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3-Manifold Spines and Bijoins

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ABSTRACT. We describe a combinatorial algorithm for constructing all orientable 3-manifolds with a given standard bidimensional spine by making use of the idea of bijoin ([BG], [Gr]) over a suitable pseudosimplicial triangulation of the spine.

1. INTRODUCTION

Throughout this paper, all spaces and maps are piecewise-linear (pl) in the sense of [Gl] or [RS]; all 3-manifolds are supposed to be compact, connected and orientable.

If M is a 3-manifold with non-empty boundary, then a bidimensional polyhedron K such that M collapses to K is said to be a *spine* of M; if M is closed, a spine of M is a spine of M-B, B being an open 3-ball in M.

Given a group presentation $\Phi=\{x_1,\dots,x_g \mid r_1,\dots,r_s\}$, denote by K_Φ the bidimensional complex constructed as follows:

- K_Φ has only one 0-cell (vertex);
- the 1-cells (resp. the 2-cells) of K_Φ are in one-to-one correspondence with the generators (resp. the relators) of Φ; denote them by α_i *(resp.* β_j);
- each 2-cell β_j is attached to the 1-skeleton by the formula given by the corresponding relator r_j.

K_Φ is said to be the *standard complex* associated to Φ; of course, the factor group of Φ is $\Pi_1(|K_\Phi|)$. We will not distinguish between a relator r_j and any

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cyclic conjugate of it or its inverse, since the associated complexes are the same. The above construction may be obviously reversed and each standard complex K induces a group presentation Φ_K of the fundamental group $\Pi_1(|K|)$.

It is well known that every 3-manifold M has a standard spine K_Φ, for some group presentation Φ, and the factor group of Φ is clearly $\Pi_1(M)$; nevertheless, not every standard complex K_Φ is a spine of a 3-manifold. Every group presentation Φ such that K_Φ is a spine of a 3-manifold (resp. of a closed 3-manifold) is said to be *geometric* (resp. *strongly geometric*).

In [N] Neuwirth gives an algorithm for testing if a balanced group presentation (same number of generators and relators) is strongly geometric. The same algorithm is restated by Osborne and Stevens ([OS_1], [S]) by making use of a graph-theoretical tool, the presentation-graph or *P-graph* P_Φ, which can be associated, by a one-to-one correspondence, to the standard complex K_Φ of a group presentation Φ. Namely, P_Φ is essentially the boundary of a regular neighbourhood of the unique vertex of K_Φ and it is easy to prove that Φ is strongly-geometric if and only if a planar imbedding condition on P_Φ holds. Moreover, as pointed out by Montesinos in [M], a Heegaard diagram of a 3-manifold M gives rise to such a planar imbedding of the P-graph P_Φ associated to a suitable group presentation Φ of $\Pi_1(M)$; in fact, this is nothing else than the *Whitehead graph* of the group presentation Φ of $\Pi_1(M)$ coming from the given Heegaard diagram of M. Thus, a group presentation Φ is geometric (resp. strongly-geometric) if and only if there exists a Heegaard diagram of a 3-manifold (resp. closed 3-manifold) M whose associated presentation for $\Pi_1(M)$ is Φ.

In [M] Montesinos describes an algorithm for checking if a given group presentation is geometric; such an algorithm seems to be completely different from Neuwirth's one, since it makes use of branched covering techniques.

In the present paper, we give a combinatorial algorithm for obtaining all 3-manifolds with a given standard spine K_Φ by making use of the *bijoin construction* ([BG], [Gr]) applied to a graph-theoretical structure representing a pseudosimplicial triangulation of K_Φ. This construction allows us to unify, in a common geometric description, both Neuwirth algorithm and Montesinos one; namely, the necessary and sufficient conditions for the geometricity (or the strong-geometricity) of a group presentation obtained in [N], [OS_1], [OS_2], [S], [M] can be all derived from the bijoin construction.

2. EDGE-COLOURED GRAPHS AND ASSOCIATED COMPLEXES

The term *pseudograph* includes loops and multiple edges, while a *multigraph* (or simply a *graph*) allows multiple edges only.

A (generalized) *coloration* on a pseudograph $\Gamma = (V(\Gamma), E(\Gamma))$ is a map γ: $E(\Gamma) \to \Delta_n = \{0, 1, ..., n\}$; if Γ is a graph, γ is said to be *proper* if $\gamma(e) \neq \gamma(f)$, for each pair e, f of adjacent edges. For each $\mathcal{F} \subset \Delta_n$, set $\Gamma_{\mathcal{F}} = (V(\Gamma), \gamma^{-1}(\mathcal{F}))$; each connected component of $\Gamma_{\mathcal{F}}$ is often called an $\mathcal{F}$*-residue*. For each $i \in \Delta_n$, set $\hat{i} = \Delta_n - \{i\}$.

The pair (Γ, γ), Γ being a graph and γ: $E(\Gamma) \to \Delta_n$ a (generalized) coloration, is said to be an *n-dimensional crystallized structure* ([G]) if, for each $i \in \Delta_n$, the $\{i\}$ — residues are cliques (complete graphs). If all these cliques are of order two, i.e. if γ is proper and Γ is regular of degree $n+1$, (Γ, γ) is simply called an *(n + 1) — coloured graph* ([F]).

An n-dimensional *pseudocomplex* K is an n-dimensional ball complex in which every h-ball, considered with all its faces, is isomorphic with the complex underlying an h-simplex; for this reason, each h-ball of K is called h-simplex. The *disjoined star* Std(s, K) of a simplex s in K is defined to be the disjoint union of the n-simplexes of K containing s, with reidentification of the $(n-1)$-faces containing s and of their faces; the subcomplex Lkd$(s, K) = \{\tau \in \text{Std}(s, K) \mid \tau \cap s = \phi\}$ is called the *disjoined link* of s in K.

As shown in [G] and [F], every n-dimensional crystallized structure (Γ, γ) represents a homogeneous n-dimensional pseudocomplex $K(\Gamma)$ constructed by the following rules:

- take an n-simplex $\sigma(v)$ for each $v \in V(\Gamma)$ and label its vertices by Δ_n;
- if $v, w \in V(\Gamma)$ are joined by an i-coloured edge, identify the $(n-1)$ — faces of $\sigma(v)$ and $\sigma(w)$ opposite to the vertices labelled by i, so that equally labelled vertices are identified together.

Every h-simplex s of $K(\Gamma)$, whose vertices are labelled by the distinct colours $c_o, ..., c_h \in \Delta_n$, corresponds to a unique $(\Delta_n - \{c_o, ..., c_h\})$ — residue $\mathfrak{R}$ of (Γ, γ) and viceversa; its associated pseudocomplex $K(\mathfrak{R})$ is Lkd$(s, K(\Gamma))$.

Moreover, (Γ, γ) is an $(n+1)$-coloured graph if and only if $|K(\Gamma)|$ is a closed pseudomanifold ([ST]), which is orientable if and only if Γ is bipartite ([CGP]).

The construction of $K(\Gamma)$ gives a coloration on the vertex set $S_o(K)$ of $K(\Gamma)$ by means of $n+1$ colours (i.e. a map ξ: $S_o(K) \to \Delta_n$ which is injective on each simplex of $K(\Gamma)$). Given a homogeneous n-dimensional pseudocomplex K with such a coloration on its vertex set $S_o(K)$, the construction can be easily reversed yielding an n-dimensional crystallized structure, denoted by $\Gamma(K)$.

It is easy to see that $\Gamma(K(\Gamma)) = (\Gamma, \gamma)$; moreover (see [G]), $K(\Gamma(K)) = K$ if and only if K satisfies the following property:

(*) the disjoined star std(s, K) of every simplex s of K is strongly-connected.

A homogeneous n-dimensional pseudocomplex satisfying (*) and admitting a coloration on its vertex set by means of $n+1$ colours is said to be a *representable n-pseudocomplex*, since it is uniquely represented by an n-dimensional crystallized structure.

An n-dimensional crystallized structure (Γ, γ) (or its associated pseudocomplex $K(\Gamma)$) is said to be *contracted* if $\Gamma_{\hat{c}}$ is connected, for each $c \in \Delta_n$ (i.e. if $K(\Gamma)$ has exactly $n+1$ vertices). A contracted $(n+1)$-coloured (bipartite) graph (Γ, γ) is said to be a *crystallization* of a closed (orientable) n-manifold M if $|K(\Gamma)| = M$. Every closed n-manifold admits a crystallization ([P]).

For a general survey on manifold representation theory by means of edge-coloured graphs, see [FGG], [BM], [V].

3. THE BIJOIN CONSTRUCTION

If Γ is an oriented pseudograph and $\gamma: E(\Gamma) \to \Delta_n$ is a (generalized) coloration, the pair $(\vec{\Gamma}, \vec{\gamma})$ is called an *n-dimensional oriented structure* ([BG]) if, for every $i \in \Delta_n$, the $\{i\}$-residues are elementary oriented cycles, possibly of length one or two.

By deleting all loops in $E(\vec{\Gamma})$ and by replacing, for every $i \in \Delta_n$, each elementary oriented i-coloured cycle in $\vec{\Gamma}_{\{i\}}$ with a clique on the same vertex set, it is easy to associate an n-dimensional crystallized structure (Γ, γ) to every n-dimensional oriented structure $(\vec{\Gamma}, \vec{\gamma})$. Of course, there are, in general, many oriented structures associated to a fixed crystallized structure; they can be easily obtained by reversing the above construction. If $(\vec{\Gamma}, \vec{\gamma})$ is an oriented structure associated to the crystallized structure (Γ, γ), we set $K(\vec{\Gamma}) = K(\Gamma)$.

The following construction, given in [BG], allows to obtain an $(n+1)$-coloured bipartite graph $(B(\vec{\Gamma}), \beta)$ from an $(n-1)$-dimensional oriented structure $(\vec{\Gamma}, \vec{\gamma})$:

- $V(B(\vec{\Gamma})) = V(\vec{\Gamma}) \times \{0, 1\}$;
- for every vertex $v \in V(\vec{\Gamma})$, join $(v, 0)$ with $(v, 1)$ by an edge e of $B(\vec{\Gamma})$ and set $\beta(e) = n$;
- if $\vec{e} \in E(\vec{\Gamma})$ and $\vec{e}(0) = v$, $\vec{e}(1) = w$, then join $(v, 0)$ with $(w, 1)$ by an edge e' of $B(\vec{\Gamma})$ and set $\beta(e') = \gamma(\vec{e})$.

Note that the choice of the opposite oriented structure, obtained by reversing the orientation of each $\{i\}$-residue, for every $i \in \Delta_{n-1}$, gives rise to the same graph. The construction in an adapting to the edge-coloured graphs of a standard method for associating a bipartite graph to an arbitrarily given oriented graph ([BHM]). The $(n+1)$-coloured graph $(B(\vec{\Gamma}), \beta)$ (and its associated pseudocomplex) is said to be the *h-bijoin* over $(\vec{\Gamma}, \vec{\gamma})$, h being the number of the $\hat{n}$-residues in $(B(\vec{\Gamma}), \beta)$; if $h=1$, $(B(\vec{\Gamma}), \beta)$ is simply called *bijoin.*

Given an $(n+1)$-coloured bipartite graph (Γ, γ), it is easy to (uniquely) construct, for each $i \in \Delta_n$, an $(n-1)$-dimensional oriented structure $({}^i\vec{\Gamma}, {}^i\vec{\gamma})$ such that $(B({}^i\vec{\Gamma}), \beta) = (\Gamma, \gamma)$ ([BG]); thus, every closed n-manifold can be obtained as a bijoin over a suitable $(n-1)$-dimensional oriented structure. A refinement of this result, in dimension three, obtained by making use of «normal crystallizations» ([BDG]), is contained in [Gr].

Extending [M], a closed orientable n-dimensional pseudomanifold N (triangulated by a pseudocomplex K) is said to be a *singular n-manifold* if the disjoined link of each k-simplex, $k>0$, is a sphere and the disjoined link of each vertex is a (closed) connected $(n-1)$-manifold. A vertex of K such that its disjoined link is (resp. is not) an $(n-1)$-sphere is said to be *regular* (resp. *singular*).

Every singular n-manifold can be obtained by capping off each boundary component of an n-manifold by a cone. In the other sense, if K is a pseudocomplex triangulating a singular n-manifold N and $W \subset S_o(K)$, let $M(K, W)$ denote the space obtained by removing from the barycentric subdivision K' of K the open stars in K' of the vertices belonging to W; then W contains all singular vertices of K if and only if $M(K, W)$ is an n-manifold whose boundary components are $Lk(v, K')$, with $v \in W$.

Note that, in dimension three, the pseudocomplex $K(\Gamma)$ associated to an arbitrary (bipartite) 4-coloured graph (Γ, γ) always triangulates a singular 3-manifold.

Proposition 1. *Let (Γ, γ) be a 4-coloured bipartite graph such that all* c-*labelled vertices of* $K(\Gamma)$ *are regular, for every* $c \in \Delta_2$. *If* W *denotes the set of all 3-labelled vertices of* $K(\Gamma)$ *and* $({}^3\vec{\Gamma}, {}^3\vec{\gamma})$ *is the 2-dimensional oriented structure such that* $(B({}^3\vec{\Gamma}), \beta) = (\Gamma, \gamma)$, *then* $K({}^3\vec{\Gamma})$ *is a spine of the 3-manifold* $M(K(\Gamma), W)$.

Proof. It directly follows from the bijoin construction that $K({}^3\vec{\Gamma})$ is the subcomplex of $K(\Gamma)$ consisting of all simplexes of $K(\Gamma)$ whose vertices are labelled by colours different from 3. Thus, for a sufficiently small $\epsilon > o$, the ϵ-neighbourhood $\mathfrak{N}_\epsilon$ of $K({}^3\vec{\Gamma})$ in $K(\Gamma)$ is an ϵ-neighbourhood of $K({}^3\vec{\Gamma})$ in $M(K(\Gamma), W)$ too. For the collapsing criterion for regular neighbourhoods ([RS], corollary 3.30), the polyhedron $|\mathfrak{N}_\epsilon| = M(K(\Gamma), W)$ collapses on $|K({}^3\vec{\Gamma})|$. ∎

Corollary 2. *If (Γ, γ) is a 4-coloured bipartite graph representing a (closed, orientable) 3-manifold M such that $K(\Gamma)$ has a unique 3-coloured vertex (in particular, if (Γ, γ) is a crystallization of M), then $|K(^3\bar{\Gamma})|$ is a spine of M.*

4. NEUWIRTH ALGORITHM VIA BIJOINS

Set $N_k = \{1, 2, ..., k\}$.

If $\Phi = \{x_1, ..., x_g \mid r_1, ..., r_s\}$ is a group presentation, denote by $\lambda(x_i)$, $i \in N_g$, the number of occurrences of the generator x_i in the relators of Φ and by $\lambda(r_j)$, $j \in N_s$, the length of each relator r_j; the length λ of Φ is defined by $\lambda = \sum_{j \in N_s} \lambda(r_j) = \sum_{i \in N_g} \lambda(x_i)$. For each relator r_j, take a 2-cell β_j and triangulate its boundary by «reading» the relator r_j. Thus, we obtain a complex H_j triangulating $\partial\beta_j$ with $\lambda(r_j)$ edges, each of which is labelled by a generator and has a suitable orientation. Label each vertex of H_j by the colour 0, take the barycentric subdivision H_j' of H_j and label all the barycenters by the colour 1. Note that each oriented x_i-labelled edge α of H_j splits into an ordered pair (α^-, α^+) of oriented x_i-labelled edges in H_j': more precisely, if b_α, u_α, v_α respectively denote the barycenter, the first and the second endpoint of α, the ordered pair (u_α, b_α) (resp. (b_α, v_α)) represents the endpoints of the oriented edge α^- (resp. α^+). By starring β_j from an inner point C_j (labelled by the colour 2) over H_j', we obtain a pseudocomplex K_j triangulating β_j with a coloration on its vertex set by the colours 0, 1, 2 (fig. 1). Now, take the disjoint union $\coprod_{j \in N_s} K_j$ and identify the oriented edges α^- (resp. α^+) of its boundary labelled by the same generator so that identified vertices have the same colour. Let $\tilde{K}_\Phi$ be the resulting representable 2-pseudocomplex and let (Γ, γ) be its associated crystallized structure.

The 0-adjacence (resp. 1-adjacence) in (Γ, γ) induces a fixed-point-free involution B (resp. A) on the set $V(\Gamma)$ and the set of the vertices belonging to the same $\{2\}$-residue of (Γ, γ) can be thought of as an orbit of a suitable permutation C on $V(\Gamma)$. These permutations are the homonimous ones associated to Φ in [N]. The assignment of such a permutation C gives rise to a particular 2-dimensional oriented structure $(\bar{\Gamma}_C, \bar{\gamma}_C)$ associated to the crystallized structure (Γ, γ); in fact, C induces a cyclic ordering in the vertices of each $\{2\}$-residue of (Γ, γ) which are the only $\{c\}$-residues of order possibly greater than two. Thus, the geometrical meaning of the choice of a particular C is to give an ordering to the 2-simplexes of $\tilde{K}_\Phi = K(\Gamma)$ containing the same 1-simplex.

We always assume this ordering system with the property that the two cyclic orderings on the $\lambda(x_i)$ 2-simplexes containing the two distinct

x_i-labelled edges of $\tilde{K}_\Phi$ are opposite; this is equivalent to require the property $BC=C^{-1}B$ for the permutation C. Let $\Omega(\Phi)$ denote the set of all permutations C on $V(\Gamma)$ whose orbits are the sets of vertices belonging to the same $\{2\}$-residue of (Γ, γ) and such that $BC=C^{-1}B$.

From now on, the symbol $|P_1, \ldots, P_n|$ will denote the orbit number of the group generated by the permutations P_h, $h \in N_n$, acting on the same set.

Note that, for every $C, C' \in \Omega(\Phi)$, $|A, C|=|A, C'|$; in fact, the number $|A, C|$ only depends upon A and the orbits of C.

The cellular structure of the pseudocomplex $\tilde{K}_\Phi$ immediately shows that $|K_\Phi|$ is the quotient of $|\tilde{K}_\Phi|$ obtained by identifying the 0-labelled vertices of $\tilde{K}_\Phi$. Moreover, the number of these vertices, i.e. the number of the $\{1,2\}$-residues in (Γ, λ), is $|A, C|$. As a consequence, we have:

Proposition 3. *The pseudocomplex $\tilde{K}_\Phi$ is a (pseudosimplicial) triangulation of the standard complex K_Φ if and only if $|A, C|=1$.*

□

Remark. Given a group presentation Φ, the number of connected components in the associated P-graph P_Φ is $|A, C|$ ([N]); moreover, it is easy to verify that every 3-manifold admits a standard spine K_Φ such that the associated P-graph P_Φ is connected.

Thus, there is no loss of generality in *restricting our study to those group presentations for which $|A, C|=1$ and in supposing that $\tilde{K}_\Phi$ triangulates K_Φ.*

□

With the above notations and assumptions, let C be a given permutation in $\Omega(\Phi)$ and let $(\vec{\Gamma}_C, \vec{\gamma}_C)$ be the 2-dimensional oriented structure associated to (Γ, γ) and generated by C.

Proposition 4. *Let (Γ_C, γ_C) be the h-bijoin over $(\vec{\Gamma}_C, \vec{\gamma}_C)$ and let W be the set of all 3-labelled vertices in $K(\Gamma_C)$.*

(a) $h=|AC, BC|$;

(b) *the space $M(K(\Gamma_C), W)$ is a 3-manifold, having K_Φ as a standard spine, if and only if $|AC|=\lambda-2g+2$;*

(c) *if (b) holds, the Euler characteristic of $K(\Gamma_C)$ is $g-s+h-1$;*

(d) *if $g=s$ (resp. $g=s+1$) and (b) holds, then $|K(\Gamma_C)|$ is a closed 3-manifold (resp. $M(K(\Gamma_C), W)$ is the exterior of a knot), having K_Φ as a standard spine, if and only if $|AC, BC|=h=1$.*

Proof. If $\mathcal{F} \subset \Delta_2$ (resp. $\mathcal{F} \subset \Delta_3$), the symbol $g_{\mathcal{F}}$ (resp. $g'_{\mathcal{F}}$) will denote the number of $\mathcal{F}$-residues in (Γ, γ) (resp. in (Γ_C, γ_C)).

Since the number of 2-simplexes in each K_j is $2\lambda(r_j)$ and $\sum_{j\in N_s} \lambda(r_j) = \lambda$, then

$$\mathrm{Card}(V(\Gamma_C)) = 2\cdot \mathrm{Card}(V(\Gamma)) = 4\lambda. \qquad [1]$$

Each $\{c\}$-residue ($c\in\Delta_1$) in (Γ,γ) is a complete graph of order two and $g_{\{o\}} = g_{\{1\}} = \lambda$; in fact, in each K_j there are exactly $\lambda(r_j)$ edges whose endpoints are labelled by the colours 2 and c and they are faces of exactly two 2-simplexes.

Hence:

$$g'_{\{0,3\}} = g_{\{0\}} = |B| = \lambda$$
$$g'_{\{1,3\}} = g_{\{1\}} = |A| = \lambda. \qquad [2]$$

For every $i\in N_g$, there are exactly two $\{2\}$-residues in (Γ,γ) which are complete graphs of order $\lambda(x_i)$. In fact, there are exactly two x_i-labelled edges in $\tilde{K}_\Phi$ and the number of 2-simplexes of which each x_i-labelled edge is a face is the number $\lambda(x_i)$ of occurrences of the generator x_i in the relators of Φ. Hence:

$$g'_{\{2,3\}} = g_{\{2\}} = |C| = 2g. \qquad [3]$$

Recall that an alternating path in an oriented graph $\vec{\Gamma}$ is a path whose adjacent edges have opposite orientations. In an oriented structure $(\vec{\Gamma}, \vec{\gamma})$, for every pair h, k of distinct colours, a weak $\{h, k\}$-cycle ([BG]) is an alternating cycle of $(\vec{\Gamma}, \vec{\gamma})$ whose edges are alternatively coloured by h and k. If $\overleftrightarrow{g}_{hk}$ denotes the number of weak $\{h, k\}$-cycles of $(\vec{\Gamma}_C, \vec{\gamma}_C)$, we have $g'_{\{h,k\}} = \overleftrightarrow{g}_{hk}$, for each $h, k\in\Delta_2$.

The number of the $\{\hat{3}\}$-residues in (Γ_C, γ_C) is the number of the orbits in the permutation group generated by AB, BC and AC. Since $AB = (AC)(BC)^{-1}$, we have:

$$h = g'_{\hat{3}} = |AB,\ BC,\ AC| = |AC,\ BC| \qquad [4]$$

and this proves (a).

If $P(c)$, $c\in\Delta_2$, denotes the permutation on $V(\vec{\Gamma}_C) = V(\Gamma)$ induced by the c-adjacence, it is easy to see that $\overleftrightarrow{g}_{hk} = |P(h)P(k)^{-1}| = |P(h)^{-1}P(k)| = \overleftrightarrow{g}_{kh}$, for each pair of distinct colours $h, k\in\Delta_2$. Thus, the following equalities hold:

$$g'_{\{0,1\}} = \overleftrightarrow{g}_{01} = |AB^{-1}| = |AB| = 2s$$
$$g'_{\{0,2\}} = \overleftrightarrow{g}_{02} = |B^{-1}C| = |BC| = \lambda \qquad [5]$$
$$g'_{\{1,2\}} = \overleftrightarrow{g}_{12} = |A^{-1}C| = |AC|.$$

Note that, $|BC| = \lambda$ since $BC = C^{-1}B$ and hence $(BC)^2 = 1$.

For each $j \in N_s$, there is one $\{0,1\}$-residue in (Γ, γ) which is a bicoloured cycle of length $2\lambda(r_j)$; in fact, the $\{0,1\}$-residues in (Γ, γ) are in one-to-one correspondence with the inner vertices C_j of β_j. Hence:

$$g'_{\hat{2}} = g_{\hat{2}} = |A, B| = s. \qquad [6]$$

For each $i \in N_g$, there is one $\{0,2\}$-residue in (Γ, γ) with $2\lambda(x_i)$ vertices, in fact, the $\{0,2\}$-residues in (Γ, γ) are in one-to-one correspondence with the barycenters in ∂K_i. Hence:

$$g'_{\hat{1}} = g_{\hat{1}} = |B, C| = g. \qquad [7]$$

Finally, since, as pointed out for the proof of Proposition 3, the 0-labelled vertices in $\tilde{K}_\Phi$ are in one-to-one correspondence with the $\{1,2\}$-residues in (Γ, γ), the assumption $|A, C| = 1$ gives:

$$g'_{\hat{0}} = g_{\hat{0}} = |A, C| = 1. \qquad [8]$$

Let us now compute the Euler characteristic $\chi(K_{\hat{d}})$ of the pseudocomplex $K_{\hat{d}} = K((\Gamma_C)_{\hat{d}})$, for each $d \in \Delta_2$, by making use of the equalities (1)-(8) and by recalling that the number of 2-simplexes (resp. 1-simplexes) in $K_{\hat{d}}$ is $\mathrm{Card}(V(\Gamma_C)) = 4\lambda$ (resp. $3\,\mathrm{Card}(V(\Gamma_C))/2 = 6\lambda$).

$$\chi(K_{\hat{0}}) = 4\lambda - 6\lambda + (g'_{\{1,2\}} + g'_{\{1,3\}} + g'_{\{2,3\}}) = -2\lambda + (|AC| + \lambda + 2g) = |AC| + 2g - \lambda;$$

$$\chi(K_{\hat{1}}) = 4\lambda - 6\lambda + (g'_{\{0,2\}} + g'_{\{0,3\}} + g'_{\{2,3\}}) = -2\lambda + (\lambda + \lambda + 2g) = 2g;$$

$$\chi(K_{\hat{2}}) = 4\lambda - 6\lambda + (g'_{\{0,3\}} + g'_{\{1,3\}} + g'_{\{0,1\}}) = -2\lambda + (\lambda + \lambda + 2s) = 2s.$$

As pointed out in section 2, each $\{\hat{d}\}$-residue in the 4-coloured graph (Γ_C, γ_C) represents the disjoined link of the represented d-labelled vertex in $K(\Gamma_C)$. Since the equality [6] (resp. [7])) states that the number of $\{\hat{2}\}$-residues (resp. $\{\hat{1}\}$-residues) in (Γ_C, γ_C) is s (resp. g), the equality $\chi(K_{\hat{2}}) = 2s$ (resp. $\chi(K_{\hat{1}}) = 2g$) proves that the disjoined link of each 2-labelled (resp. 1-labelled) vertex in $K(\Gamma_C)$ is a 2-sphere. Hence all 1-labelled and 2-labelled vertices of $K(\Gamma_C)$ are regular. Moreover, the disjoined link $K_{\hat{0}}$ of the unique 0-labelled vertex of $K(\Gamma_C)$ is a 2-sphere if and only if $\chi(K_{\hat{0}}) = |AC| + 2g - \lambda = 2$, that is if and only if $|AC| = \lambda - 2g + 2$. This result, together with Proposition 1, proves (b).

The Euler characteristic computation of $K(\Gamma_C)$ gives:

$$\chi(K(\Gamma_C)) = (g'_{\hat{0}} + g'_{\hat{1}} + g'_{\hat{2}} + g'_{\hat{3}}) - (g'_{\{0,1\}} + g'_{\{0,2\}} + g'_{\{0,3\}} + g'_{\{1,2\}} + g'_{\{1,3\}} + g'_{\{2,3\}}) +$$
$$+ \mathrm{Card}(E(\Gamma_C)) - \mathrm{Card}(V(\Gamma_C)) =$$
$$= (1 + g + s + h) - (2s + \lambda + \lambda + |AC| + \lambda + 2g) + 8\lambda - 4\lambda = \lambda + h + 1 - s - g - |AC|.$$

Thus, if (b) holds, $\chi(K(\Gamma_C)) = g - s + h - 1$.

Finally, if $g=s$ (resp. $g=s+1$) and (b) holds, $\chi(K(\Gamma_C))=h-1$ (resp. $\chi(K(\Gamma_C)=h)$ and hence $|K(\Gamma_C)|$ is a closed 3-manifold (resp. $M(K(\Gamma_C), W)$ is the exterior of a knot) if and only if $h=1$; proposition 1, corollary 2 and equality [4] complete the proof of (d).

□

Proposition 5. *Let M be a 3-manifold having K_Φ as a standard spine. There exists a permutation $C \in \Omega(\Phi)$ such that $M = M(K(\Gamma_C), W)$.*

Proof. If α is a 1-simplex of $\tilde{K}_\Phi$, the imbedding of its star st $(\alpha, \tilde{K}_\Phi)$ in the (arbitrarily oriented) 3-manifold M induces a cyclic ordering of the 2-simplexes of $\tilde{K}_\Phi$ containing α. Thus, a permutation C on $V(\Gamma)$ or, equivalently, an oriented structure $(\vec{\Gamma}_C, \vec{\gamma}_C)$ can be associated to the crystallized structure (Γ, γ) representing $\tilde{K}_\Phi$.

Note that the imbedding of $\tilde{K}_\Phi$ in M directly gives $BC=C^{-1}B$ and hence $C \in \Omega(\Phi)$. Let (Γ_C, γ_C) be the h-bijoin over $(\vec{\Gamma}_C, \vec{\gamma}_C)$; note that the choice of the opposite orientation in M gives rise to the opposite oriented structure but to the same graph (Γ_C, γ_C), as pointed out in section 3. If $\hat{M}$ denotes the singular 3-manifold obtained by capping off each boundary component of M by a cone, them $\hat{M} = |K(\Gamma_C)|$ and hence $M = M(K(\Gamma_C), W)$, W being the set of all 3-labelled vertices in $K(\Gamma_C)$.

□

If $\Omega'(\Phi)$ denotes the subset of $\Omega(\Phi)$ consisting of all $C \in \Omega(\Phi)$ such that $|AC| = \lambda - 2g + 2$, then proposition 4 and proposition 5 lead to the following result:

Corollary 6. *The complex K_Φ is a standard spine of a 3-manifold M if and only if there exists a permutation $C \in \Omega'(\Phi)$ such that $M = M(K(\Gamma_C), W)$.*

□

The above result directly produces an effective algorithm for testing the geometricity of a group presentation, extending Neuwirth's one to non-balanced presentations.

Example. Let $\Phi = \langle x, y \mid x^3 y^2 \rangle$. In this case, $g=2$, $s=1$ and, with the notations of fig. 1, the permutations A, B can be written in the following way:

$$A = (\bar{1}\,2)\,(\bar{2}\,3)\,(\bar{3}\,4)\,(\bar{4}\,5)\,(\bar{5}\,1),\quad B = (1\,\bar{1})\,(2\,\bar{2})\,(3\,\bar{3})\,(4\,\bar{4})\,(5\,\bar{5}).$$

Moreover, the orbits of the permutation C are $\{1,2,3\}$, $\{\bar{1},\bar{2},\bar{3}\}$, $\{4,5\}$, $\{\bar{4},\bar{5}\}$.

The choice of $C=(1\,2\,3)\,(\bar{3}\,\bar{2}\,\bar{1})\,(4\,5)\,(\bar{5}\,\bar{4})\in\Omega'(\Phi)$ produces the 4-coloured graph (Γ_C, γ_C) drawn in fig. 1 and $M(K(\Gamma_C), W)$ is the exterior of the trefoil knot.

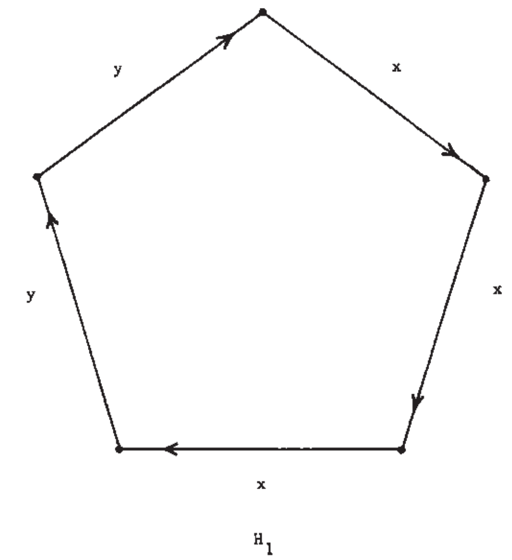


Figure 1a.

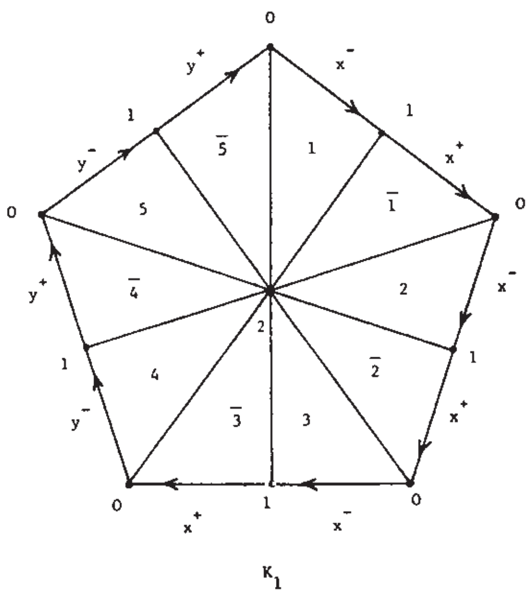
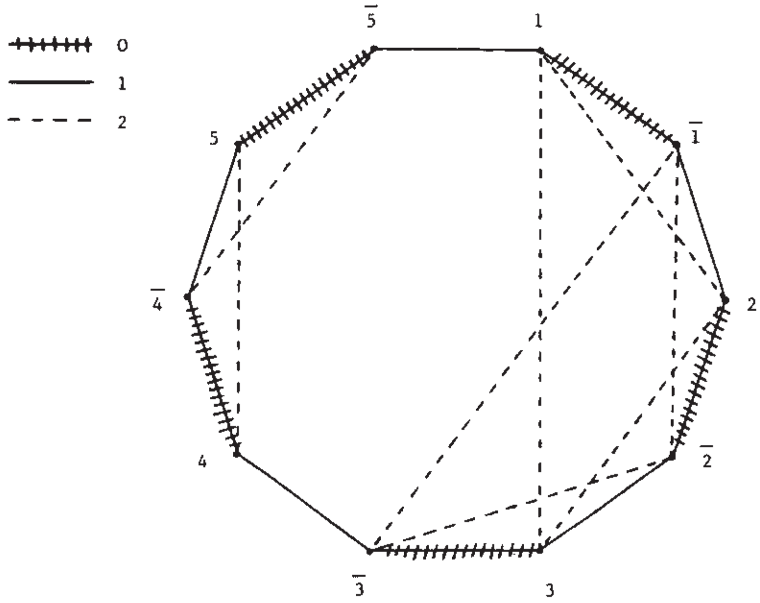
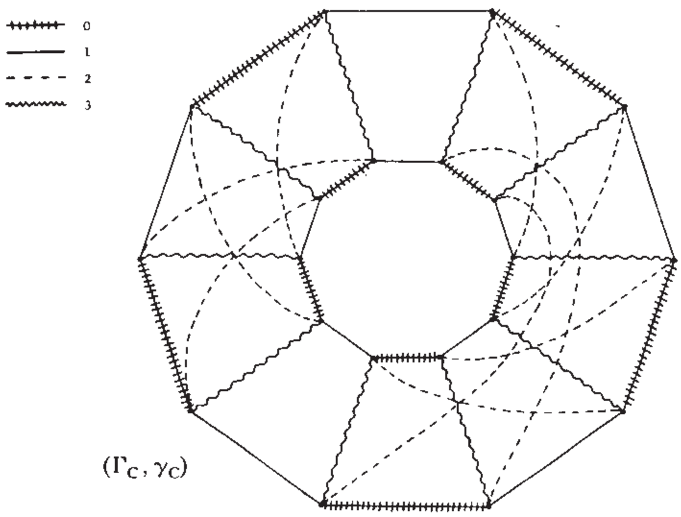


Figure 1b.



(Γ, γ)

Figure Ic.



(Γ_C, γ_C)

Figure Id.

5. MONTESINOS ALGORITHM VIA BIJOINS

We sketch Montesinos algorithm described in [M].

With the notations of the previous section, let Φ be a given group presentation whose associated P-graph P_Φ is connected; make Φ positive and call the new presentation Φ again.

Take the permutation $\tau=(1,2,...,\lambda(r_1))\cdot(\lambda(r_1)+1,...,\lambda(r_1)+\lambda(r_2))\cdots(...,\lambda)$ on N_λ and the set of all permutations σ on N_λ whose orbits $d_1,...,d_g$ are defined as follows: the number $j\in N_\lambda$ belongs to d_i if and only if there is a relator r_k whose $(j-\lambda(r_{k-1}))$ – th letter is x_i. Let $\Sigma(\Phi)$ denote the subset of all such σ satisfying $|\sigma,\tau\sigma\tau^{-1}|=1$ and $|[\sigma,\tau]|=\lambda-2g+2$.

If $\Sigma(\Phi)\neq\varnothing$, then, for each $\sigma\in\Sigma(\Phi)$, construct the singular 3-manifold $N(\sigma,\tau)$ by taking λ copies $\{t_1,...,t_\lambda\}$ of the standard tetrahedron t whose bidimensional faces are denoted by S, $\tilde{S}$, T, $\tilde{T}$. Label the faces S, $\tilde{S}$, T, $\tilde{T}$ in the copy t_i as $S_{i\sigma(i)}$, $\tilde{S}_{i\sigma^{-1}(i)}$, $T_{i\tau(i)}$, $\tilde{T}_{i\tau^{-1}(i)}$ respectively; identify S_{ij} with $\tilde{S}_{ji}$ and T_{ij} with $\tilde{T}_{ji}$ by an orientation-reversing linear homeomorphism respecting the edges $S\cap\tilde{S}$ and $T\cap\tilde{T}$.

If W denotes the set of all singular vertices of $N(\sigma,\tau)$, then $\{M(N(\sigma,\tau),W)\mid\sigma\in\Sigma(\Phi)\}$ is the set of all 3-manifolds M^3 admitting a Heegaard diagram whose associated presentation for $\Pi_1(M^3)$ is Φ.

Since P_Φ is connected, the representable 2-pseudocomplex $\tilde{K}_\Phi$ triangulates K_Φ; now, it is possible to label the vertices of the crystallized structure (Γ,γ) associated to $\tilde{K}_\Phi$ by the set $\bar{N}_\lambda=\{1,2,...,\lambda,\bar{1},\bar{2},...,\bar{\lambda}\}$ so that:

$$A=(\bar{1}2)\cdot(\bar{2}3)\ldots(\overline{\lambda(r_1)-1}\ \lambda(r_1))\cdot(\overline{\lambda(r_1)}\ 1)\cdot(\overline{\lambda(r_1)+1}\ \lambda(r_1)+2)\ldots$$
$$(\overline{\lambda(r_1)+\lambda(r_2)}\ \lambda(r_1)+1)\cdot$$
$$\cdots(\bar{\lambda}\ \lambda(r_1)+\ldots+\lambda(r_{\lambda-1})+1)\qquad\text{and}\qquad B=\prod_{h\in N_\lambda}(h\bar{h}).$$

Moreover, if C is a permutation on $\bar{N}_\lambda$ satisfying the following properties:

- j (resp. $\bar{j}$) belongs to the orbit d_i (resp. $\bar{d}_i$) of C if and only if there is a relator r_k whose $(j-\lambda(r_{k-1}))$ – th letter is x_i,
- the ordering of the elements j in d_i is opposite to the ordering of the elements $\bar{j}$ in $\bar{d}_i$,

then $C\in\Omega(\Phi)$.

Thus, the choice of σ induces an associated C_σ (and hence an oriented structure $(\vec{\Gamma}_\sigma,\vec{\gamma}_\sigma)=(\vec{\Gamma}_{C_\sigma},\vec{\gamma}_{C_\sigma})$) in a standard way and viceversa.

Proposition 7. *The singular 3-manifold $N(\sigma, \tau)$ is pl-homeomorphic with $|K(\Gamma_\sigma)|$, $(\Gamma_\sigma, \gamma_\sigma)$ being the h-bijoin over $(\bar{\Gamma}_\sigma, \bar{\gamma}_\sigma)$.*

Proof. If t is the standard tetrahedron, subdivide it into four tetrahedra in the following way. If V_S (resp. V_T) is the barycenter of $S \cap \tilde{S}$ (resp. $T \cap \tilde{T}$), join V_S with V_T by an edge whose interior is contained in the interior of t and subdivide S, $\tilde{S}$ (resp. T, $\tilde{T}$) by joining V_S (resp. V_T) with the endpoints of $T \cap \tilde{T}$ (resp. $S \cap \tilde{S}$). Label V_S by 1, V_T by 2 and the endpoints of $S \cap \tilde{S}$ (resp. $T \cap \tilde{T}$) by 0 (resp. 3) (fig. 2).

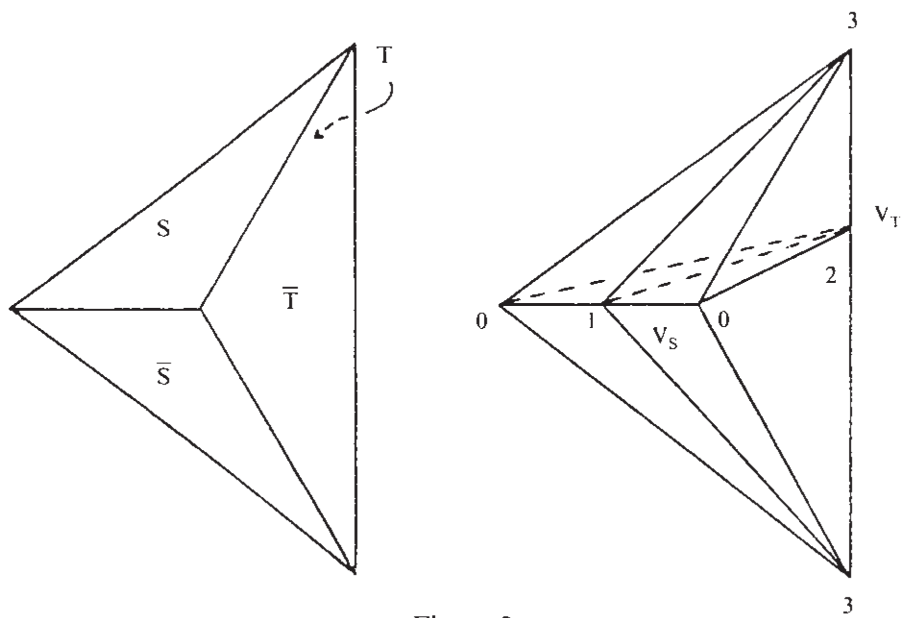


Figure 2

In this way, $N(\sigma, \tau)$ is triangulated by a representable 4-pseudocomplex K' in which each t_i splits into four tetrahedra.

If (Γ', γ') is the 4-coloured graph representing K', it is straightforward that the oriented structure $({}^3\bar{\Gamma}', {}^3\bar{\gamma}')$ is isomorphic with $(\bar{\Gamma}_\sigma, \bar{\gamma}_\sigma)$ and hence $N(\sigma, \tau) = |K(\Gamma_\sigma)|$.

□

Since $|[\sigma, \tau]| = g'_{\{1,2\}} = |AC|$, $|\sigma| = g$, $|\tau| = s$, $|\sigma, \tau\sigma\tau^{-1}| = g'_{\hat{0}} = |A, C| = 1$, $|\tau, \sigma\tau\sigma^{-1}| = g'_{\hat{3}} = h = |AC, BC|$, all results in [M] can be restated in terms of spines or in terms of bijoins and edge-coloured graphs.

It appears as evident that the graph-theoretical bijoin construction is the idea which unifies both Neuwirth and Montesinos algorithm.

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