1-bundles of the covering manifold.

This procedure was already used in the author's thesis to give an algorithm for describing cyclic branched coverings. The problem of describing non abelian covers is less easy in its first step, when computing the monodromy representations induced on the fundamental group of each S^1-bundle of the link exterior. In the abelian case one does not need to care about conjugation, and the work can be greatly simplified ([S]).

The inductive presentation of the fundamental group of the link exterior derived in §3 can also be used for finding representations of iterated torus link groups into symmetric groups in an algorithmic way. It has to be remarked that there is no systematic procedure for finding representations of a knot or link group, apart from the cyclic and metacyclic ones ([F2]). This may help to represent locally an algebraic complex surface as a branched covering of $\mathbf{C}^2$.

This work is an expansion of part of the doctoral dissertation, written under the direction of Prof. José Montesinos, whose guidance the author deeply appreciates. The author would like to thank also Prof. C. Weber and E. Artal for helpful suggestions, and J. Mas for help when programming the algorithm.

2. ITERATED TORUS LINKS

We take the concept of iterated torus link (ITL, for short) as it appears in [M - W]. Let us recall the definition in order to establish notation.

(2.1) For integers λ, μ, n, where $n>0$, λ and μ coprime, $L(\lambda, \mu; n)$ denotes an $n+2$ component link $E \cup I \cup N_1 \cup \ldots \cup N_n$. Here E and I are the components of a Hopf link in an oriented S^3 such that the linking number $Lk(E, I) = +1$ and N_i, $1 \leq i \leq n$, are parallel torus knots lying on unknotted torus separating E and I, such that

$$Lk(E, N_i) = \lambda$$
$$Lk(I, N_i) = \mu$$

In [M - W] it is always assumed that $\lambda > 0$ and $\mu \neq 0$.

(2.2) For a link L in oriented S^3 we will denote by $N(L)$ a closed tubular neighborhood of L, and $N^0(L)$ its interior. For an oriented knot K, a meridian m is assumed to be oriented in such a way that $Lk(K, m) = +1$.

The components E and I of a link $L(\lambda, \mu; n)$ are each a trivial knot. We will take always their canonical framings. On the other hand, $S^3 - N^0(E \cup I)$ is an S^1-bundle over an annulus with fibers isotopic to N_i, $i = 1, \ldots, r$. So $\partial N(N_i)$ is also fibered, and we will take any of these fibers as a framing for N_i.

Let K be an oriented knot in S^3 and l a framing of K on $\partial N(K)$. Set

$$X = S^3 - N^0(K)$$
$$X' = S^3 - N^0(L(\lambda, \mu; n))$$

Then ∂X and $\partial N(E)$ are tori endowed each with meridian-longitude pairs (m, l) and (m', l') respectively. There exists an essentially unique homeomorphism $\varphi\colon \partial N(E) \to \partial X$ permuting meridians and longitudes, and $X \cup_\varphi X'$ is the complement in S^3 of a link K' which is said to be obtained by satellization of $L(\lambda, \mu; n)$ on K.

(2.3) An *iterated torus link* is defined to be any link obtained by perfoming the following operations:

1) A finite number r of satellizations are made succesively. The first one is a satellization of a link $L(\lambda_1, \mu_1; n_1)$ on the trivial knot. For $2 \leq i \leq r$, the ith-satellization can be perfomed with a link $L(\lambda_i, \mu_i; n_i)$ on any component of the link already obtained.

2) After having performed all the satellizations, some link components can be deleted or some orientation reversed.

(2.4) For our purposes it is convenient to perform the satellizations using $L(\lambda, \mu) := L(\lambda, \mu, 1)$. We will continue assuming $\lambda > 0$ and $\mu \neq 0$. Obviously, the same class of *ITL* is obtained.

After a satellization of $L(\lambda, \mu)$ on a knot K, the components I and N of $L(\lambda, \mu)$ become components of the new link which we will denote K again and $K(\lambda, \mu)$ respectively. The images of the framings selected in (2.2) for I and N after the satellization will be called *toral framings* of K and $K(\lambda, \mu)$, respectively. *It will be assumed throughout this paper that satellizations in ITL are perfomed using these framings.* This choice of the framings in *ITL* is the most suitable for a topological study of this class of links, and seems to have been used first by Eisenbud and Neumann. Other framings are sometimes used (see [M-W]).

(2.5) Once the choice of framings has been made, four vectors of $r+1$ entries suffice to reconstruct a given *ITL* obtained by a sequence of r iterations performed using $L(\lambda_i, \mu_i)$, $1 \leq i \leq r$. These vectors will be named *it*, *la*, *mu* and *ep*, and will be next defined. We will refer to them as the *toral sequence* of the *ITL* and will use them as the data for introducing the *ITL* to a computer.

Definition

1)
$it(0) := 0$

$$it(i) := \begin{cases} j & \textit{if the ith-satellization is made on the component introduced in the jth-satellization} \\ 0 & \textit{if the ith-satellization is made on the trivial knot.} \end{cases}$$

2) $la(0) := 1$, $la(i) := \lambda_i$ *for* $1 \leq i \leq r$

3) $mu(0) := 0$, $mu(i) := \mu_i$ *for* $1 \leq i \leq r$

4) $ep(i) := 0$, 1 *or* -1 *according to whether the component introduced in the ith-satellization (or the starting trivial knot, if* $i = 0$*) is after all deleted, conserved with the same orientation or with reversed orientation.*

Algebraic links are a specially important family of iterated torus links. In [M-W] it is given a criterion to decide whether an *ITL* is algebraic or not.

3. GRAPH-STRUCTURE OF THE EXTERIOR OF AN *ITL*

(3.1) Let L be an *ITL* given by a toral sequence.

If L is not the trivial knot there is a sequence of links $L_0, L_1, \ldots, L_r$ such that L_0 is the trivial knot and L_i is the result of a satellization of $L(\lambda_i, \mu_i)$ on some component of L_{i-1}, for $1 \leq i \leq r$, and L is obtained from L_r by possibly neglecting some components and reversing the orientation of some others. The sequence of satellizations provides expressions

$$S^3 - N^0(L_r) = X_0 \cup W_1 \cup \ldots \cup W_r$$
$$S^3 - N^0(L) = X_0 \cup W_1 \cup \ldots \cup W_r \cup U$$

where X_0 is the exterior of the trivial knot in S^3, each W_i is a copy of the exterior of $L(\lambda_i, \mu_i)$ in S^3 and U is a disjoint union of solid tori, tubular neighborhoods of the components of L_r neglected to get L. For $i = 1, \ldots, r$, W_i is an S^1-bundle over an annulus with a hole. Therefore the union written above shows a graph-manifold structure of the exterior of the *ITL*. We will refer to it as the *graph-structure associated to the toral sequence* of the link.

(3.2) Let us look at the exterior of $L(\lambda, \mu)$ in S^3, which we will denote W. Since $I \cup N$ is a closed braid with braid axis E, W is homeomorphic to a mapping torus

$$H_\eta = \frac{H \times I}{(x, 0) \equiv (\eta(x), 1)}$$

of a disc H with $\lambda + 1$ holes, where $I = [0, 1]$ and η denotes the automorphism of H defined by the braid $(\sigma_\lambda^2 \sigma_{\lambda-1} \ldots \sigma_2 \sigma_1)^\mu$. We follow the notation of [B] for braids. Let g be a basepoint for H and $x_1, \ldots, x_\lambda, x_{\lambda+1}$ be a basis for $\pi_1(H, g)$ such that

$$\sigma(x_j) = x_j \qquad \text{if } j \neq i, i+1$$
$$\sigma_i(x_i) = x_i x_{i+1} \bar{x}_i$$
$$\sigma_i(x_{i+1}) = x_i$$

Let l denote the homotopy class of the loop

$$I \longrightarrow H_\eta$$
$$t \longrightarrow [(g, t)]$$

Then a presentation of $\pi_1(H_\eta, [(g,0)])$ is the following one:

$$|x_1,\dots,x_\lambda,x_{\lambda+1},l:\bar{l}x_il=x_i\eta,\quad i=1,\dots,\lambda+1|$$

where $x_i\eta$ denotes the image of x_i under (the homomorphism) η. The braid group acts on the right on $\pi_1(H,g)$. If O denotes the set $\{x_1,x_2,\dots,x_\lambda,x_{\lambda+1},l\}$ of the generators and T the set $\{\bar{l}x_il=x_i\eta,\ i=1,\dots,\lambda+1\}$ of relations, the above presentation of $\pi_1(H_\eta,g)$ is briefly written $|O:T|$.

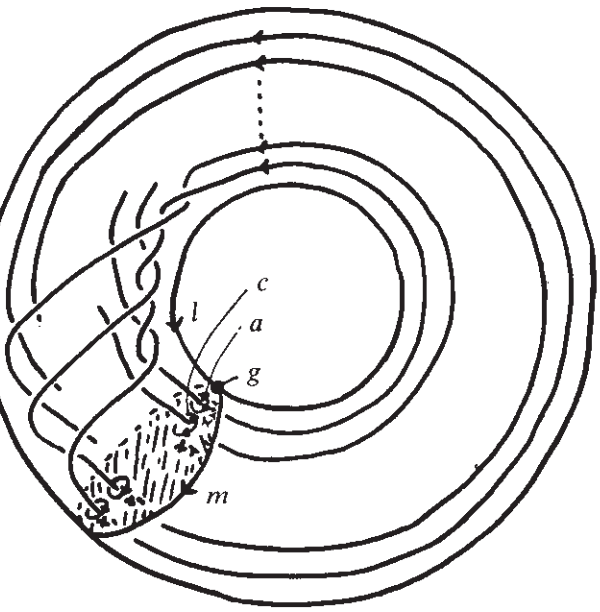


Figure 3.1 represents W. $\mu=3$ in the picture.

(3.3) Figure 3.1 represents W. Through a homeomorphism from H_η to W, the basepoint g of H, l and $x_1\dots x_\lambda\ x_{\lambda+1}$ go to the points and loops denoted the same way in figure 3.1. This is how the generators of the presentation $|O:T|$ of $\pi_1(W,g)$ must be read.

We will denote by T, T_a and T_c the boundary tori of W, which are, respectively, $\partial N(E)$, $\partial N(l)$ and $\partial N(N)$. We take g, a and c in figure 3.1 as basepoints for T, T_a and T_c. Fix paths α and α' in W joining respectively a and c to g. Let

$$\delta:\pi_1(T,g)\longrightarrow\pi_1(W,g)$$
$$\delta_a:\pi_1(T_a,a)\longrightarrow\pi_1(W,g)$$
$$\delta_c:\pi_1(T_c,c)\longrightarrow\pi_1(W,g)$$

denote inclusion induced homomorphisms followed by the isomorphisms of change of basepoint associated to the fixed paths. Let m be as in figure 3.1. Then m and l form a coordinate system of T. Let l_a be a toral (i.e. canonical) framing of l based at a on T_a, and m_a be a meridian of l based at a such that $x_{\lambda+1} = \alpha m_a \alpha^{-1}$. Then m_a and l_a form a basis of $\pi_1(T_a, a)$. Finally, m_c and l_c provide a basis for $\pi_1(T_c, c)$, where m_c is a meridian of N such that $x_\lambda = \alpha' m_c \alpha'^{-1}$ and l_c is a toral framing of N on T_c based at c.

Lemma

i) $\delta(m) = x_1 \ldots x_\lambda x_{\lambda+1}$

$\delta(l) = l$

ii) $\delta_a(m_a) = x_{\lambda+1}$

$$\delta_a(l_a) = \begin{cases} \bar{x}_{\lambda+1}^{t+1}(x_{\lambda-a+1} \ldots x_\lambda x_{\lambda+1})(x_1 \ldots x_\lambda x_{\lambda+1})^t l & \text{if } \mu > 0 \\ l(\bar{x}_{\lambda+1} \ldots \bar{x}_1)^t (\bar{x}_{\lambda+1} \ldots \bar{x}_{\lambda-a+1}) x_{\lambda+1}^{t+1} & \text{if } \mu < 0 \end{cases}$$

where $|\mu| = t\lambda + \alpha,\ t \geq 0,\quad 0 < \alpha < \lambda$

iii) $\delta_c(m_c) = x_\lambda$

$\delta_c(l_c) = l^\lambda (x_1 \ldots x_\lambda x_{\lambda+1})^\mu$

Proof. We will derive only the formula for $\delta_a(l_a)$, since the others are immediate. We can think of $\delta_a(l_a)$ as the pictorical longitude of the l component of $L(\lambda, \mu)$ projected as the closure of the braid $(\sigma_\lambda^2 \sigma_{\lambda-1} \ldots \sigma_2 \sigma_1)^\mu$ together with its axis. We will look at the closure of this braid inside a solid torus. Assume first $\mu > 0$. Then $\delta_a(l_a) = \alpha_1 \alpha_2 \ldots \alpha_\mu l$, where α_i are the following:

$\alpha_1 = \bar{x}_{\lambda+1} x_\lambda x_{\lambda+1}$

For $1 < r \leq \lambda$, $\alpha_r = (\bar{\alpha}_{r-1} \bar{\alpha}_{r-2} \ldots \bar{\alpha}_1 \bar{x}_{\lambda+1}) x_{\lambda-r+1} (x_{\lambda+1} \alpha_1 \ldots \alpha_{r-2} \alpha_{r-1})$

For $\lambda < r$, $\alpha_r = (\bar{\alpha}_{r-1} \bar{\alpha}_{r-2} \ldots \bar{\alpha}_1 \bar{x}_{\lambda+1}) \alpha_1 \ldots \alpha_{r-\lambda+1} \alpha_{r-\lambda} \bar{\alpha}_{r-\lambda+1} \ldots$

$\ldots \bar{\alpha}_1 (x_{\lambda+1} \alpha_1 \ldots \alpha_{r-2} \alpha_{r-1})$

See figure 3.2

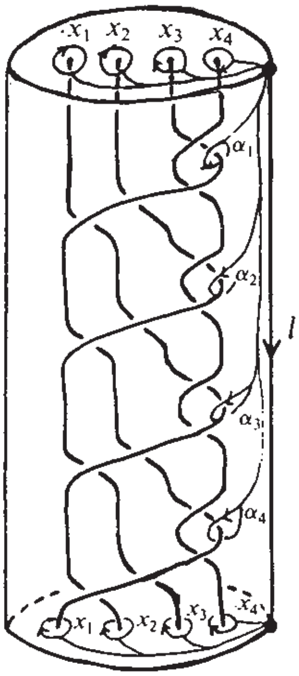


Figure 3.2. Top and botton are identified by translation.

If $r = t\lambda + a$, $\quad 0 < a \leq \lambda$, then

$$\alpha_r = (\bar{\alpha}_{r-1} \dots \bar{\alpha}_1)\, \bar{x}_{\lambda+1}^{t-1}\, x_{\lambda-a+1}\, x_{\lambda+1}^{t-1}\, (\alpha_1 \dots \alpha_{r-1}).$$

This follows from an easy induction, using the fact that

$$(\alpha_1 \dots \alpha_{a-1})\, \alpha_a\, (\bar{\alpha}_{a-1} \dots \bar{\alpha}_1) = \bar{x}_{\lambda+1}\, x_{\lambda-a+1}\, x_{\lambda+1} \text{ for } 0 < a \leq \lambda.$$

Therefore, if $\mu = t\lambda + a > 0$, $0 < a \leq \lambda$,

$$\begin{aligned}
\alpha_1 \alpha_2 \dots \alpha_\mu &= (\alpha_1 \dots \alpha_{\mu-1})\, (\bar{\alpha}_{\mu-1} \dots \bar{\alpha}_1)\, \bar{x}_{\lambda+1}^{t+1}\, x_{\lambda-a+1}\, x_{\lambda+1}^{t+1}\, (\alpha_1 \dots \alpha_{\mu-1}) \\
&= \bar{x}_{\lambda+1}^{t+1}\, x_{\lambda-a+1}\, x_{\lambda+1}^{t+1}\, (\alpha_1 \dots \alpha_{\mu-1}) = \\
&= \bar{x}_{\lambda+1}^{t+1}\, (x_{\lambda-a+1} \dots x_\lambda)\, x_{\lambda+1}^{t+1}\, \bar{x}_{\lambda+1}^{t}\, (x_1 \dots x_\lambda)\, x_{\lambda+1}^{t} \dots \\
&\quad \bar{x}_{\lambda+1}^{2}\, (x_1 \dots x_\lambda)\, x_{\lambda+1}^{2}\, \bar{x}_{\lambda+1}\, (x_1 \dots x_\lambda)\, x_{\lambda+1} = \\
&= \bar{x}_{\lambda+1}^{t+1}\, (x_{\lambda-a+1} \dots x_\lambda\, x_{\lambda+1})\, (x_1 \dots x_\lambda\, x_{\lambda+1})^t
\end{aligned}$$

Finally, $\delta_a(l_a) = \bar{x}_{\lambda+1}^{t+1}\, (x_{\lambda-a+1} \dots x_\lambda\, x_{\lambda+1})\, (x_1 \dots x_\lambda\, x_{\lambda+1})^t\, l.$

If $\mu<0$, $\delta_a(l_a)=l(\gamma_1 \ldots \gamma_\mu)^{-1}$, where

$$\gamma_1=\bar{x}_{\lambda+1}\, x_\lambda\, x_{\lambda+1}$$

$$\gamma_r=(\bar{\gamma}_{r-1} \ldots \bar{\gamma}_1\, \bar{x}_{\lambda+1})\, x_{\lambda-r+1}\, (x_{\lambda+1}\, \gamma_1 \ldots \gamma_{r-1}) \quad \text{for } 1<r\leq\lambda$$

$$\gamma_r=(\bar{\gamma}_{r-1} \ldots \bar{\gamma}_1\, \bar{x}_{\lambda+1})\, \gamma_1 \ldots \gamma_{r-\lambda-1}\, \gamma_{r-\lambda}\, \bar{\gamma}_{r-\lambda-1} \ldots$$
$$\ldots \bar{\gamma}_1\, (x_{\lambda+1}\, \gamma_1 \ldots \gamma_{r-1}) \qquad \text{for } \lambda<r.$$

(See figure 3.3).

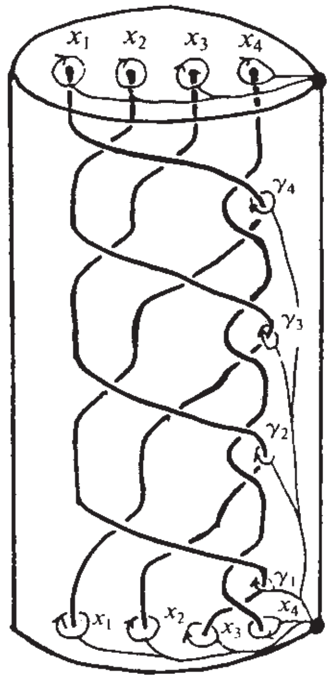


Figure 3.3. $(\lambda, \mu)=(3,-4)$. Top and botton are identified by translation.

As before, if $r=t\lambda+a>0$, $0<a\leq\lambda$,

$$\gamma_r=(\bar{\gamma}_{r-1} \ldots \bar{\gamma}_1)\, \bar{x}_{\lambda+1}^{t+1}\, x_{\lambda-a+1}\, x_{\lambda+1}^{t+1}\, (\gamma_1 \ldots \gamma_{r-1})$$

$$\gamma_1 \ldots \gamma_{|\mu|}=\gamma_1 \ldots \gamma_{|\mu|-1}\, \bar{\gamma}_{|\mu|-1} \ldots \bar{\gamma}_1\, \bar{x}_{\lambda+1}^{t+1}\, x_{\lambda-a+1}\, x_{\lambda+1}^{t+1}\, (\gamma_1 \ldots \gamma_{|\mu|-1})=$$
$$=\bar{x}_{\lambda+1}^{t+1}\, x_{\lambda-a+1}\, x_{\lambda+1}^{t+1}\, (\gamma_1 \ldots \gamma_{|\mu|-1}),$$

where $|\mu|=t\lambda+a$, $0<a\leq\lambda$. As we can see, the formulas are the same as those for α_i's. Hence

$$\gamma_1 \ldots \gamma_{|\mu|}=\bar{x}_{\lambda+1}^{t+1}\, (x_{\lambda-a+1} \ldots x_\lambda\, x_{\lambda+1})\, (x_1 \ldots x_\lambda\, x_{\lambda+1})^t$$

Finally,

$$\delta_a(l_a)=l(\gamma_1\ldots\gamma_{|\mu|})^{-1}=l(\bar{x}_{\lambda+1}\bar{x}_\lambda\ldots\bar{x}_1)^t\,(\bar{x}_{\lambda+1}\bar{x}_\lambda\ldots\bar{x}_{\lambda-a+1})\,x_{\lambda+1}^{t+1}$$

∎

(3.3) Going back to the *ITL* L of (3.1), for $i=1,\ldots,r$ $S^3-N^0(L_i)=S^3-N^0(L_{i-1})\cup W_i$, where W_i is a copy of the exterior of $L(\lambda_i,\mu_i)$ in S^3. The pattern of W_i has been described in (3.2). All the elements introduced in (3.2) will be used for W_i after labelling them with the subscript or superscript i. Thus, were write $\partial W_i=T^i\cup T_a^i\cup T_c^i$; the basepoints in T^i, T_a^i, T_c^i will be respectively g_i, a_i, c_i and so on. The group $\pi_1(W_i,g_i)$ is presented by

$$|x_{i1},x_{i2},\ldots,x_{i\lambda_i},x_{i(\lambda_i+1)},l_i\colon \bar{l}_ix_{ij}l_i=x_{ij}\eta_i,\ j=1,\ldots,\lambda_i+1|$$

Let $0_i=\{x_{i_1},\ldots,x_{i\lambda_i},x_{i(\lambda_i+1)},l_i\}$ and let $T_i=\{\bar{l}_ix_{ij}l_i\overline{(x_{ij}\eta_i)},\ \ j=1,\ldots,\lambda_i+1\}$. Then $\pi_1(W_i)\cong|0_i\colon T_i|$.

For $i=1,\ldots,r$, $S^3-N^0(L_i)=X_0\cup W_1\cup\ldots\cup W_i$. The piece W_i is attached through T_i to a boundary torus of X_0 (when $i=1$) or $X_0\cup W_1\cup\ldots\cup W_{i-1}$. It is possible to determine to which torus it is attached. In order to do this, we define a function

$$\mathrm{ff}\colon\{1,2,\ldots,r\}\to\{0,1,2,\ldots,r-1\}$$

recursively as follows, with the aid of an auxiliary function $\mathrm{cf}\colon\{0,1,\ldots,r\}\to\{1,2,\ldots,r\}$.

Define $\mathrm{cf}(0)=0$. To define $\mathrm{ff}(i)$, assume $\mathrm{cf}(i')$ is defined for all $i'<i$. Then

$$\mathrm{ff}(i)=\mathrm{cf}\ (\mathrm{it}\ (i))$$

Redefine cf on i and $\mathrm{it}(i)$ by

$$\mathrm{cf}(i)=i\quad \mathrm{cf}(\mathrm{it}(i))=i$$

and continue the inductive definition of ff.

Now we claim that if $\mathrm{ff}(i)=j$, the bundle W_i is attached to the bundle W_j through T_c^j if $\mathrm{it}(i)$ does not coincide with $\mathrm{it}(i')$ for any $i'<i$, and through T_a^j otherwise. Note that $\mathrm{cf}(i)=j$ if and only if $\partial N(K_i)\subset\partial W_j$. Moreover,

$$\partial N(K_i)=\begin{cases}T_a^j & \text{if } i<j\\ T_c^j & \text{if } i=j\end{cases}$$

Set $G_0=\pi_1(S^3-N^0(L_0))$ and $G_i=\pi_1(S^3-N^0(L_i))$, for $i=1,\dots,r$. A presentation of G_i can be obtained inductively as follows. The group G_0 is isomorphic to $\mathbf{Z}$. Let x be a generator of G_0 (a meridian of L_0), and $S_0=\{x\}$. Since $S^3-N^0(L_i)=(S^3-N^0(L_{i-1}))\cup W_i$, G_i can be obtained from G_{i-1} and $\pi_1(W_i)$ by the Seifert-Van Kampen theorem (assume the selected basepoints in the attaching tori are matched). For $1\le i\le r$, define $S_i=S_{i=1}\cup O_i$. Then S_i is a set of generators of G_i, with defining relations R_i, which is next described inductively:

$R_0=\emptyset$

$R_1=T_1\cup\{l_1=1,\ x_{11}\dots x_{1\lambda_1}x_{1(\lambda_1+1)}=x\}$

$R_i=R_{i-1}\cup T_i\cup\{l_i=\delta_a^j(l_a^j),\ x_{i1}\dots x_{i\lambda_i}x_{i(\lambda_i+1)}=x_{j(\lambda_j+1)}\}$

if W_i is attached to W_j through T_a^j, and

$R_i=R_{i-1}\cup T_i\cup\{l_i=\delta_c^j(l_c^j),\ x_{i1}\dots x_{i\lambda_i}x_{i(\lambda_i+1)}=x_{j\lambda_j}$

if W_i is attached to W_j through T_c^j.

In particular, we obtain that $\pi_1(S^3-N^0(L_r))\cong|S_r:\ R_r|$. Let S be the subset of S_r which consists of the elements representing meridians of L_r-L. Then $|S_r:\ R_r,\ S|$ is a presentation of $\pi_1(S^3-N^0(L))$.

4. REPRESENTING GRAPH-MANIFOLDS

A connected compact oriented 3-manifold M is called a *graph-manifold* if it contains a system $\{T_1,\dots,T_r\}$ of disjoint properly embedded tori such that each component of $M-N^0(T)$, where $T=T_1\cup\dots\cup T_r$, is a fiber bundle with fiber S^1 and base a connected surface. The system $\{T_1,\dots,T_r\}$ is a *graph-structure* for M. The definition is due to Waldhausen [W].

We will call *m-graph* any connected graph with vertices $\mu_1,\dots,\mu_n$ and edges $\tau_1,\dots,\tau_r$, where each vertex μ_i carries a triple of integer numbers (g_i,r_i,e_i), $r_i\ge 0$, and each edge τ_j is oriented and carries a matrix $A_j\in GL(2,\mathbf{Z})-SL(2,\mathbf{Z})$. (The prefix *m-* is to suggest a graph with matrices).

Given a graph-manifold, construct an *m*-graph as follows:

1. Represent each component M_i, $i=1,\dots,n$, of $M-N^0(T)$ by a vertex μ_i.

2. Each torus T_j, $j=1,\dots,r$ is adjacent to two components (possibly the same) of $M-N^0(T)$. Call them, arbitrarily, M_{j+} and M_{j-}. This means that the boundary of $N(T_j)$ $(N(T_j)\cong T^2\times I)$ shares one torus T_j^+ with ∂M_{j+} and the other, T_j^-, with ∂M_{j-}. Then represent the torus T_j by an edge τ_j joining μ_{j+} and μ_{j-}, oriented towards μ_{j+}.

Fix an orientation for the fiber of each S^1-bundle whose base is an orientable surface and choose a coordinate system (a_{j+}, f_{j+}) on T_j^+ such that f_{j+} is a fiber of M_{j+} (with the fixed orientation if the base of M_{j+} is orientable, and with an arbitrary orientation otherwise), and $f_{j+} \bullet a_{j+} = 1$ where $\bullet$ denotes the intersection pairing on ∂M_{j+} (M_{j+} inherits an orientation from M, which induces an orientation on ∂M_{j+}). Choose (a_{j-}, f_{j-}) in the analogous way. Under the natural isomorphism, both coordinate systems are related in $H_1(T_j)$ by a matrix of $GL(2, \mathbf{Z}) - SL(2, \mathbf{Z})$,

$$\begin{bmatrix} a_{j+} \\ f_{j+} \end{bmatrix} = \begin{pmatrix} \gamma_j & \delta_j \\ \alpha_j & \beta_j \end{pmatrix} \begin{bmatrix} a_{j-} \\ f_{j-} \end{bmatrix}$$

Put the matrix $\begin{pmatrix} \gamma_j & \delta_j \\ \alpha_j & \beta_j \end{pmatrix}$ on the edge τ_j.

3. Let g_i be the genus of the base of the S^1-bundle M_i, which is negative if this base is nonorientable. Let r_i be the number of boundary components of M lying in M_i. If $r_i = 0$, let e_i be the Euler number of the S^1-bundle obtained from M_i after collapsing the curves parallel to $a_{j+(-)}$ whenever $j +_{(-)} = i$. Then, value the vertex μ_i with the triple $(g_i, 0, e_i)$ $i = 1, \dots, n$. If $r_i \neq 0$, value it with the triple $(g_i, r_i, -)$.

Remark. Suppose $r_i = 0$. Note that, if $a_1, \dots, a_t$ are the horizontal curves of the selected coordinate systems in the boundary tori of M_i, then $a_1 + \dots + a_t = -e_i f_i$ in $H_1(M_i)$.

Actually, the m-graph constructed represents the manifold M which, up to o.p. homeomorphisms, is the graph-manifold constructed as follows. For each vertex μ_i take an oriented S^1-bundle over a closed connected surface of genus g_i, with Euler number e_i. Choose a pseudosection S_i. If $g_i \geq 0$, choose an orientation for it, which induces an orientation for the fiber declaring that the orientation of S_i followed by the orientation of the fiber coincides with the orientation of the S^1-bundle. Then remove the interior of a tubular neighborhood of $r_i + n_i$ disjoint fibers, where n_i is the number of edges of the m-graph incident at the vertex μ_i (loops are counted twice). The result is an S^1-bundle M_i. Then, for each edge τ_j joining vertices μ_i, μ_k choose a torus T_{ij} in ∂M_i and a torus T_{kj} in ∂M_k. Glue M_i, M_k along these tori according to the homeomorphism defined by

$$\begin{bmatrix} a_{ij} \\ f_i \end{bmatrix} = \begin{pmatrix} \gamma_j & \delta_j \\ \alpha_j & \beta_j \end{pmatrix} \begin{bmatrix} a_{kj} \\ f_j \end{bmatrix}$$

where $\begin{pmatrix} \gamma_j & \delta_j \\ \alpha_j & \beta_j \end{pmatrix} \in GL(2, \mathbf{Z}) - SL(2, \mathbf{Z})$ is the matrix assigned to the edge τ_j and (a_{ij}, f_j) is a coordinate system on T_{ij} such that f_i is a fiber of M_i (oriented if

$g_i \geq 0$), and a_{ij} is the curve $S_i \cap T_{ij}$ oriented in such a way that $f_i \bullet a_{ij} = 1$ on ∂M_i. The system (a_{kj}, f_k) is chosen on T_{kj} in the analogous way.

This way of representing graph-manifolds justs generalizes the way Waldhausen represented almost all reduced graph-manifolds in [W]. Waldhausen defined reduced graph-manifolds by excluding a number of cases which can occur in a graph-structure. But for the purposes of this paper it is convenient to be able to represent any graph-manifold. Part (6.3.3) of theorem (6.3) of [W] statres the following: «For a graph-manifold (M, T) and a choice of homeomorphisms of each component of $M - T$ with S^1-bundles, one can construct in a finite number of steps the manifolds $N_1, ..., N_m$ such that for $j = 1, ..., n$ N_j is either homeomorphic to a reduced graph-manifold or a lens space or $S^2 \times S^1$». We devote the next section to work out an algorithm to apply efectively this theorem for the family of manifolds which is the object of our study, namely the covers of S^3 branched over ITL.

5. WALDHAUSEN GRAPHS

The covers of S^3 branched over iterated torus links are in a natural way graph-manifolds. Suppose that L is an ITL and pr: $M \to S^3$ is a covering branched over L. If

$$S^3 - N^0(L) = X_0 \cup W_1 \cup ... \cup W_r \cup U$$

is, with the notation of §3, the graph-structure associated to a toral sequence of the link, then

$$M = \mathrm{pr}^{-1}(X_0) \cup \mathrm{pr}^{-1}(W_1) \cup ... \cup \mathrm{pr}^{-1}(W_r) \cup \mathrm{pr}^{-1}(U) \cup \mathrm{pr}^{-1}(N(L))$$

exhibits M (which is connected, closed and orientable) as a union of S^1-bundles. The base of each of these S^1-bundles is an orientable surface. Therefore, and thanks to theorem (6.3) of [W], parts (6.3.1) and (6.3.2), for these graph-structures the cases to be excluded from reduced graph-structures are the following. Here (M, T) denotes the graph-manifold and M_1, M_2 denote S^1-bundles of the graph-manifold adjacent to a torus T_1 of the graph-structure.

R1. $M_1 \neq M_2$, and M_1 is homeomorphic to the product of an annulus and the cercle S^1.

R2. The fiber of M_1 is homologous on T_1 to the fiber of M_2.

R3. M_1 is a solid torus, and a meridian curve of M_1 intersects the fiber of M_2 transversally once.

R4. M_1 is a solid torus, and a meridian curve is homologous to the fiber of M_2.

R5. $M_1 = M_2$ is homeomorphic to the product of a torus and $I=[0,1]$, and there exists an element of $H_1(T_1)$ sent to itself or its inverse by the natural gluing automorphism of $H_1(T_1)$.

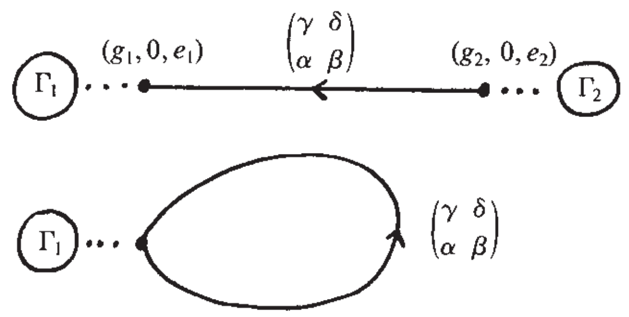
R6. M_1 and M_2 an solid tori.

R7. $T \neq 0$ and M is an S^1-bundle with base S^2.

Suppose that G is an m-graph representing (M, T); μ_1 and μ_2 the vertices of G representing M_1 and M_2, respectively, valued $(g_1, 0, e_1)$, $(g_2, 0, e_2)$; τ_1 the edge representing T_1. Suppose $M_1 = M_{1+}$ and $M_2 = M_{1-}$, with the notation of §4; (a_1, f_1) and (a_2, f_2) coordinate systems on T_1^+, T_1^-, respectively, such that $f_1 \cdot a_1 = 1$ on ∂M_1 and $f_2 \cdot a_2 = 1$ on ∂M_2: In $H_1(T_1)$ both systems are related by a matrix,

$$\begin{bmatrix} a_1 \\ f_1 \end{bmatrix} = \begin{pmatrix} \gamma & \delta \\ \alpha & \beta \end{pmatrix} \begin{bmatrix} a_2 \\ f_2 \end{bmatrix}$$

and the matrix $\begin{pmatrix} \gamma & \delta \\ \alpha & \beta \end{pmatrix} \in GL(2, \mathbf{Z}) - SL(2, \mathbf{Z})$ valuates τ_1, oriented towards μ_1. Thus, G has the form



The condition $Ri)$ is equivalent to the following condition $Gi)$ on the graph, $i = 1, 2, 3, 4, 6, 7$.

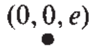
G1. $\mu_1 \neq \mu_2, g_1 = 0$ and μ_1 has valence 2.

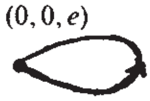
G2. $\alpha = 0$ if $\mu_1 \neq \mu_2$; $\alpha = 0$ and $\beta = 1$ if $\mu_1 = \mu_2$.

G3. μ_1 has valence $1, g_1 = 0$ and whenever $e_1 = 0$, then $\gamma = \epsilon$, $\epsilon \in \{1, -1\}$.

G4. μ_1 has valence 1, $g_1 = 0$ and whenever $e_1 = 0$, then $\gamma = 0$.

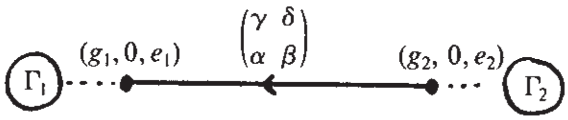
G6. $\mu_1 \neq \mu_2$, both have valence 1 and $g_1 = g_2 = 0$

G7. G is 

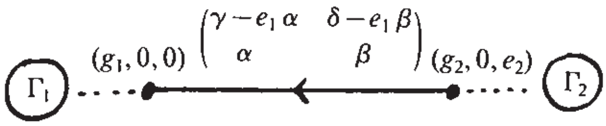
When R5 occurs, G is 

$\begin{pmatrix} \gamma & \delta \\ \alpha & \beta \end{pmatrix}$. Then M is a torus bundle, and when $e = 0$, the matrix is a monodromy matrix. The classification of torus bundles up to homeomorphism from their monodromy matrices is well known.

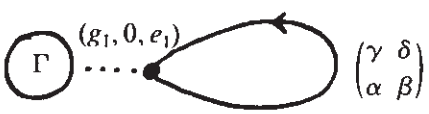
Different choices of coordinate systems lead to different m-graphs for (M, T). We would like to decide whether the graph-manifold is or not reduced by looking at an m-graph. One can realize that checking conditions G1 to G7 is independent of the coordinate systems chosen, and that one can easily get an m-graph where $e_1 = 0$. In fact



and



represent the same graph-manifold. Also the m-graphs



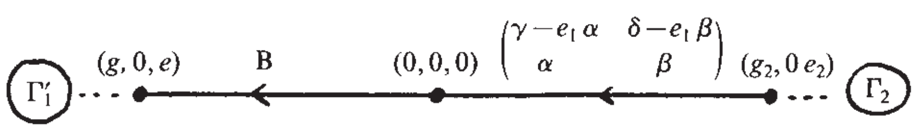
and

represent the same graph-manifold. Finally, in an m-graph there is a choice in the orientation of the edges, which corresponds to the choice of M_1 as M_{1+}. Only checking condition G2 is afected by this choice, and one can verify that G2 holds for $\begin{pmatrix} \gamma & \delta \\ \alpha & \beta \end{pmatrix}$ if and only if it holds for its inverse $\begin{pmatrix} -\beta & \delta \\ \alpha & -\delta \end{pmatrix}$. Therefore, from any m-graph representing (M, T), either we see that the manifold is a torus bundle and know a monodromy matrix, or otherwise we can immediately decide whether the graph-structure is reduced by obtaining first an m-graph where all vertices of valence 1 are valued $(0,0,0)$ and checking on it conditions G1, 2, 3, 4, 6, 7.

For the graph-structure of M induced by the branched covering pr: $M \to S^3$ over an ITL, we know that R1, R5 and R7 do not occur. However, we may find that the graph-manifold is not reduced. In this case we try to eliminate superfluous tori of the graph-structure and, ultimately, to obtain the decomposition of the manifold into connected summands which are either $S^2 \times S^1$, or lenses, or reduced graph-manifolds, as Waldhausen's theorem (6.3) in [W] guarantees. We can do this algorithmically as next described. And, in this reduction process, cases R1, R5 and R7 may appear.

Reduction rules. They are stated without proof because it is contained in the proof of Waldhausen's theorem. For the formula in 4) see [Ne], prop. 2. 1.

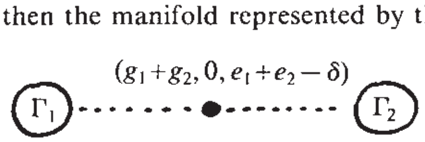
1) If G1 occurs, obtain from G an m-graph of (M, T) of the from



with $B \in GL(2,\mathbf{Z}) - SL(2,\mathbf{Z})$. Then the manifold represented by

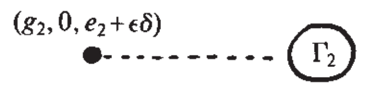
where $B' = B \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma - e_1\alpha & \delta - e_1\beta \\ \alpha & \beta \end{pmatrix}$, is homeomorphic to M.

2) If G2 occurs, then the manifold represented by the m-graph



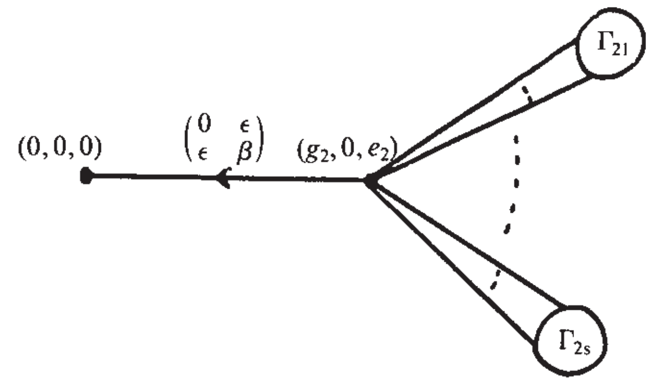
is homeomorphic to M.

3) If G3 occurs, then the manifold represented by the m-graph



is homeomorphic to M.

4) If G4 occurs, obtain a graph of the form



representing *(M, T)*. Here Γ_{2j}, $j = 1, \ldots, k$, is a connected m-graph, joined to μ_2 by k_j edges. Then M is homeomorphic to $N_1 \# \ldots \# N_j \# (\#^k S^2 \times S^1)$, where N_j is the manifold represented by the m-graph Γ_{2j}, and

$$k = 2g_2 + \sum_1^s (k_j - 1)$$

6) If G6 occurs, then M is a lens space. Compute a fraction determining it.

7) If G7 occurs, then $M \cong L(e, 1)$.

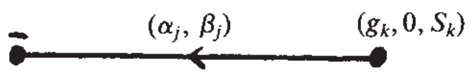
Notice that these reduction steps reduce the number of tori of the graph-structure and keep the orientable character of the base of the S^1-bundles. Therefore other cases than R1-R7 from Waldhausen's list never appear. In a finite number of steps we will get a disjoint union of m-graphs, each of them representing either a lens space, or $S^1 \times S^2$, or a reduced graph-structure. And this is easily decidable from an m-graph which does not fall in any of the cases G1-G4. For the only graph-structures of a lens space such that the bas s of the S^1-bundles are orientable are Seifert structures with base S^2 with at most two exceptional fibers ([Se], [W]).

Consider an m-graph representing a reduced graph-manifold which is neither a torus bundle over S^1 nor a Seifert fibration over S^2 with three exceptional fibers. Then, according to [W], §9, we can associate to the graph-manifold a valued graph which classifies the manifold up to homeomorphism. And this graph can be obtained from the m-graph as follows. Look at an edge τ_j of the m-graph joining vertices μ_{j+} and μ_{j-}. This edge represents a torus T_j adjacent to S^1-bundles M_{j+} and M_{j-}.

Case 1. One of the S^1-bundles M_{j+} or M_{j-} is a solid torus (only one, since the graph-manifold is reduced). Then the solid torus is chosen to be M_{j+}. Let h be a meridian curve of M_{j+}. Among the systems (a_{j-}, f_{j-}) chosen as in §4, there are exactly two (and one of them is obtained from the other by changing the orientation of both curves) such that

$$h = \epsilon_j(\alpha_j a_{j-} + \beta_j f_{j-})$$

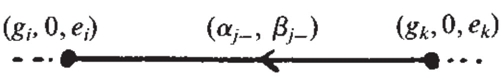
where α_j and β_j are integers such that $0 < \beta_j < \alpha_j$, and $\epsilon_j = \pm 1$. In Waldhausen's notation, μ_{j+}, μ_{j-} and τ_j appear as follows ($k = j_-$):



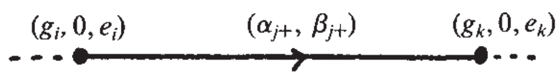
Case 2. Neither M_{j+} nor M_{j-} are solid tori. Then there are (unique up to change of orientation of both curves in the system) coordinate systems (a_{j-}, f_{j-}) and (a_{j+}, f_{j+}) on T_j^-, T_j^+, respectively, such that, in $H_i(T_j)$,

$$f_{j+} = \epsilon_{j-}(\alpha_{j-} a_{j-} + \beta_{j-} f_{j-})$$
$$f_{j-} = \epsilon_{j+}(\alpha_{j+} a_{j+} + \beta_{j+} f_{j+})$$

where $\epsilon_{j(\overset{+}{-})} \in \{1, -1\}$ and $0 \leq \beta_{j(\overset{+}{-})} < \alpha_{j(\overset{+}{-})}$. Then μ_{j+}, μ_{j-} and τ_j appear as follows, if $j_- = k$ and $j_+ = i$.



or



As Waldhausen derives, $\epsilon_{j+} = \epsilon_{j-} =: \epsilon_j$, and $\alpha_{j+} = \alpha_{j-} =: \alpha_j$ and β_{j-} is the inverse of β_{j+} mod α_j.

Call A the graph obtained. A homomorphism $H_1(A) \to \mathbf{Z}_2$ is defined by assigning to a cycle formed by edges $\tau_{j_1}, \ldots, \tau_{j_m}$ the product $\epsilon_{j_1} \cdot \ldots \cdot \epsilon_{j_m}$. The

Waldhausen graph consists of the graph A with valued vertices, oriented and valued edges, and the homomorphism $H_1(A) \to \mathbf{Z}_2$.

6. COVERINGS OF S^3 BRANCHED OVER *ITL*

Let L be a link in S^3, and pr: $M \to S^3$ be a finite covering branched over L. It is determined by the unbranched covering of the exterior of the link, which very often is given by a transitive representation ω: $\pi_1(S^3 - N^0(L)) \to \Sigma n$. Here Σn denotes the symmetric group on n letters, n the number of sheets of the covering.

It is a folklore result that any finite covering of a graph-manifold branched over a union of fibers induces a graph-structure in the cover. The analogous result for Seifert manifolds was proved by Gordon-Heil [G-H], and the result for graph-manifolds follows from the proof of their theorem. As a consequence of this and of (3.1), if L is an *ITL* the cover is a graph-manifold. The aim of this section is to develop an algorithm to compute an m-graph for it.

(6.1) Consider an *ITL* L given by a toral sequence. Take for L the notation of §3. From the toral sequence we derive a closed braid presentation of L_r by the following inductive procedure. Start with the one-string braid presentation of the trivial knot L_0. Once a braid with closure L_{i-1} is obtained, look at the strings which close to the component K_j of L_{i-1}, $j = \text{it}\,(i)$. For all of them but one do the following: replace it by $\lambda_i + 1$ parallel strings. Replace the other string by the (λ_i+1)-braid Δ^k $(\sigma_{\lambda_i}^2\ \sigma_{\lambda_i-1} \ldots \sigma_2\sigma_1)^{\mu_i}$, where $\Delta = (\sigma_{\lambda_i} \ldots \sigma_2\sigma_1)^{\lambda_i+1}$ (a full twist in a (λ_i+1)-braid), and k is the integer verifying that the toral minus the canonical framing of K_j is k times the meridian of K_j in $H_1(S^3 - K_j)$. Then what is obtained is a braid whose closure is L_i. By repeating this, finally arrive to a braid whose closure is L_r. Associate to each string of the braid an (oriented) meridian of L_r. Then a representation of $\pi_1(S^3 - L_r)$ into a symmetric group is determined by a suitable assignation of a permutation to each string of the braid. This assignation will provide also a representation of $\pi_1(S^3 - L)$ if, for any string corresponding to a component of L_r which is to be deleted do get L, the permutation assigned is the identity. This procedure to obtain a braid whose closure is L_r reflects the gluing of the S^1-bundles of the graph-structure described in §3, and the strings of the resulting braid are grouped in a nice way which will allow to obtain, from the permutations associated to the strings of the braid, the permutation associated to each element of the generator system S_r defined in §3.

(6.2) The starting data of our algorithm will be the toral sequence of the link L together with the assignation of a permutation to each string of the braid (whose closure is L_r) corresponding to a component of L. We will assume that this assignation comes from a transitive representation $\omega\colon \pi_1(S^3-L)\to\Sigma n$. The toral sequence of our ITL consists of vectors of $r+1$ entries, $r\geq 1$, and we will use the notation of §3 for the graph-structure of the exterior of L. The cover M associed to ω is then the union $M=\mathrm{pr}^{-1}(X_0)\cup \mathrm{pr}^{-1}(W_1)\cup\ldots\cup \mathrm{pr}^{-1}(W_r)\cup \mathrm{pr}^{-1}(N(L_r))$. The pieces $\mathrm{pr}^{-1}(X_0)$, $\mathrm{pr}^{-1}(W_i)$, $i=1,\ldots,r$, and $\mathrm{pr}^{-1}(N(L_r))$ are S^1-bundles which most often will be non connected. They are pasted through tori, the preimages under pr of the tori of the graph-structure downstairs. The vertices of a graph representing this graph-structure of M are in one-to-one correspondence with the set of connected components of $\mathrm{pr}^{-1}(X_0)$, $\mathrm{pr}^{-1}(W_i)$, $i=1,\ldots,r$, and $\mathrm{pr}^{-1}(N(L_r))$. The edges are in one to-one correspondence with the connected components of the preimages of the tori of $\{T^i, T^i_a, T^i_c, i=1,\ldots,r\}$

(6.3) The monodromy representation ω will be our tool to construct the graph of M. We will repeteadly apply the following

Basic fact. If we are given a homomorphism $W\colon \pi_1(X,x)\to\Sigma n$, where X is a connected space, say a manifold for simplicity, and $x\in X$, there is a (possibly non connected) covering pr: $\tilde{X}\to X$ associated to ω. The group Im ω acts on $\{1,2,\ldots,n\}$ determining a partition $\mathscr{P}_\omega=\{P_1,\ldots,P_p\}$ of this set. Then $\mathrm{pr}^{-1}(X)$ has p connected components, each labelled by an element of $\mathscr{P}_\omega$. No matter which basepoint is taken in X, the covering is determined by ω up to equivalence of coverings. Howewer, fundamental groups of X based at different points are isomorphic but not in a canonical way; two isomorphisms differ in an inner automorphism. And a subgroup of Σn conjugate to Im ω yields in general a different partition when acting on $\{1,2,\ldots,n\}$.

If Y is a (connected) subspace of X and $y\in Y$, the inclusion $\epsilon\colon(Y,y)\to(X,y)$ induces

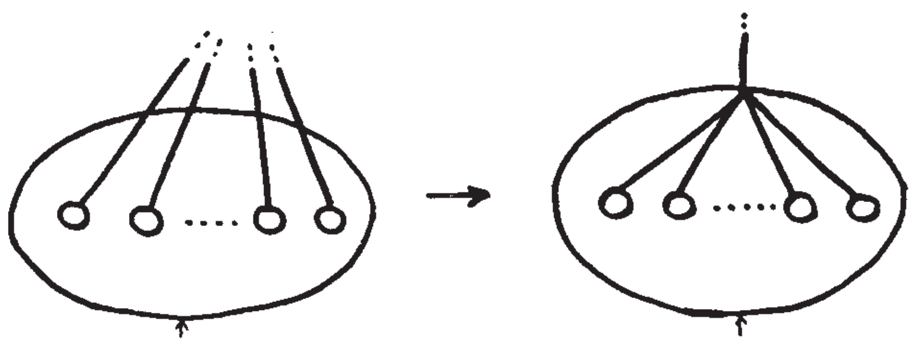
$$\omega\epsilon_*\colon \pi_1(Y,y)\to\pi_1(X,y)\to\Sigma n$$

which is the monodromy representation of the covering

$$\mathrm{pr}|\colon \mathrm{pr}^{-1}(Y)\to Y$$

The partition $\mathscr{P}_{\omega\epsilon_*}$; of $\{1,2,\ldots,n\}$ associated to $\omega\epsilon_*$ is a refinement of $\mathscr{P}_\omega$ in the sense that each set of $\mathscr{P}_\omega$ is a union of elements of $\mathscr{P}_{\omega\epsilon_*}$. A component $\bar{Y}$ of $\mathrm{pr}^{-1}(Y)$ is a subspace of a component $\bar{X}$ of $\mathrm{pr}^{-1}(X)$ if and only if the element of $\mathscr{P}_{\omega\epsilon_*}$ which labels $\bar{Y}$ is a subset of the element of $\mathscr{P}_\omega$ which labels $\bar{X}$. Note that the procedure to obtain the partition $\mathscr{P}_\omega$ is algorithmic if one starts from a set $\sigma_1,\ldots,\sigma_m$ of permutations which generate Im ω.

(6.4) Look at the closed braid projection obtained for L_r in (6.2). We can visualize on it the graph-structure of the exterior of L_r. For as L_r is obtained from L_{r-1} by replacing a component of it by a (λ_r+1)-braid, it appears a solid torus containing this braid, such that L_{r-1} is obtained from L_r by colapsing this solid torus to its core. The solid torus minus an open tubular neighborhood of the (λ_r+1)-braid is W_r. The same way one sees $W_{r-1},\ldots, W_1$, X_0. A natural way to choose basepoints on each torus of the graph-structure, and paths joining them, is the following. Fix a point $*$ over the plane of projection of L_r. Fix a level of the closed braid that does not contain any multiple point of the projection. Join the point $*$ to each string of the closed braid at the fixed level by a linear path. These linear paths intersect each inner most torus (the tori of $\partial(S^3-N_0(L_r))$ in one point, which will be taken as basepoint on this torus. Then proceed inductively as indicated in figure 6.1:



Of all sections of T^i, take the closest to the braid axis of L_r and, on this curve, the basepoint for T^i. The paths joining the basepoints on the tori will be the linear segments joining them. Then isomorphisms between the fundamental groups of $S^3-N^0(L_r)$ based at the different basepoints we handle have been fixed. We are in the situation of (3.3) and know exactly what the elements of S_r represent. It is clear how to compute $\{(x,\omega(x)),\ x\in S_r\}$, using the relations of R_r for the longitudes.

(6.5) Since paths joining the basepoints at the bundles or tori of the graph-structure of $S^3-N^0(L)$ have been fixed, the inclusion induced homomorphisms followed by ω are unambiguously defined. We will look first at the associated unbranched covering of $S^3-N^0(L_r)$, and at the end we will look at the branch locus. Thus, for W_i and T^i, T^i_a and T^i_c, $i=1,\ldots,r$ we have respective partitions Q_i, $\mathscr{P}^i$, $\mathscr{P}^i_a$ and $\mathscr{P}^i_c$, $i=1,\ldots,r$, of $\{1,2,\ldots,n\}$ whose elements (subsets of $\{1,2,\ldots,n\}$) label the components of the preimage under pr of the piece considered.

Compute the partitions $\mathscr{P}^i$, $\mathscr{P}^i_a$ and $\mathscr{P}^i_c$, respectively, from the sets of permutations

$$A^i=\{\omega(\delta^i(m_i)),\omega(l_i)\}$$
$$A_a^i=\{\omega(\delta_a^i(m_a^i)),\omega(\delta_a^i(l_a^i))\}$$
$$A_c^i=\{\omega(\delta_c^i(m_c^i)),\omega(\delta_c^i(l_c^i))\}$$

where $\delta^i(m_i)$, $\delta_a^i(m_a^i)$, $\delta_a^i(l_a^i)$, $\delta_c^i(m_c^i)$ and $\delta_c^i(l_c^i)$ are to be understood as words on letters of $O_i\subset S_i\subset S_r$ given by Lemma 3.3. This is a calculation from the data $\{(x,\omega(x)), x\in S_r\}$. The sets A^i, A_a^i and A_c^i generate the image under ω of $\pi_1(T^i, g_i)$, $\pi_1(T_a^i, a_i)$ and $\pi_1(T_c^i, c_i)$, respectively. Since W_i is a trivial bundle, $A^i\cup A_a^i\cup A_c^i$ generate the image under ω of $\pi_1(W_i,g_i)$ and so Q_i can be computed from this set of permutations. Since X_0 and each component of $N(L_r)$ are solid tori, the partition associated to them coincides with the one associated to their respective boundary torus.

Let

$$\mathscr{P}^i=\{E_1^i,\ldots,E_{r(i,0)}^i\}$$
$$\mathscr{P}_a^i=\{E_1^{ia},\ldots,E_{r(i,a)}^{ia}\}$$
$$\mathscr{P}_c^i=\{E_1^{ic},\ldots,E_{r(i,c)}^{ic}\}$$
$$\mathscr{Q}^i=\{V_1^i,\ldots,V_{r(i)}^i\}$$

for $i=1,\ldots,r$. Then $r(i,0)$, $r(i,a)$, $r(i,c)$ and $r(i)$ are the number of components of $\mathrm{pr}^{-1}(T^i)$, $\mathrm{pr}^{-1}(T_a^i)$, $\mathrm{pr}^{-1}(T_c^i)$ and $\mathrm{pr}^{-1}(W_i)$, respectively.

Construct a simplicial graph supporting the valued graph representing M as follows:

1. For $i=1,\ldots,r$, draw vertices $v_{i1}\ldots v_{ir(i)}$ corresponding to the sets $V_1^i,\ldots,V_{r(i)}^i$. Draw also vertices $v_{01},\ldots,v_{0r(1,0)}$, corresponding to the elements of $\mathscr{P}^1$, to represent the components of $\mathrm{pr}^{-1}(X_0)$.

2. For $i=1,\ldots,r$, W_i is attached to $W_{\mathrm{ff}(i)}$, where ff is the function defined in (3.4). for $1\leq l\leq r(i)$, $1\leq l'\leq \mathrm{ff}(i)$, join v_{il} to $v_{il'}$ by k edges if $V_{il}\cap V_{jl'}$ is the union of k sets of $\mathscr{P}^i$.

3. The link L_r has $r+1$ components. There are $r+1$ partitions of type $\mathscr{P}_c^i$ which were not involved in point 2. For each of these, draw $r(i,c)$ vertices and join k of them to v_{il}, $1\leq l\leq r(i)$ if V_{il} is the union of k elements of $\mathscr{P}_c^i$.

(6.6) Fix i, $1\leq i\leq r$. Let W_{il} denote the component of $\mathrm{pr}^{-1}(W_i)$ labelled by $V_l^i\in Q_i$. The number of boundary components of W_{il} is $n_{il}=k_1+k_2+k_3$ if V_{il} the union of k_1 (respectively k_2, k_3) sets of $\mathscr{P}^i$ (respect. $\mathscr{P}_a^i$, $\mathscr{P}_c^i$).

The curve l_c^i is a fiber of W_i. Write the permutation $\omega(\delta_c^i(l_c^i))$ as a product of disjoint cycles, d_{ie} of which have figures in V_l^i. Then $\mathrm{pr}^{-1}(l_c^i)$ has d_{ie} connected components in W_{il}, each of them is a fiber.

Lemma. *Let F be an S^1-bundle over a sphere with b holes. Let* pr: $\bar{F} \to F$ *be a covering such that, if f is a fiber of F,* $\mathrm{pr}^{-1}(f)$ *has d components. The base $\bar{B}$ of the S^1-bundle $\bar{F}$ has Euler characteristic*

$$\chi(\bar{B}) = (2-b)\cdot d$$

Proof. The surface $\bar{B}$ is homotopically equivalent to a 1-complex where the number of 0-cells is d, and the number of 1-cells is $(b-1)\cdot d$. ∎

The base of the S^1-bundle W_{il} is an orientable surface whose Euler characteristic is, according to the Lemma, $(2-3)\cdot d_{il} = -d_{il}$. Hence its genus is

$$g_{il} = 1 + \frac{1}{2}(d_{il} - n_{il})$$

(6.7) The choice of coordinate systems for the boundary tori of W_i, $i = 1, 2, \ldots, r$ as in (3.4) was made to make easy the computation of the homomorphism ω on the fundamental groups of these tori. Howewer these coordinate systems are not appropiate to represent the gluing of the S^1-bundles of the graph-structure of $S^3 - N^0(L)$ the way of Waldhausen. Appropiate coordinate systems are obtained as follows.

Let P_i be a cross-section of the (trivialisable) S^1-bundle W_i, oriented in such a way that $\partial P_i \cap T_c^i$ is the oriented curve $-m_c^i$, in the notation of (3.4). Orient the fibers of W_i so that the orientation of P_i followed by the orientation of the fibers gives the orientation of W_i, induced by that of S^3. Let $u_i = \partial P_i \cap T_a^i$ and $v_i = \partial P_i \cap T_a^i$ (oriented curves). Then

$$u_i + v_i - m_c^i = \partial P_i$$

and therefore, if the homology classes of the curves are denoted the same way as the curves themselves,

$$u_i + v_i - m_c^i = 0 \quad \text{in } H_1(W_i)$$

Let f_i denote the fiber of W_i. In an unambiguous way we can denote by f_i this fiber in any of the tori T^i, T_a^i, T_c^i. Note that f_i is homologous to l_c^i on T_c^i. Henceforth we will take (u_i, f_i), (v_i, f_i) and $(-m_c^i, f_i)$ on T^i, T_a^i and T_c^i, respectively. The following relations are verified

$$\begin{bmatrix} u_i \\ f_i \end{bmatrix} = \begin{pmatrix} -\gamma_i & -\delta_i \\ \mu_i & \lambda_i \end{pmatrix} \begin{bmatrix} m_i \\ l_i \end{bmatrix} \quad \text{in } H_1(T^i)$$

$$\begin{bmatrix} v_i \\ f_i \end{bmatrix} = \begin{pmatrix} \gamma_i & \delta_i \\ \mu_i & \lambda_i \end{pmatrix} \begin{bmatrix} m_a^i \\ l_a^i \end{bmatrix} \quad \text{in } H_1(T_a^i) \tag{1}$$

for some intergers γ_i, δ_i such that $\gamma_i\lambda_i-\mu_i\delta_i=-1$, $1\leq i\leq r$. Hence the first matrix has determinant 1, and the second has determinant -1. Since the fundamental group of a torus is abelian, these relations suffice to compute the permutations $\omega(u_i)$, $\omega(v_i)$.

We know that W_1 is attached to the solid torus X_0 in such a way that l_1 becomes a meridian of X_0. On the other hand, for $i\geq 2$ T^i coincides with either T^j_a or T^j_c, $j=\mathrm{ff}(i)$ (see (3.4)). Since satellization permutes meridians and longitudes of the gluing tori (cf. (2.2)), in the first case the relation

$$\begin{bmatrix} l_i \\ m_i \end{bmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{bmatrix} m^j_a \\ l^j_a \end{bmatrix}$$

is satisfied, and in the second case

$$\begin{bmatrix} l_i \\ m_i \end{bmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{bmatrix} m^j_c \\ l^j_a \end{bmatrix}$$

Hence $\begin{bmatrix} m_i \\ l_i \end{bmatrix} = \begin{bmatrix} m^j_a \\ l^j_a \end{bmatrix}$ or $\begin{bmatrix} m_i \\ l_i \end{bmatrix} = \begin{bmatrix} m^j_c \\ l^j_c \end{bmatrix}$, and the following relations derive from these and relations (1) given above in this paragraph:

$$\begin{gathered}
\begin{bmatrix} u_i \\ f_i \end{bmatrix} = \begin{pmatrix} -\gamma_i & -\delta_i \\ \mu_i & \lambda_i \end{pmatrix} \begin{bmatrix} m^j_a \\ l^j_a \end{bmatrix} = \begin{pmatrix} -\gamma_i & -\delta_i \\ \mu_i & \lambda_i \end{pmatrix} \begin{pmatrix} \gamma_j & \delta_j \\ \mu_j & \lambda_j \end{pmatrix}^{-1} \begin{bmatrix} v_j \\ f_j \end{bmatrix} \\
\begin{bmatrix} u_i \\ f_i \end{bmatrix} = \begin{pmatrix} -\gamma_i & -\delta_i \\ \mu_i & \lambda_i \end{pmatrix} \begin{bmatrix} m^j_c \\ l^j_c \end{bmatrix} = \begin{pmatrix} \gamma_i & -\delta_i \\ -\mu_i & \lambda_i \end{pmatrix} \begin{bmatrix} -m^j_c \\ f_j \end{bmatrix}
\end{gathered} \tag{2}$$

The first equation holds in the first case, the second in the second case.

(6.8) Let T be an oriented torus, and pr: $\bar{T}\to T$ a covering associated to a transitive representation ω: $H_1(T)\to\Sigma s$. Let (a,f) be a coordinate system on T such that $a\cdot l=1$, where the dot denotes intersection on T. Suppose we are given $\sigma_a=\omega(a)$ and $\sigma_f=\omega(f)$, and σ_f is the product of disjoint cycles of length ρ. Then ρ divides s. Let pr_*: $H_1(\bar{T})\to H_1(T)$ be induced by pr.

Lemma

i) $\mathrm{pr}_*(\bar{f})=\rho f$

ii) There is a simple closed curve $\bar{a}$ on $\bar{T}$ such that, if $\cdot$ denotes the intersection pairing on $\bar{T}$ with the orientation inherited by that of T and pr,

$$\bar{a}\cdot\bar{f}=1$$

$$\mathrm{pr}_*(\bar{a})=\frac{s}{\rho}\,a-\alpha f$$

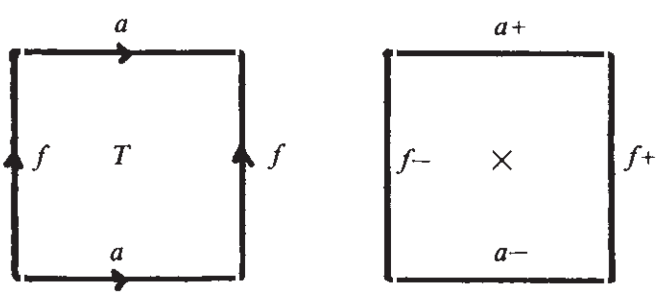
where α *is determined by the conditions* $\sigma_a^{\frac{s}{\rho}}(1)=\sigma_f^{\rho}(1)$ *and* $0\leq\alpha<\rho$

iii) *If* c *is a closed curve on* T *such that* $c=xa+y\ f$ in $H_1(T)$, then

$$\mathrm{pr}^{-1}(c)=\rho x\bar{a}+(\frac{s}{\rho}\,y+\alpha x)\,\bar{f}\ \ in\ H_1(\bar{T}).$$

Proof: The proof is based on constructing the covering pr: $\bar{T}\to T$ using the subcomplex $a\cup l$ as a splitting complex for T. Call X the square obtained as the result of cutting T along this splitting complex (figure 6.2)

Fig. 6.2



Then $\bar{T}$ is the union of s copies $X_1,\ldots,X_s$ of X glued in this way: the edge a_+ of X_k is glued to the edge a_- of $X_{\sigma l(k)}$, and the edge f_+ of X_k is glued to the edge f_- of $X_{\sigma_a(K)}$. The result is a big square like the one in figure 6.3, where each column is made of ρ squares, and there are $\frac{s}{\rho}$ columns. Top and bottom are identified by translation, and the vertical edges of the big square represent $\bar{f}$. These vertical edges are to be identified taking the side f_+ of X_r, where $r=\sigma_a^{\frac{s}{\rho}-1}(1)$ to the side f_- of $X_{r'}$, where $r'=\sigma_a^{\frac{s}{\rho}}(1)$.

An orient straight line from a point on the edge f_- of $X_{r'}$ to its corresponding point on the edge f_+ of X_r provides an oriented simple closed curve $\bar{a}$ on $\bar{T}$ satisfying the statement of the lemma. Actually, a is covered by the curves $\rho\bar{a}+\alpha\bar{f}$. Therefore

$$sa=\mathrm{pr}_*(\rho\bar{a}+\alpha\bar{f})=\rho\ \mathrm{pr}_*(a)+\alpha\ \mathrm{pr}_*(f)=$$
$$=\rho\,\mathrm{pr}_*(\bar{a})+\alpha\rho f$$

Hence $\mathrm{pr}_*(\bar{a})=\frac{s}{\rho}\,a-\alpha f$. Part iii) is a simple calculation derived from i) and ii). ∎

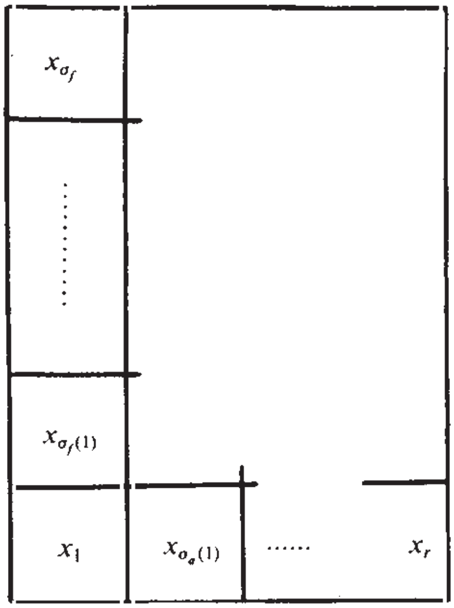


Fig. 6.3

(6.9) Up to now we have determined, for each $i=1,\dots,r$, partitions $\mathcal{Q}_i$, $\mathcal{P}^i$, $\mathcal{P}^i_a$ and $\mathcal{P}^i_c$ associated to W_i, T^i, T^i_a and T^i_c, respectively. We have described how to find from them a simplicial graph representing the S^1-bundles and tori of the graph-structure of M. Moreover we have selected in T^i (resp. T^i_a, T^i_c) appropriate coordinate system (u_i, f_i) (resp. (v_i, f_i), $(-m^i_c, f_i)$) and the permutations associated to them. We can decide, from the functions ff and cf, whether one of these tori is a component of $\partial(S^3-n^0((L_r))$ or it is the same torus as T^j, T^j_a or T^j_c for some $j\neq i$.

Now let us fix an index i and look at an element of $\mathcal{Q}_i$, say V^i_l; it is a subset of $\{1,2,\dots,n\}$ and labels a component W_{il} of $\mathrm{pr}^{-1}(W_i)$. We can algorithmically find the elements of $\mathcal{P}^i$, $\mathcal{P}^i_a$ and $\mathcal{P}^i_c$ that are subsets of V^i_l. Consider an element E of one of these partitions such that $E\subset V^i_l\subset\{1,2,\dots,n\}$. Then E labels a component of the preimage of a torus of ∂W_i, say T. Let us denote $\bar{T}$ the component of $\mathrm{pr}^{-1}(T)$ labelled by E. Next we are going to find a coordinate system on $\bar{T}$ and what is needed to valuate the edge of the graph of M representing the torus $\bar{T}$. Three cases can occur: either $T\subset\partial(S^3-N^0(L_r))$, or $T=T_1$, or T coincides with a torus $T'=T^j$, T^j_a or T^j_c for some $j\neq i$. There is a selected coordinate system on T, call it (a, f). Truncate the permutations $\omega(a)$, $\omega(f)$, written as a product of disjoint cycles, taking the cycles whose figures lie in E. Then a permutation σ_a is obtained from $\omega(a)$, and σ_f from $\omega(f)$.

Case 1. $T\subset\partial(S^3-N^0(L_r))$. Then either $T=T^i_a$ or $T=T^i_c$. Set $m=m^i_a$ or m^i_c. Express m as a linear combination of (a, f), using equations (1) in (6.7).

Apply lemma (6.8) to obtain a transversal $\bar{a}$ in $\bar{T}$, and to express $(\mathrm{pr}/\bar{T})^{-1}(m)$ as a linear sum of $\bar{a}$ and $\bar{f}$. Let $\bar{m}$ be a connected component of $(\mathrm{pr}/\bar{T})^{-1}(m)$. Normalize if necessary $\bar{a}$ to another transversal $\bar{a}n$ such that $\bar{m}=\epsilon(\alpha\bar{a}n+\beta\bar{f})$ with $0\leq\beta<\alpha$, $\epsilon=\pm1$. The integer α is different from 0 since $\bar{m}$ is not homologous to $\bar{f}$ (otherwise m would be homologous to f). To label the edge, representing $\bar{T}$, which is incident to v_{il}, oriented outwards v_{il} with a matrix complete (α,β) to $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in GL(2,\mathbf{Z})-SL(2,\mathbf{Z})$. Compute $\mathrm{pr}_*(\bar{a}n)=Aa+Bf$. Keep associated to E the pairs (α,β), (A, B).

Case 2. $T=T^1$. Equations (1) in (6.7) allow to express l_1 as a function of (u_1, f_1). Take $m=l_1$ and proceed as in case 1.

Case 3. $T=T'$. Let f' denote the fiber in T'. Equations (2) in (6.7) allow to express f' as a linear combination of (a, f). Take $m=f'$ and proceed as in case 1. Keep associated to T the pairs (α,β), (A, B) and ϵ. Analogously, the selected coordinate system (a', f') in T', express f as a linear combination of (a', f') and obtain the corresponding (α', β'), (A', B'). Then $\alpha'=\alpha$, $\beta'=-\gamma$, and one can obtain δ such that $\begin{pmatrix}\gamma & \delta\\ \alpha & \beta\end{pmatrix}\in GL(2,\mathbf{Z})-SL(2,\mathbf{Z})$. This matrix valuates the edge associated to E, oriented towards the vertex representing W_{il}.

(6.10) Once step (6.9) has been carried out for all parts $\mathcal{P}^i$, $\mathcal{P}^i_a$ and $\mathcal{P}^i_c$ which are subsets of V^i_l, we have got a series of transversals, one for each boundary torus of W_{il}, and corresponding to them, a series of pairs (A_1, B_1), ..., $(A_{n(i,l)}, B_{n(i,l)})$. Due to the relation $u_i+v_i-m^i_c=0$ in $H_1(W_i)$, the sum $B_1+\ldots+B_{n(i,l)}=-\rho_{il}e_{il}$, where e_{il} is the Euler number associated to $v_{il}(W_{il})$ in the graph of M. (see the remark in section 4; ρ_{il} is the integer such that $\mathrm{pr}_*\bar{f}=\rho_{il}f$).

(6.11) Next proposition generalizes theorem 1 of [Ne].

Proposition. *Any cover of S^3 branched over an ITL es irreducible.*

Proof: Let M be a branched cover of S^3 over an *ITL* L. Let $\tilde{T}$ be the graph-structure on M induced by the covering and the graph-structure $\tilde{T}$ associated to some toral sequence of L. In order to prove that M is not a connected sum, we begin by checking that R4 is never satisfied by $(M, \tilde{T})$.

Suppose that T' is a torus of $\tilde{T}$ which separates M in a solid torus V and the union of the other S^1-bundles of the graph-manifold. Then $T=\mathrm{pr}(T')$ is a torus of $\bar{T}$, the graph-structure of the link exterior, and T is either a component of $\partial(S^3-N^0(L_r))$, or $T=T^1$, with the notation on §3 and the present section. Let (a, f) be the coordinate system on T chosen as in (6.9).

Case 1. $T\subset\partial(S^3-N^0(L_r))$. Then either $T=T_a^i$ or $T=T_c^i$. Let m be the meridian of the corresponding component of L_r. Since m is not homologous to the fiber of W_i, $\mathrm{pr}^{-1}(m)$ is neither a multiple of the fiber of $\mathrm{pr}^{-1}(W_i)$, and therefore the meridian of V will never be homologous on T' to a fiber of the adjacent S^1-bundle.

Case 2. $T=T^1$. Let m be a meridian of X_0. It is not homologous to the fiber of W_1 because $\mu_1\neq 0$. Again, the meridian of V, a component of $\mathrm{pr}^{-1}(m)$, is not homologous to the fiber of the adjacent S^1-bundle, which is a component of $\mathrm{pr}^{-1}(W_1)$.

Now, if L is not the trivial knot, the proof of theorem (7.1) of [W] serves as a proof that M is irreducible. On the other hand, all branched covers of S^3 over the trivial knot are S^3, which is irreducible. ∎

Corollary. *$S^2\times S^1$ never appears as a branched cover of S^3 over an ITL.* ∎

Corollary. *Any branched cover M of S^3 over an ITL is either a lens space or a manifold homeomorphic to a reduced graph-manifold. Moreover, the reduction process described in §5, applied to an m-graph of the branched cover obtained by the algorithm of this section, decides wheter M is a lens spaces, or a torus bundle, or a Seifert fibered space over S^2 with three exceptional fibers (and give the invariants), or otherwise, ends in a Waldhausen graph of a reduced graph-structure of M.* ∎

By an irreducible link we mean a link whose exterior is an irreducible manifold. The next corollary is a well known result

Corollary. *Any ITL is prime and irreducible.*

Proof: It is known that a link is prime and irreducible if and only if its 2-fold branched cover (or any virtually regular branched cover) is an irreducible manifold (see [Lo-S]). ∎

7. FINDING REPRESENTATIONS

The algorithmic procedure described in §3 for finding a presentation of the fundamental group of the exterior of an *ITL* has been used in §6 to develop an algorithm or computing branched covers of S^3 over *ITL*. Another application is an algorithm for finding representations of the groups of iterated torus links into symmetric groups.

Suppose we are given an *ITL* and we are asked for all coverings of S^3 branched over it with a fixed number of sheets, say n. Let us take the notation of §3 for the *ITL*, that is, let us call L the *ITL*, and $L_0, L_1, \ldots, L_r$ the sequence of links of the satellization process carried out to obtain L, given a toral sequence. The group $\pi_1(S^3 - N^0(L_i))$ is generated by S_i with relators R_i, $i \geq 0$. The procedure consists in checking inductively which maps from S_i into Σ_n for $i = 0, 1, \ldots, r$, subject to the restriction that $x_{i1}, x_{i2}, \ldots, x_{i\lambda_i}$ be mapped to conjugate permutations (i.e., permutations of the same type), preserve the relators of R_i. Those maps which pass the test for S_i, R_i are tried to be extended to S_{i+1}. The maps of S_r into Σ_r which preserve the relators of R_r and S yield representations of the group of L into Σ_n. The ones that are transitive correspond to n-fold coverings of S^3 branched over L. A non transitive representation gives rise to several branched covers whose number of sheets divides n.

On the other hand, one may be interested in determining the iterated torus link which admit a branched covering of a certain type, for instance of a given branch index type. Then we work in the reverse direction. For simplicity, let us look first for iterated torus knots. Such a knot K is uniquely determined by a sequence $(\lambda_1, \mu_1), \ldots, (\lambda_r, \mu_r)$. Then $S^3 - N^0(K) = X_0 \cup W_1 \cup \ldots \cup W_r \cup U$, where W_i is attached to W_{i-1} through T_c^{i-1}, for $i > 0$. Fix the order of the symmetric group, Σ_n. We begin looking for pairs (λ_r, μ_r) and maps from O_r into Σ_n which map $x_{r(\lambda_r+1)}$ to the identity permutation, send $x_{r1} \ldots x_{r\lambda_r}$ to conjugate permutations of the type required by the branch index conditions, and which preserve the relators T_r. Then the inductive step is to find the pairs (λ_i, μ_i) for which it is possible to extend the map from $S_r - S_i$ to $S_r - S_{i-1}$.

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