3-Manifold Spines and Bijoins

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ABSTRACT. We describe a combinatorial algorithm for constructing all orientable 3-manifolds with a given standard bidimensional spine by making use of the idea of bijoin ([BG], [Gr]) over a suitable pseudosimplicial triangulation of the spine.

1. INTRODUCTION

Throughout this paper, all spaces and maps are piecewise-linear (pl) in the sense of [GI] or [RS]; all 3-manifolds are supposed to be compact, connected and orientable.

If $M$ is a 3-manifold with non-empty boundary, then a bidimensional polyhedron $K$ such that $M$ collapses to $K$ is said to be a spine of $M$; if $M$ is closed, a spine of $M$ is a spine of $M-B$, $B$ being an open 3-ball in $M$.

Given a group presentation $\Phi = \{x_1, \ldots, x_r, r_1, \ldots, r_s\}$, denote by $K_\Phi$ the bidimensional complex constructed as follows:

- $K_\Phi$ has only one 0-cell (vertex);
- the 1-cells (resp. the 2-cells) of $K_\Phi$ are in one-to-one correspondence with the generators (resp. the relators) of $\Phi$; denote them by $\alpha_i$ (resp. $\beta_i$);
- each 2-cell $\beta_i$ is attached to the 1-skeleton by the formula given by the corresponding relator $r_i$.

$K_\Phi$ is said to be the standard complex associated to $\Phi$; of course, the factor group of $\Phi$ is $\Pi_1(\{K_\Phi\})$. We will not distinguish between a relator $r_i$ and any

__Work performed under the auspices of the G.N.S.A.G.A. of the C.N.R. (National Research Council of Italy) and within the project "Geometria delle Varieta' Differenziali" of the M.P.I. (Italy).__

cyclic conjugate of it or its inverse, since the associated complexes are the same. The above construction may be obviously reversed and each standard complex $K$ induces a group presentation $\Phi_K$ of the fundamental group $\Pi_1(\partial K)$.

It is well known that every 3-manifold $M$ has a standard spine $K_0$, for some group presentation $\Phi$, and the factor group of $\Phi$ is clearly $\Pi_1(\partial M)$; nevertheless, not every standard complex $K_0$ is a spine of a 3-manifold. Every group presentation $\Phi$ such that $K_0$ is a spine of a 3-manifold (resp. of a closed 3-manifold) is said to be geometric (resp. strongly geometric).

In [N] Neuwirth gives an algorithm for testing if a balanced group presentation (same number of generators and relations) is strongly geometric. The same algorithm is restated by Osborne and Stevens ([OS], [S]) by making use of a graph-theoretical tool, the presentation-graph or $P$-graph $P_\Phi$, which can be associated, by a one-to-one correspondence, to the standard complex $K_\Phi$ of a group presentation $\Phi$. Namely, $P_\Phi$ is essentially the boundary of a regular neighbourhood of the unique vertex of $K_\Phi$ and it is easy to prove that $\Phi$ is strongly-geometric if and only if a planar imbedding condition on $P_\Phi$ holds. Moreover, as pointed out by Montesinos in [M], a Heegaard diagram of a 3-manifold $M$ gives rise to such a planar imbedding of the $P$-graph $P_\Phi$ associated to a suitable group presentation $\Phi$ of $\Pi_1(\partial M)$; in fact, this is nothing else than the Whitehead graph of the group presentation $\Phi$ of $\Pi_1(\partial M)$ coming from the given Heegaard diagram of $M$. Thus, a group presentation $\Phi$ is geometric (resp. strongly-geometric) if and only if there exists a Heegaard diagram of a 3-manifold (resp. closed 3-manifold) $M$ whose associated presentation for $\Pi_1(\partial M)$ is $\Phi$.

In [M] Montesinos describes an algorithm for checking if a given group presentation is geometric; such an algorithm seems to be completely different from Neuwirth’s one, since it makes use of branched covering techniques.

In the present paper, we give a combinatorial algorithm for obtaining all 3-manifolds with a given standard spine $K_0$ by making use of the bijoin construction ([BG], [Gr]) applied to a graph-theoretical structure representing a pseudosimplicial triangulation of $K_0$. This construction allows us to unify, in a common geometric description, both Neuwirth algorithm and Montesinos one; namely, the necessary and sufficient conditions for the geometricity (or the strong-geometricity) of a group presentation obtained in [N], [OS], [OS2], [S], [M] can be all derived from the bijoin construction.

2. EDGE-COLOURED GRAPHS AND ASSOCIATED COMPLEXES

The term pseudograph includes loops and multiple edges, while a multigraph (or simply a graph) allows multiple edges only.
A (generalized) coloration on a pseudograph \( \Gamma = (V(\Gamma), E(\Gamma)) \) is a map \( \gamma: E(\Gamma) \to \Delta_n = \{0, 1, \ldots, n\} \); if \( \Gamma \) is a graph, \( \gamma \) is said to be proper if \( \gamma(e) \neq \gamma(f) \), for each pair \( e, f \) of adjacent edges. For each \( \gamma \subset \Delta_n \), set \( \Gamma_\gamma = (V(\Gamma), \gamma^{-1}(\Delta_n)) \); each connected component of \( \Gamma_\gamma \) is often called an \( \gamma \)-residue. For each \( i \in \Delta_n \), set \( i = \Delta_n - \{i\} \).

The pair \((\Gamma, \gamma)\), \( \Gamma \) being a graph and \( \gamma: E(\Gamma) \to \Delta_n \) a (generalized) coloration, is said to be an \( n \)-dimensional crystallized structure ([G]) if, for each \( i \in \Delta_n \), the \( i \)-residues are cliques (complete graphs). If all these cliques are of order two, i.e. if \( \gamma \) is proper and \( \Gamma \) is regular of degree \( n + 1 \), \((\Gamma, \gamma)\) is simply called an \((n + 1)\)-coloured graph ([F]).

An \( n \)-dimensional pseudocomplex \( K \) is an \( n \)-dimensional ball complex in which every \( h \)-ball, considered with all its faces, is isomorphic with the complex underlying an \( h \)-simplex; for this reason, each \( h \)-ball of \( K \) is called \( h \)-simplex. The disjoined star \( \Std(s, K) \) of a simplex \( s \) in \( K \) is defined to be the disjoint union of the \( n \)-simplexes of \( K \) containing \( s \), with reidentification of the \((n-1)\)-faces containing \( s \) and of their faces; the subcomplex \( \Lk(s, K) = \{ \tau \in \Std(s, K) \mid \tau \cap s = \emptyset \} \) is called the disjoined link of \( s \) in \( K \).

As shown in [G] and [F], every \( n \)-dimensional crystallized structure \((\Gamma, \gamma)\) represents a homogeneous \( n \)-dimensional pseudocomplex \( K(\Gamma) \) constructed by the following rules:

- take an \( n \)-simplex \( \sigma(v) \) for each \( v \in V(\Gamma) \) and label its vertices by \( \Delta_n \);
- if \( v, w \in V(\Gamma) \) are joined by an \( i \)-coloured edge, identify the \((n-1)\)-faces of \( \sigma(v) \) and \( \sigma(w) \) opposite to the vertices labelled by \( i \), so that equally labelled vertices are identified together.

Every \( h \)-simplex \( s \) of \( K(\Gamma) \), whose vertices are labelled by the distinct colours \( c_1, \ldots, c_h \in \Delta_n \), corresponds to a unique \((\Delta_n - \{c_1, \ldots, c_h\})\)-residue \( \mathcal{R} \) of \((\Gamma, \gamma)\) and vice versa; its associated pseudocomplex \( K(\mathcal{R}) \) is \( \Lk(s, K(\Gamma)) \).

Moreover, \((\Gamma, \gamma)\) is an \((n+1)\)-coloured graph if and only if \(|K(\Gamma)|\) is a closed pseudomanifold ([ST]), which is orientable if and only if \( \Gamma \) is bipartite ([CGP]).

The construction of \( K(\Gamma) \) gives a coloration on the vertex set \( S_n(K) \) of \( K(\Gamma) \) by means of \( n + 1 \) colours (i.e. a map \( \xi: S_n(K) \to \Delta_n \) which is injective on each simplex of \( K(\Gamma) \)). Given a homogeneous \( n \)-dimensional pseudocomplex \( K \) with such a coloration on its vertex set \( S_n(K) \), the construction can be easily reversed yielding an \( n \)-dimensional crystallized structure, denoted by \( \Gamma(K) \).

It is easy to see that \( \Gamma(K(\Gamma)) = (\Gamma, \gamma) \); moreover (see [G]), \( K(\Gamma(K)) = K \) if and only if \( K \) satisfies the following property:

(*) the disjoined star \( \Std(s, K) \) of every simplex \( s \) of \( K \) is strongly-connected.
A homogeneous $n$-dimensional pseudocomplex satisfying (*) and admitting a coloration on its vertex set by means of $n+1$ colours is said to be a **representable $n$-pseudocomplex**, since it is uniquely represented by an $n$-dimensional crystallized structure.

An $n$-dimensional crystallized structure $(\Gamma, \gamma)$ (or its associated pseudocomplex $K(\Gamma)$) is said to be **contracted** if $\Gamma$ is connected, for each $e \in \Delta_n$ (i.e. if $K(\Gamma)$ has exactly $n+1$ vertices). A contracted $(n+1)$-coloured (bipartite) graph $(\Gamma, \gamma)$ is said to be a **crystallization** of a closed (orientable) $n$-manifold $M$ if $|K(\Gamma)| = M$. Every closed $n$-manifold admits a crystallization ([P]).

For a general survey on manifold representation theory by means of edge-coloured graphs, see [FGG], [BM], [V].

### 3. THE BIJOIN CONSTRUCTION

If $\Gamma$ is an oriented pseudograph and $\gamma: E(\Gamma) \to \Delta_n$ is a (generalized) coloration, the pair $(\overline{\Gamma}, \overline{\gamma})$ is called an $n$-dimensional oriented structure ([BG]) if, for every $i \in \Delta_n$, the $\{i\}$-residues are elementary oriented cycles, possibly of length one or two.

By deleting all loops in $E(\overline{\Gamma})$ and by replacing, for every $i \in \Delta_n$, each elementary oriented $i$-coloured cycle in $\Gamma_{\{i\}}$ with a clique on the same vertex set, it is easy to associate an $n$-dimensional crystallized structure $(\Gamma, \gamma)$ to every $n$-dimensional oriented structure $(\overline{\Gamma}, \overline{\gamma})$. Of course, there are, in general, many oriented structures associated to a fixed crystallized structure; they can be easily obtained by reversing the above construction. If $(\overline{\Gamma}, \overline{\gamma})$ is an oriented structure associated to the crystallized structure $(\Gamma, \gamma)$, we set $K(\overline{\Gamma}) = K(\Gamma)$.

The following construction, given in [BG], allows to obtain an $(n+1)$-coloured bipartite graph $(B(\overline{\Gamma}), \beta)$ from an $(n-1)$-dimensional oriented structure $(\overline{\Gamma}, \overline{\gamma})$:

- $V(B(\overline{\Gamma})) = V(\overline{\Gamma}) \times \{0, 1\}$;
- for every vertex $v \in V(\overline{\Gamma})$, join $(v, 0)$ with $(v, 1)$ by an edge $e$ of $B(\overline{\Gamma})$ and set $\beta(e) = \gamma(e)$;
- if $\overline{e} \in E(\overline{\Gamma})$ and $\overline{e}(0) = v$, $\overline{e}(1) = w$, then join $(v, 0)$ with $(w, 1)$ by an edge $e'$ of $B(\overline{\Gamma})$ and set $\beta(e') = \gamma(\overline{e})$.

Note that the choice of the opposite oriented structure, obtained by reversing the orientation of each $\{i\}$-residue, for every $i \in \Delta_{n-1}$, gives rise to the same graph. The construction in an adapting to the edge-coloured graphs of a standard method for associating a bipartite graph to an arbitrarily given oriented graph ([BHM]). The $(n+1)$-coloured graph $(B(\overline{\Gamma}), \beta)$ (and its
associated pseudocomplex) is said to be the $h$-bijoin over $(\Gamma, \gamma)$, $h$ being the number of the $n$-residues in $(B(\Gamma), \beta)$; if $h = 1$, $(B(\Gamma), \beta)$ is simply called bijoin.

Given an $(n+1)$-coloured bipartite graph $(\Gamma, \gamma)$, it is easy to (uniquely) construct, for each $i \in \Delta_n$, an $(n-1)$-dimensional oriented structure $(\Gamma^i, \gamma^i)$ such that $(B(\Gamma^i), \beta^i) = (\Gamma, \gamma)$ ([BG]); thus, every closed $n$-manifold can be obtained as a bijoin over a suitable $(n-1)$-dimensional oriented structure. A refinement of this result, in dimension three, obtained by making use of «normal crystallizations» ([BDG]), is contained in [Gr].

Extending [M], a closed orientable $n$-dimensional pseudomanifold $N$ (triangulated by a pseudocomplex $K$) is said to be a singular $n$-manifold if the disjoined link of each $k$-simplex, $k > 0$, is a sphere and the disjoined link of each vertex is a (closed) connected $(n-1)$-manifold. A vertex of $K$ such that its disjoined link is (resp. is not) an $(n-1)$-sphere is said to be regular (resp. singular).

Every singular $n$-manifold can be obtained by capping off each boundary component of an $n$-manifold by a cone. In the other sense, if $K$ is a pseudocomplex triangulating a singular $n$-manifold $N$ and $W \subseteq S_n(K)$, let $M(K, W)$ denote the space obtained by removing from the barycentric subdivision $K'$ of $K$ the open stars in $K'$ of the vertices belonging to $W$; then $W$ contains all singular vertices of $K$ if and only if $M(K, W)$ is an $n$-manifold whose boundary components are $Lk(v, K')$, with $v \in W$.

Note that, in dimension three, the pseudocomplex $K(\Gamma')$ associated to an arbitrary (bipartite) 4-coloured graph $(\Gamma', \gamma)$ always triangulates a singular 3-manifold.

**Proposition 1.** Let $(\Gamma, \gamma)$ be a 4-coloured bipartite graph such that all $c$-labelled vertices of $K(\Gamma)$ are regular, for every $c \in \Delta_3$. If $W$ denotes the set of all 3-labelled vertices of $K(\Gamma)$ and $(\Gamma', \gamma')$ is the 2-dimensional oriented structure such that $(B(\Gamma'), \beta) = (\Gamma, \gamma)$, then $K(\Gamma')$ is a spine of the 3-manifold $M(K(\Gamma), W)$.

**Proof.** It directly follows from the bijoin construction that $K(\Gamma')$ is the subcomplex of $K(\Gamma)$ consisting of all simplexes of $K(\Gamma)$ whose vertices are labelled by colours different from 3. Thus, for a sufficiently small $\varepsilon > 0$, the $\varepsilon$-neighbourhood $\mathcal{B}_\varepsilon$ of $K(\Gamma)$ in $K(\Gamma')$ is an $\varepsilon$-neighbourhood of $K(\Gamma')$ in $M(K(\Gamma), W)$ too. For the collapsing criterion for regular neighbourhoods ([RS], corollary 3.30), the polyhedron $|\mathcal{B}_\varepsilon| = M(K(\Gamma), W)$ collapses on $|K(\Gamma')|$.  


Corollary 2. If \((\Gamma, \gamma)\) is a 4-coloured bipartite graph representing a (closed, orientable) 3-manifold \(M\) such that \(K(\Gamma)\) has a unique 3-coloured vertex (in particular, if \((\Gamma, \gamma)\) is a crystallization of \(M\)), then \(|K(\Gamma')|\) is a spine of \(M\).

4. NEUWIRTH ALGORITHM VIA BIJOINS

Set \(N_k = \{1, 2, \ldots, k\}\).

If \(\Phi = \{x_1, \ldots, x_r\} | r_1, \ldots, r_g\) is a group presentation, denote by \(\lambda(x_i), i \in N_g\), the number of occurrences of the generator \(x_i\) in the relators of \(\Phi\) and by \(\lambda(r_j), j \in N_r\), the length of each relator \(r_j\); the length \(\lambda\) of \(\Phi\) is defined by \(\lambda = \sum_{i \in N_g} \lambda(x_i) = \sum_{j \in N_r} \lambda(r_j)\). For each relator \(r_j\), take a 2-cell \(\beta_j\) and triangulate its boundary by «reading» the relator \(r_j\). Thus, we obtain a complex \(H_j\) triangulating \(\partial \beta_j\) with \(\lambda(r_j)\) labelled vertices, each of which is labelled by a generator and has a suitable orientation. Label each vertex of \(H_j\) by the colour 0, take the barycentric subdivision \(H'_j\) of \(H_j\) and label all the barycenters by the colour 1. Note that each oriented \(x_i\)-labelled edge \(\alpha\) of \(H_j\) splits into an ordered pair \((\alpha^-, \alpha^+)\) of oriented \(x_i\)-labelled edges in \(H'_j\); more precisely, if \(b_0, u_0, v_0\) respectively denote the barycenter, the first and the second endpoint of \(\alpha\), the ordered pair \((u_0, b_0)\) (resp. \((b_0, v_0)\)) represents the endpoints of the oriented edge \(\alpha^+\) (resp. \(\alpha^-\)). By starring \(\alpha\) from an inner point \(C_\alpha\) (labelled by the colour 2) over \(H'_j\), we obtain a pseudocomplex \(K_\alpha\) triangulating \(\beta_j\) with a coloration on its vertex set by the colours 0, 1, 2 (fig. 1). Now, take the disjoint union \(\bigsqcup_{\alpha \in N_\Phi} K_\alpha\) and identify the oriented edges \(\alpha^-\) (resp. \(\alpha^+\)) of its boundary labelled by the same generator so that identified vertices have the same colour. Let \(\bar{K}_\Phi\) be the resulting representable 2-pseudocomplex and let \((\Gamma', \gamma)\) be its associated crystallized structure.

The 0-adjacency (resp. 1-adjacency) in \((\Gamma', \gamma)\) induces a fixed-point-free involution \(B\) (resp. \(A\)) on the set \(V(\Gamma')\) and the set of the vertices belonging to the same \([2]\)-residue of \((\Gamma', \gamma)\) can be thought of as an orbit of a suitable permutation \(C\) on \(V(\Gamma')\). These permutations are the homonimous ones associated to \(\Phi\) in \([N]\). The assignment of such a permutation \(C\) gives rise to a particular 2-dimensional oriented structure \((\Gamma_C, \gamma_C)\) associated to the crystallized structure \((\Gamma', \gamma)\); in fact, \(C\) induces a cyclic ordering in the vertices of each \([2]\)-residue of \((\Gamma', \gamma)\) which are the only \([c]\)-residues of order possibly greater than two. Thus, the geometrical meaning of the choice of a particular \(C\) is to give an ordering to the 2-simplexes of \(\bar{K}_\Phi = K(\Gamma')\) containing the same 1-simplex.

We always assume this ordering system with the property that the two cyclic orderings on the \(\lambda(x_i)\) 2-simplexes containing the two distinct
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$x_r$-labelled edges of $\bar{K}_\Phi$ are opposite; this is equivalent to require the property $BC = C^{-1}B$ for the permutation $C$. Let $\Omega(\Phi)$ denote the set of all permutations $C$ on $V(\Gamma)$ whose orbits are the sets of vertices belonging to the same $[2]$-residue of $(\Gamma, \gamma)$ and such that $BC = C^{-1}B$.

From now on, the symbol $|P_1, \ldots, P_m|$ will denote the orbit number of the group generated by the permutations $P_h$, $h \in N_m$, acting on the same set.

Note that, for every $C, C' \in \Omega(\Phi)$, $|A, C| = |A, C'|$; in fact, the number $|A, C|$ only depends upon $A$ and the orbits of $C$.

The cellular structure of the pseudocomplex $\bar{K}_\Phi$ immediately shows that $|K_\Phi|$ is the quotient of $|\bar{K}_\Phi|$ obtained by identifying the 0-labelled vertices of $\bar{K}_\Phi$. Moreover, the number of these vertices, i.e. the number of the $\{1, 2\}$-residues in $(E, \lambda)$, is $|A, C|$. As a consequence, we have:

**Proposition 3.** The pseudocomplex $\bar{K}_\Phi$ is a (pseudosimplicial) triangulation of the standard complex $K_\Phi$ if and only if $|A, C| = 1$.

**Remark.** Given a group presentation $\Phi$, the number of connected components in the associated $P$-graph $P_\Phi$ is $|A, C|$ ($|\Phi|$); moreover, it is easy to verify that every 3-manifold admits a standard spine $K_\Phi$ such that the associated $P$-graph $P_\Phi$ is connected.

Thus, there is no loss of generality in restricting our study to those group presentations for which $|A, C| = 1$ and in supposing that $\bar{K}_\Phi$ triangulates $K_\Phi$.

With the above notations and assumptions, let $C$ be a given permutation in $\Omega(\Phi)$ and let $(\bar{\Gamma}_C, \bar{\gamma}_C)$ be the 2-dimensional oriented structure associated to $(\Gamma, \gamma)$ and generated by $C$.

**Proposition 4.** Let $(\bar{\Gamma}_C, \bar{\gamma}_C)$ be the $h$-bijoin over $(\bar{\Gamma}_C, \bar{\gamma}_C)$ and let $W$ be the set of all 3-labelled vertices in $K(\bar{\Gamma}_C)$.

(a) $h = |AC, BC|$;
(b) the space $M(K(\bar{\Gamma}_C), W)$ is a 3-manifold, having $K_\Phi$ as a standard spine, if and only if $|AC| = \lambda - 2g + 2$;
(c) if (b) holds, the Euler characteristic of $K(\bar{\Gamma}_C)$ is $g - s + h - 1$;
(d) if $g = s$ (resp. $g = s + 1$) and (b) holds, then $|K(\bar{\Gamma}_C)|$ is a closed 3-manifold (resp. $M(K(\bar{\Gamma}_C), W)$ is the exterior of a knot), having $K_\Phi$ as a standard spine, if and only if $|AC, BC| = 1$.

**Proof.** If $\mathcal{F} \subset \Delta_2$ (resp. $\mathcal{F} \subset \Delta_3$), the symbol $g_\mathcal{F}$ (resp. $g_\mathcal{F}'$) will denote the number of $\mathcal{F}$-residues in $(\Gamma, \gamma)$ (resp. in $(\bar{\Gamma}_C, \bar{\gamma}_C)$).
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Since the number of 2-simplexes in each $K_i$ is $2\lambda(r_i)$ and $\sum_{i} \lambda(r_i) = \lambda$, then

$$\text{Card}(V(\Gamma_c)) = 2 \cdot \text{Card}(V(\Gamma)) = 4\lambda. \quad [1]$$

Each $\{c\}$-residue $(c \in \Delta_i)$ in $(\Gamma, \gamma)$ is a complete graph of order two and $g_{\{c\}} = g_{\{1\}} = \lambda$; in fact, in each $K_i$ there are exactly $\lambda(r_i)$ edges whose endpoints are labelled by the colours 2 and $c$ and they are faces of exactly two 2-simplexes.

Hence:

$$g'_{\{0, 1\}} = g_{\{0\}} = |B| = \lambda \quad [2]$$

$$g'_{\{1, 3\}} = g_{\{1\}} = |A| = \lambda.$$

For every $i \in N_g$, there are exactly two $\{2\}$-residues in $(\Gamma, \gamma)$ which are complete graphs of order $\lambda(x_i)$. In fact, there are exactly two $x_i$-labelled edges in $\overline{\mathcal{K}}_b$ and the number of 2-simplexes of which each $x_i$-labelled edge is a face is the number $\lambda(x_i)$ of occurrences of the generator $x_i$ in the relators of $\Phi$. Hence:

$$g'_{\{2, 3\}} = g_{\{2\}} = |C| = 2g. \quad [3]$$

Recall that an alternating path in an oriented graph $\overline{\Gamma}$ is a path whose adjacent edges have opposite orientations. In an oriented structure $(\overline{\Gamma}, \overline{\gamma})$, for every pair $h, k$ of distinct colours, a weak $\{h, k\}$-cycle ([BG]) is an alternating cycle of $(\overline{\Gamma}, \overline{\gamma})$ whose edges are alternatively coloured by $h$ and $k$. If $\overline{g}_{hk}$ denotes the number of weak $\{h, k\}$-cycles of $(\overline{\Gamma}_c, \overline{\gamma}_c)$, we have $g'_{\{h, k\}} = \overline{g}_{hk}$, for each $h, k \in \Delta_2$.

The number of the $\{3\}$-residues in $(\Gamma_c, \gamma_c)$ is the number of the orbits in the permutation group generated by $AB$, $BC$ and $AC$. Since $AB = (AC)(BC)^{-1}$, we have:

$$h = g'_{\{3\}} = |AB, BC, AC| = |AC, BC| \quad [4]$$

and this proves (a).

If $P(c)$, $c \in \Delta_2$, denotes the permutation on $V(\overline{\Gamma}_c) = V(\Gamma)$ induced by the $c$-adjacency, it is easy to see that $\overline{g}_{hk} = |P(h)^{-1} P(k)| = |P(h)^{-1} P(k)| = \overline{g}_{hk}$, for each pair of distinct colours $h, k \in \Delta_2$. Thus, the following equalities hold:

$$g'_{\{0, 1\}} = \overline{g}_{01} = |AB^{-1}| = |AB| = 2\lambda \quad [5]$$

$$g'_{\{0, 2\}} = \overline{g}_{02} = |B^{-1} C| = |BC| = \lambda$$

$$g'_{\{1, 2\}} = \overline{g}_{12} = |A^{-1} C| = |AC|.$$
Note that, \(|BC| = \lambda\) since \(BC = C^{-1}B\) and hence \((BC)^2 = 1\).

For each \(j \in N_a\) there is one \([0,1]\)-residue in \((\Gamma, \gamma)\) which is a bicoloured cycle of length \(2\lambda (r_j)\); in fact, the \([0,1]\)-residues in \((\Gamma, \gamma)\) are in one-to-one correspondence with the inner vertices \(C_j\) of \(\beta_j\). Hence:

\[ g'_1 = g_1 = |A, B| = s. \]  

[6]

For each \(i \in N_c\), there is one \([0,2]\)-residue in \((\Gamma, \gamma)\) with \(2\lambda (x_i)\) vertices, in fact, the \([0,2]\)-residues in \((\Gamma, \gamma)\) are in one-to-one correspondence with the barycenters in \(\partial K_c\). Hence:

\[ g'_0 = g_0 = |A, C| = g. \]  

[7]

Finally, since, as pointed out for the proof of Proposition 3, the 0-labelled vertices in \(K_c\) are in one-to-one correspondence with the \([1,2]\)-residues in \((\Gamma, \gamma)\), the assumption \(|A, C| = 1\) gives:

\[ g'_0 = g_0 = |A, C| = 1. \]  

[8]

Let us now compute the Euler characteristic \(\chi(K_d)\) of the pseudocomplex \(K_d = K((\Gamma_c)_{\lambda})\), for each \(d \leq \Delta_c\), by making use of the equalities (1) - (8) and by recalling that the number of 2-simplexes (resp. 1-simplexes) in \(K_d\) is \(\text{Card}(\bar{V}(\Gamma_c)) = 4\lambda\) (resp. \(3 \text{Card}(\bar{V}(\Gamma_c)) = 6\lambda\)).

\[ \chi(K_d) = 4\lambda - 6\lambda + (g'_{[0,1]} + g'_{[1,2]} + g'_{[2,3]}) = -2\lambda + (|AC| + \lambda + 2g) = |AC| + 2g - \lambda; \]

\[ \chi(K_1) = 4\lambda - 6\lambda + (g'_{[0,1]} + g'_{[1,2]} + g'_{[2,3]}) = -2\lambda + (\lambda + \lambda + 2g) = 2g; \]

\[ \chi(K_0) = 4\lambda - 6\lambda + (g'_{[0,1]} + g'_{[1,2]} + g'_{[0,1]}) = -2\lambda + (\lambda + \lambda + 2s) = 2s. \]

As pointed out in section 2, each \([d]\)-residue in the 4-coloured graph \((\Gamma_c, \gamma_c)\) represents the disjoined link of the represented \(d\)-labelled vertex in \(K(\Gamma_c)\). Since the equality [6] (resp. [7]) states that the number of \([2]\)-residues (resp. \([1]\)-residues) in \((\Gamma_c, \gamma_c)\) is \(s\) (resp. \(g\)), the equality \(\chi(K_d) = 2s\) (resp. \(\chi(K_0) = 2g\)) proves that the disjoined link of each \(2\)-labelled (resp. \(1\)-labelled) vertex in \(K(\Gamma_c)\) is a 2-sphere. Hence all 1-labelled and 2-labelled vertices of \(K(\Gamma_c)\) are regular. Moreover, the disjoined link \(K_0\) of the unique 0-labelled vertex of \(K(\Gamma_c)\) is a 2-sphere if and only if \(\chi(K_0) = |AC| = 2g - \lambda = 2\), that is if and only if \(|AC| = \lambda - 2g + 2\). This result, together with Proposition 1, proves (b).

The Euler characteristic computation of \(K(\Gamma_c)\) gives:

\[ \chi(K(\Gamma_c)) = (g'_0 + g'_1 + g'_2 + g'_3) - (g'_{[0,1]} + g'_{[0,2]} + g'_{[0,3]} + g'_{[1,2]} + g'_{[1,3]} + g'_{[2,3]}) + \text{Card}(\bar{V}(\Gamma_c)) = \]

\[ (1 + g + s + h) - (2s + \lambda + \lambda + |AC| + \lambda + 2g) + 8\lambda - 4\lambda = \lambda + h + 1 - s - g - |AC|. \]

Thus, if (b) holds, \(\chi(K(\Gamma_c)) = g - s + h - 1\).
Finally, if \( g = s \) (resp. \( g = s + 1 \)) and (b) holds, \( \chi(K(\Gamma_0)) = h - 1 \) (resp. \( \chi(K(\Gamma_0)) = h \)) and hence \( |K(\Gamma_0)| \) is a closed 3-manifold (resp. \( M(K(\Gamma_0), W) \) is the exterior of a knot) if and only if \( h = 1 \); proposition 1, corollary 2 and equality [4] complete the proof of (d).

\[ \square \]

**Proposition 5.** Let \( M \) be a 3-manifold having \( K_8 \) as a standard spine. There exists a permutation \( C \in \Omega(\Phi) \) such that \( M = M(K(\Gamma_0), W) \).

**Proof.** If \( \alpha \) is a 1-simplex of \( K_8 \), the embedding of its star \( (\alpha, K_8) \) in the (arbitrarily oriented) 3-manifold \( M \) induces a cyclic ordering of the 2-simplexes of \( K_8 \) containing \( \alpha \). Thus, a permutation \( C \) on \( V(\Gamma) \) or, equivalently, an oriented structure \((\Gamma_C, \gamma_C)\) can be associated to the crystallized structure \((\Gamma, \gamma)\) representing \( K_8 \).

Note that the embedding of \( K_8 \) in \( M \) directly gives \( BC = C^{-1}B \) and hence \( C \in \Omega(\Phi) \). Let \((\Gamma_C, \gamma_C)\) be the \( h \)-bijoin over \((\Gamma_C, \gamma_C)\); note that the choice of the opposite orientation in \( M \) gives rise to the opposite oriented structure but to the same graph \((\Gamma_C, \gamma_C)\), as pointed out in section 3. If \( M \) denotes the singular 3-manifold obtained by capping off each boundary component of \( M \) by a cone, then \( M = [K(\Gamma_0)] \) and hence \( M = M(K(\Gamma_0), W) \), \( W \) being the set of all 3-labelled vertices in \( K(\Gamma_0) \).

\[ \square \]

If \( \Omega'(\Phi) \) denotes the subset of \( \Omega(\Phi) \) consisting of all \( C \in \Omega(\Phi) \) such that \( |AC| = \lambda - 2g + 2 \), then proposition 4 and proposition 5 lead to the following result:

**Corollary 6.** The complex \( K_8 \) is a standard spine of a 3-manifold \( M \) if and only if there exists a permutation \( C \in \Omega'(\Phi) \) such that \( M = M(K(\Gamma_0), W) \).

\[ \square \]

The above result directly produces an effective algorithm for testing the geometricity of a group presentation, extending Neuwirth's one to non-balanced presentations.

**Example.** Let \( \Phi = <x, y \mid x^3 y^2> \). In this case, \( g = 2 \), \( s = 1 \) and, with the notations of fig. 1, the permutations \( A, B \) can be written in the following way:

\[ A = (1 \ 2 \ 3 \ 4 \ 5), \ B = (1 \ 3 \ 5 \ 2) \ (3 \ 1 \ 4) \ (4 \ 5 \ 3) \ (5 \ 4) \ (5 \ 3). \]

Moreover, the orbits of the permutation \( C \) are \( \{1, 2, 3\}, \{1, 2, 3\}, \{4, 5\}, \{4, 5\} \).
The choice of \( C = (1 \, 2 \, 3) \, (3 \, \bar{2} \, \bar{1}) \, (4 \, 5) \, (\bar{3} \, \bar{4}) \in \Omega(\Phi) \) produces the 4-coloured graph \((\Gamma_C, \gamma_c)\) drawn in fig. 1 and \( M(K(\Gamma_C), W) \) is the exterior of the trefoil knot.

Figure 1a.

Figure 1b.
Figure 1c.

$(\Gamma', \gamma)$

Figure 1d.

$(\Gamma'_c, \gamma_c)$
5. MONTESINOS ALGORITHM VIA BIJOINS

We sketch Montesinos algorithm described in [M].

With the notations of the previous section, let $\Phi$ be a given group presentation whose associated $P$-graph $P_\Phi$ is connected; make $\Phi$ positive and call the new presentation $\Phi$ again.

Take the permutation $\tau = (1, 2, \ldots, \lambda (r_1)) \cdot (\lambda (r_1) + 1, \ldots, \lambda (r_1) + \lambda (r_2)) \cdot \ldots \cdot (\ldots, \lambda (r_k) \cdot (\lambda (r_k) + 1, \ldots, \lambda (r_k) + \lambda (r_{k+1}) \cdot \ldots \cdot (\ldots, \lambda (r_s) \cdot (\lambda (r_s) + 1, \ldots, \lambda (r_s) + \lambda (r_{s+1}))))$.

(\ldots, \lambda (n))$ on $N$, and the set of all permutations $\sigma$ on $N_n$ whose orbits $d_1, \ldots, d_k$ are defined as follows: the number $j \in N$ belongs to $d_i$ if and only if there is a relator $r_k$ whose $(j - \lambda (r_{k-1}))$-th letter is $x_i$. Let $\Sigma (\Phi)$ denote the subset of all such $\sigma$ satisfying $|\sigma, \tau \sigma \tau^{-1}| = 1$ and $[\sigma, \tau] = \lambda - 2g + 2$.

If $\Sigma (\Phi) \neq \emptyset$, then, for each $\sigma \in \Sigma (\Phi)$, construct the singular 3-manifold $N(\sigma, \tau)$ by taking $\lambda$ copies $t_1, \ldots, t_\lambda$ of the standard tetrahedron $t$ whose bidimensional faces are denoted by $S, \tilde{S}, T, \tilde{T}$. Label the faces $S, \tilde{S}, T, \tilde{T}$ in the copy $t_1$ as $S_{i\alpha(\tilde{t})}, S_{i\alpha(\tilde{t})}, T_{i\alpha(\tilde{t})}, T_{i\alpha(\tilde{t})}$ respectively; identify $S_{t_i}$ with $S_{t_i}$ and $T_{t_i}$ with $T_{t_i}$ by an orientation-reversing linear homeomorphism respecting the edges $S \cap S$ and $T \cap T$.

If $W$ denotes the set of all singular vertices of $N(\sigma, \tau)$, then $\{ M(N(\sigma, \tau), W) \mid \sigma \in \Sigma (\Phi) \}$ is the set of all 3-manifolds $M^3$ admitting a Heegaard diagram whose associated presentation for $\Pi_1 (M^3)$ is $\Phi$.

Since $P_\Phi$ is connected, the representable 2-pseudocomplex $\tilde{K}_\Phi$ triangulates $K_\Phi$; now, it is possible to label the vertices of the crystallized structure $(\Gamma, \gamma)$ associated to $K_\Phi$ by the set $N_\lambda = \{ 1, 2, \ldots, \lambda, \tilde{1}, \tilde{2}, \ldots, \tilde{\lambda} \}$ so that:

$$A = (12) \cdot (23) \cdot (\ldots, (\lambda (r_1) - 1, \lambda (r_1) - 1) \cdot (\lambda (r_1) + 1, \lambda (r_1) + 1) \cdot \ldots \cdot \lambda (r_s) \cdot (\lambda (r_s) + 1) \cdot \ldots$$

$$\cdot \lambda (r_s) + 1, \lambda (r_s) + 1) \cdot (\lambda (r_s) + 1, \lambda (r_s) + 1) \cdot \ldots \cdot (\lambda (r_s) + 1, \lambda (r_s) + 1).$$

Moreover, if $C$ is a permutation on $N_\lambda$ satisfying the following properties:

- $j$ (resp. $j$) belongs to the orbit $d_i$ (resp. $d_j$) of $C$ if and only if there is a relator $r_k$ whose $(j - \lambda (r_{k-1}))$-th letter is $x_i$,
- the ordering of the elements $j$ in $d_i$ is opposite to the ordering of the elements $j$ in $d_j$,

then $C \in \Omega (\Phi)$.

Thus, the choice of $\sigma$ induces an associated $C_\sigma$ (and hence an oriented structure $(\bar{\Gamma}_\sigma, \bar{\gamma}_\sigma) = (\bar{\Gamma}_c, \bar{\gamma}_C)$) in a standard way and vice versa.
Proposition 7. The singular 3-manifold $N(\alpha, \tau)$ is $PL$-homeomorphic with $|K(T_0)|$, $(T_0, \gamma_0)$ being the $h$-bijoin over $(T_0, \gamma_0)$.

Proof. If $t$ is the standard tetrahedron, subdivide it into four tetrahedra in the following way. If $V_5$ (resp. $V_7$) is the barycenter of $S \cap \bar{S}$ (resp. $T \cap \bar{T}$), join $V_5$ with $V_7$ by an edge whose interior is contained in the interior of $t$ and subdivide $S, \bar{S}$ (resp. $T, \bar{T}$) by joining $V_5$ (resp. $V_7$) with the endpoints of $T \cap \bar{T}$ (resp. $S \cap \bar{S}$). Label $V_5$ by 1, $V_7$ by 2 and the endpoints of $S \cap \bar{S}$ (resp. $T \cap \bar{T}$) by 0 (resp. 3) (fig. 2).

![Diagram](image)

In this way, $N(\alpha, \tau)$ is triangulated by a representable 4-pseudocomplex $K'$ in which each $t_i$ splits into four tetrahedra.

If $(E', \gamma')$ is the 4-coloured graph representing $K'$, it is straightforward that the oriented structure $(E', \gamma')$ is isomorphic with $(\bar{E}_0, \bar{\gamma}_0)$ and hence $N(\alpha, \tau) = |K(T_0)|$.

Since $|\sigma, \tau| = g' = g$, $|\sigma| = g$, $|\tau| = s$, $|\sigma, \tau\sigma^{-1}| = g'_0 = |A, C|$, $|\tau, \sigma\tau^{-1}| = g'_3 = h = |AC, BC|$, all results in [M] can be restated in terms of spines or in terms of bijoins and edge-coloured graphs.

It appears as evident that the graph-theoretical bijoin construction is the idea which unifies both Neuwirth and Montesinos algorithm.
3-Manifold Spines and Bijoins

References


