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Projective Limits of Vector Measures

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ABSTRACT. A necessary and sufficient condition for the existence of the projective limit of measures with values in a locally convex space is given. A similar theorem for measures with values in different locally convex spaces (under certain conditions) is given too (in this case, the projective limit is valued in the projective limit of these spaces). Finally, a result about the projective limit of vector measures is stated.

1. INTRODUCTION AND NOTATION

In [26] L. Schwartz has proved the Prokhorov's theorem about the existence of the projective limit of a projective system of finite (scalar) Radon measures (of type $(\mathscr{K})$) on Hausdorff topological spaces. This result has been extended in [16] for arbitrary (scalar) Radon measures of type $(\mathscr{U})$ on topological spaces.

As it is well known the Prokhorov's theorem has a very important role in the study of cylindrical measures and in general in probability theory.

The main object of this paper is to prove a Prokhorov's type theorem for vector Radon measures. This has been made here for Radon measures of type $(\mathscr{U})$ on an arbitrary topological space E with values in a complete locally convex Hausdorff space X. Of special interest are the following particular cases: (1) E is a Hausdorff topological space and $\mathscr{U}$ is the class of all compact subsets of E, and (2) X is a Banach space.

In the last section we give the relation between the projective limit of a system of product measures $(\mu_i \otimes \nu_i)_{i\in I}$ and the tensor product of the limits of the systems $(\mu_i)_{i\in I}$ and $(\nu_i)_{i\in I}$. This result remains valid in general without any assumption about the regularity of the measures. A theorem of this type for scalar Radon measures has been proved in [12].

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Let X be a complete locally convex Hausdorff space whose topology is defined by a saturated family $\mathscr{P}$ of seminorms, and denote by E, $\mathscr{G}$, $\mathscr{F}$ and $\mathscr{B}$ a topological space and the classes of its open, closed and Borel subsets respectively.

If $\mu:\mathscr{B}\longrightarrow X$ is a (σ-additive) vector measure, and $p\in\mathscr{P}$, the p-semivariation of μ will be as usual the mapping $||\mu||_p:\mathscr{B}\longrightarrow\mathbb{R}^+\cup\{+\infty\}$ defined by

$$||\mu||_p(B)=\sup p(\sum_{j=1}^{n} t_j\mu(A_j)) \qquad (B\in\mathscr{B}) \quad ,$$

where the supremum is taken over all partitions of B into a finite number of disjoint sets $\{A_j\}_{j=1}^{n}\subset\mathscr{B}$ and all finite collections of elements $\{t_j\}_{j=1}^{n}\subset\mathbb{R}$ with $|t_j|\leq 1$ for every $j=1,...,n$. It is easily proved that

$$||\mu||_p(B)=\sup\ (\sum_{j=1}^{n}|x'\mu(A_j)|) \qquad (B\in\mathscr{B}) \quad ,$$

where the supremum is taken over all finite partitions $\{A_j\}_{j=1}^{n}\subset\mathscr{B}$ of B and all $x'\in X'$ such that $|x'(y)|\leq p(y)$ for every $y\in X$, where X' denotes the dual space of X ; and

$$||\mu||_p(B)\leq 2\sup\{p(\mu(A)): A\in\mathscr{B}, A\subset B\} \quad ,$$

for every borel subset $B\in\mathscr{B}$.

2. EXISTENCE THEOREMS FOR PROJECTIVE LIMITS OF VECTOR MEASURES

Definition 1. *A borel subset $B\in\mathscr{B}$ is said to be μ-compact if for every open cover $\{G_i\}_{i\in I}$ of B, every seminorm $p\in\mathscr{P}$ and $\epsilon>0$, there exists a finite subset $J\subset I$ such that*

$$||\mu||_p(B-\bigcup_{i\in J}G_i)\leq\epsilon \qquad [1.1]$$

Definition 2. *Let $\mathscr{A}$ be a family of closed subsets of E. We say that μ is a Radon measure of type ($\mathscr{A}$) if the following statements hold :*

2.1. Every $H\in\mathscr{A}$ is μ-compact.

2.2. For every $B\in\mathscr{B}$, $p\in\mathscr{P}$ and $\epsilon>0$, there exists $H\in\mathscr{A}$ such that $H\subset B$ and $||\mu||_p(B-H)\leq\epsilon$.

If μ is a Radon measure of type ($\mathscr{A}$) then it is easily proved that every Borel subset of E is μ-compact.

Definition 3. *Let E be a projective limit of the projective system $(E_i, \Pi_{ij})_{i,j\in I}$ of topological spaces and denote by Π_i the corresponding projection from E into E_i $(i\in I)$. If for every $i\in I$, $\mu_i:\mathscr{B}_i \longrightarrow X$ is a Radon measure of type $(\mathscr{K}_i)$ ($\mathscr{B}_i$ denotes the Borel family of E_i; and $\mathscr{K}_i$, a family of closed subsets of E_i, which is closed under finite unions), we say that $(\mu_i)_{i\in I}$ is a projective system of Radon measures of type $(\mathscr{K}_i)$, if $\Pi_{ij}(\mu_j)=\mu_i$ (i.e. $\mu_i(B_i)=\mu_j(\Pi_{ij}^{-1}(B_i))$ for all $B_i\in\mathscr{B}_i$) for every $i,j\in I$ with $i\leq j$; and a Radon measure of type $(\mathscr{K})$ $\mu:\mathscr{B}\longrightarrow X$ is said to be the projective limit of the measures $(\mu_i)_{i\in I}$ (it is immediately proved that if the projective limit measure exists then it is unique) when $\Pi_i(\mu)=\mu_i$ holds for every $i\in I$ (i.e. $\mu_i(B_i)= =\mu(\Pi_i^{-1}(B_i))$ for every $B_i\in\mathscr{B}_i$ and $i\in I$).*

Let us introduce the following conditions:

3.1. $\mathscr{K}$ is closed under finite unions, $H\cap F\in\mathscr{K}$ for every $H\in\mathscr{K}$ and $F\in\mathscr{F}$, $\overline{\Pi_i(H)}\in\mathscr{K}_i$ for every $H\in\mathscr{K}$ and $i\in I$, and for every $H\in\mathscr{K}$ there exists $i_H\in I$ such that $\Pi_i(H)\in\mathscr{B}_i$ for every $i\geq i_H$.

3.2. For every $i\in I$ and $p\in\mathscr{P}$, there exists a non negative and finite measure ν_i^p: $\mathscr{B}_i\longrightarrow \mathbb{R}^+$ such that

3.2.1. $\Pi_{ij}(\nu_j^p)=\nu_i^p$ for every $p\in\mathscr{P}$ and $i, j\in I$ with $i\leq j$.

3.2.2. $\nu_i^p(B_i)=\inf\{\nu_i^p(G_i): B_i\subset G_i\in\mathscr{G}_i\}$, for every $i\in I$, $p\in\mathscr{P}$ and $B_i\in\mathscr{B}_i$.

3.2.3. For every $\epsilon>0$ and $p\in\mathscr{P}$ there exists $i_o\in I$ and $\eta>0$ such that $||\mu_i||_p(B_i)\leq\epsilon$ for all $i\geq i_o$ and $B_i\in\mathscr{B}_i$ with $\nu_i^p(B_i)\leq\eta$.

Lemma 4. *If the Radon measure of type $(\mathscr{K})$ $\mu:\mathscr{B}\longrightarrow X$ is the projective limit of a projective system $(\mu_i)_{i\in I}$ of Radon measures of type $(\mathscr{K}_i)$ and conditions 3.1 and 3.2 are verified, then*

$$\mu(H)=\lim_{j\geq i_H} \mu_j(\Pi_j(H)) \tag{4.1}$$

for every $H\in\mathscr{K}$. Moreover, μ is of bounded semivariation if and only if the semivariations of the measures μ_i $(i\in I)$ are uniformly bounded.

Proof. It follows from conditions 3.1 and 3.2 that for every $H\in\mathscr{K}$, $(\mu_i(\Pi_i(H)))_{i\geq i_H}$ is a Cauchy net in X.

Let us set

$$\lambda(H)=\lim_{i\geq i_H} \mu_i(\Pi_i(H)). \tag{4.2}$$

(If the semivariations $(\|\mu_i\|_p)_{i\epsilon I}$ are uniformly bounded (for every $p\,\epsilon\,\mathscr{P}$), then $(\mu_i(\Pi_i(H)))_{i\geq i_H}$ is a bounded Cauchy net in X and so the limit [4.2] exists also assuming only that the space X is quasi-complete.)

Let $H\,\epsilon\,\mathscr{A}$, $p\,\epsilon\,\mathscr{P}$ and $\epsilon>0$. Then

$$H=\bigcap_{\substack{i\epsilon I\\ i\geq i_H}} \Pi_i^{-1}\,\Pi_i(H)=\bigcap_{\substack{i\epsilon I\\ i\geq i_H}} \Pi_i^{-1}\,(\overline{\Pi_i(H)}),$$

and there exists $i_1,\ldots,i_r\,\epsilon\,I$ such that

$$\|\mu\|_p((\Pi_{i_H}^{-1}(\overline{\Pi_{i_H}(H)})-H)\cap(\bigcap_{h=1}^{r}\Pi_{i_h}^{-1}(\overline{\Pi_{i_h}(H)}))\ )\leq\epsilon.$$

Therefore, if $i_o\,\epsilon\,I$ is such that $i_H\leq i_o$ and $i_h\leq i_o$ for $h=1,\ldots,r$, then

$$\|\mu\|_p(\Pi_i^{-1}\,\Pi_i(H)-H)\leq\epsilon$$

for every $i\,\epsilon\,I$ with $i\geq i_o$, and

$$p(\mu_i(\Pi_i(H))-\mu(H))=p(\mu(\Pi_i^{-1}\,\Pi_i(H))-\mu(H))=$$

$$=p(\mu(\Pi_i^{-1}\,\Pi_i(H)-H))\leq\|\mu\|_p(\Pi_i^{-1}\,\Pi_i(H)-H)\leq\epsilon$$

for every $i\,\epsilon\,I$ with $i_o\leq i$. So, $p(\lambda(H)-\mu(H))\leq\epsilon$;

from where it follows immediately that $\mu(H)=\lambda(H)$.

Moreover, if μ is of bounded semivariarion then the measures $\mu_i(i\,\epsilon\,I)$ are of uniformly bounded semivariation since

$$\|\mu_i\|_p(B_i)\leq\|\mu\|_p(\Pi_i^{-1}(B_i))\ ,$$

for all $p\,\epsilon\,\mathscr{P}$ and $B_i\,\epsilon\,\mathscr{B}_i$. Conversely, if for every $p\,\epsilon\,\mathscr{P}$ there exists $K>0$ such that $\|\mu_i\|_p(E_i)\leq K$ for every $i\,\epsilon\,I$, then for every $\epsilon>0$ and $B\,\epsilon\,\mathscr{B}$, there exists $H\,\epsilon\,\mathscr{A}$ such that $H\subset B$ and $\|\mu\|_p(B-H)\leq\epsilon$; therefore ,

$$p(\mu(B))\leq p(\mu(B-H))+p(\mu(H))\leq\|\mu\|_p(B-H)+$$

$$+p(\lambda(H))\leq\epsilon+\sup_{\substack{i\epsilon I\\ i\geq i_H}}\|\mu_i\|_p(E_i)\leq\epsilon+K\ ;$$

and $\|\mu\|_p(E)\leq 2\cdot\sup\{p(\mu(B)):B\,\epsilon\,\mathscr{B}\}\leq 2K<+\infty$.

Theorem 5. *Let $(\mu_i)_{i\epsilon I}$ be a projective system of Radon measures of type $(\mathscr{A}_i)$, and assume that conditions 3.1 and 3.2 are satisfied. Then the projective* *limit μ of the system $(\mu_i)_{i\in I}$ exists and $\mu:\mathscr{B}\longrightarrow X$ is a Radon measure of type $(\mathscr{A})$, if and only if the following statements hold:*

5.1. For every $p\in\mathscr{P}$, $\epsilon>0$, $H\in\mathscr{A}$ and open cover $\{G_j\}_{j\in J}$ of H, there exists a finite subset $J'\subset J$ and $i_o\in I$ such that $i_o\geq i_{H_1}$, with $H_1=H-\bigcup_{j\in J'}G_j$, and

$$||\mu_i||_p(\Pi_i(H-\bigcup_{j\in J'}G_j))<\epsilon$$

for every $i\geq i_o$.

5.2. For every $p\in\mathscr{P}$, $\epsilon>0$, $i\in I$ and $H_i\in\mathscr{A}_i$, there exists $H\in\mathscr{A}$ such that $H\subset\Pi_i^{-1}(H_i)$ and

$$||\mu_{i'}||_p(\Pi_{ii'}^{-1}(H_i)-\Pi_{i'}(H))\leq\epsilon,$$

for every $i'\in I$ with $i'\geq i$ and $i'\geq i_H$.

Proof. The condition is necessary. Consider $p\in\mathscr{P}$, $\epsilon>0$ and $H\in\mathscr{A}$. If $\{G_j\}_{j\in J}$ is an open cover of H, there exists a finite subset $J'\subset J$ such that

$$||\mu||_p(H-\bigcup_{j\in J}G_j)\leq\epsilon/2.$$

Set $H_1=H-\bigcup_{j\in J'}G_j\in\mathscr{A}$; then as we have seen in the proof of Lemma 4, there exists $i_o\in I$ such that $i_o\geqq i_{H_1}$ and

$$||\mu||_p(\Pi_i^{-1}\Pi_i(H_1)-H_1)\leq\epsilon/2$$

for all $i\in I$ with $i\geq i_o$. Therefore, we get

$$||\mu_i||_p(\Pi_i(H-\bigcup_{j\in J'}G_j))\leq||\mu||_p(\Pi_i^{-1}\Pi_i(H_l))\leq$$
$$\leq||\mu||_p(\Pi_i^{-1}\Pi_i(H_1)-H_1)+||\mu||_p(H_1)\leq\epsilon,$$

for all $i\geq i_o(i\in I)$, and 5.1 holds.

Moreover, for $i\in I$, $H_i\in\mathscr{A}_i$, $p\in\mathscr{P}$ and $\epsilon>0$, there exists $H\in\mathscr{A}$ such that $H\subset\Pi_i^{-1}(H_i)$ and

$$||\mu||_p(\Pi_i^{-1}(H_i)-H)\leq\epsilon.$$

Therefore we get, if $j\geq i$ and $j\geq i_H$ $(j\in I)$,

$$||\mu_j||_p(\Pi_{ij}^{-1}(H_i)-\Pi_j(H))\leq||\mu||_p\,(\Pi_j^{-1}\,(\Pi_{ij}^{-1}(H_i)-\Pi_j(H))\,)=$$

$=||\mu||_p(\Pi_i^{-1}(H_i)-\Pi_j^{-1}\Pi_j(H))\leq||\mu||_p(\Pi_i^{-1}(H_i)-H)\leq\epsilon$, and 5.2 holds.

The condition is sufficient. Let λ be defined as in [4.2]; then we have:

i) If $H_1, H_2 \in \mathscr{A}$ satisfy $H_1 \cap H_2 = \emptyset$, then

$$\lambda(H_1 \cup H_2) = \lambda(H_1) + \lambda(H_2). \qquad [5.1]$$

In fact, $H_1 \subset E - H_2 = \bigcup_{i \in I}(E - \Pi_i^{-1}(\overline{\Pi_i(H_2)}))$, and it follows from 5.1 that for every seminorm $p \in \mathscr{P}$ and $\epsilon > 0$ there exists $i_o \in I$ such that $||\mu_j||_p(\Pi_j(B)) \leq \epsilon$ for every $B \in \mathscr{B}$ and $j \in I$ such that $j \geq i_o$, $B \subset H_1 \cap \Pi_j^{-1}(\overline{\Pi_j(H_2)})$ and $\Pi_j(B) \in \mathscr{B}_j$. Therefore,

$$p(\lambda(H_1) + \lambda(H_2) - \lambda(H_1 \cup H_2)) =$$

$$= p(\lim_{\substack{i \in I \\ i \geq i_{H_1}, i_{H_2}, i_{H_1 \cup H_2}}} (\mu_i(\Pi_i(H_1)) + \mu_i(\Pi_i(H_2)) - \mu_i(\Pi_i(H_1 \cup H_2)))) =$$

$$= \lim_{\substack{i \in I \\ i \geq i_{H_1}, i_{H_2}, i_o}} p(\mu_i(\Pi_i(H_1) \cap \Pi_i(H_2))) \leq \sup_{i \geq i_o, i_{H_1}, i_{H_2}} ||\mu_i||_p(\Pi_i(H_1) \cap \Pi_i(H_2)) =$$

$$= \sup_{i \geq i_o, i_{H_1}, i_{H_2}} ||\mu_i||_p(\Pi_i(H_1 \cap \Pi_i^{-1}\, \Pi_i(H_2))) \leq \epsilon,$$

and the equality [5.1] holds.

ii) For every $H \in \mathscr{A}$, $p \in \mathscr{P}$ and $\epsilon > 0$, there exists $G \in \mathscr{G}$ such that $H \subset G$ and

$$p(\lambda(H') - \lambda(H'') - \lambda(H)) \leq \epsilon$$

$$\text{if } H', H'' \in \mathscr{A},\ H \subset H' \text{ and } H' - G \subset H'' \subset H' - H.$$

To prove this, let us remark first that for every $H \in \mathscr{A}$, $p \in \mathscr{P}$ and $\epsilon > 0$, there exists $i_o \geq i_H$ $(i_o \in I)$ such that if $k \geq j \geq i_o$ $(k, j \in I)$, then

$$||\mu_k||_p(\Pi_{jk}^{-1}(\Pi_j(H)) - \Pi_k(H)) \leq \epsilon/2.$$

Moreover, there exists $\eta > 0$ and $i_1 \in I$ such that $||\mu_i||_p(B_i) \leq \epsilon/2$ for every $i \geq i_1$ $(i \in I)$ and $B_i \in \mathscr{B}_i$ with $\nu_i^p(B_i) \leq \eta$.

Let us consider $j \geq i_o, i_1$. Since $i_o \geq i_H$, $\Pi_j(H) \in \mathscr{B}_j$ and then

$$\nu_j^p(\Pi_j(H)) = \inf\{\nu_j^p(G) : \Pi_j(H) \subset G \in \mathscr{G}_j\},$$

and there exists $G_j \in \mathscr{G}_j$ such that $\Pi_j(H) \subset G_j$ and $\nu_j^p(G_i - \Pi_j(H)) \leq \eta$.

So we have

$$\nu_i^p(\Pi_{ji}^{-1}(G_j) - \Pi_{ji}^{-1}(\Pi_j(H))) = \nu_j^p(G_j - \Pi_j(H)) \leq \eta,$$

and

$||\mu_i||_p(\Pi_{ji}^{-1}(G_j)-\Pi_i(H))\leq$

$\leq ||\mu_i||_p(\Pi_{ji}^{-1}(G_j)-\Pi_{ji}^{-1}\Pi_i(H))+||\mu_i||_p(\Pi_{ji}^{-1}\Pi_i(H)-\Pi_i(H))\leq$

$\leq \epsilon/2+\epsilon/2=\epsilon$,

if $i\geq j \quad (i\epsilon I)$.

Set $G=\Pi_j^{-1}(G_j)$; then $H\subset G$ and

$\Pi_i(H')-\Pi_i(H'')\subset\Pi_{ji}^{-1}(G_j)$ if $H', H''\epsilon\mathscr{U}$, $H\subset H'$,

$H'-G\subset H''\subset H'-H$ and $i\geq j$. Therefore,

$p(\lambda(H')-\lambda(H'')-\lambda(H))=p(\lambda(H')-\lambda(H''\cup H))=$

$=p(\lim_{i\geq j, i_{H'}, i_{H\cup H''}}(\mu_i(\Pi_i(H'))-\mu_i(\Pi_i(H''\cup H))))=$

$=\lim_{i\geq j, i_{H'}, i_{H\cup H''}} p(\mu_i(\Pi_i(H')-\Pi_i(H''\cup H)))\leq$

$\leq\sup_{i\geq j, i_{H'}, i_{H\cup H''}} ||\mu_i||_p(\Pi_i(H')-(\Pi_i(H'')\cup\Pi_i(H)))\leq$

$\leq\sup_{i\geq j, i_H} ||\mu_i||_p(\Pi_{ji}^{-1}(G_j)-\Pi_i(H))\leq\epsilon$,

as stated.

iii) If $(H_j)_{j\epsilon J}\subset\mathscr{U}$ is a decreasing filtering net then

$$\lambda(\bigcap_{j\epsilon J}H_j)=\lim_j\lambda(H_j) \ .$$

Let us suppose first that $\bigcap_{j\epsilon J}H_j=\emptyset$. In this case $H_r\subset\bigcup_{j\epsilon J}(E-H_j)$ for every $r\epsilon J$; and it follows from 5.1 that for every $p\epsilon\mathscr{P}$ and $\epsilon>0$ there exists $k\epsilon J$ and $i_o\epsilon I$ such that

$$||\mu_i||_p(\Pi_i(H_r))\leq\epsilon$$

for every $r\geq k$ $(r\epsilon J)$ and $i\geq i_o$, i_{H_r} $(i\epsilon I)$. Therefore,

$p(\lambda(H_r))=\lim_{i\geq i_o, i_{H_r}} p(\mu_i(\Pi_i(H_r)))\leq\sup_{i\geq i_o, i_{H_r}} ||\mu_i||_p(\Pi_i(H_r))\leq\epsilon$

for every $r\geq k$ $(r\epsilon J)$; hence

$$\lim_j\lambda(H_j)=0=\lambda(\emptyset).$$

Let us consider the general case. Let $p \in \mathscr{P}$ and $\epsilon > 0$. As we have proved before, since $\bigcap_{j\in J} H_j \in \mathscr{A}$, there exists $G \in \mathscr{G}$ such that $\bigcap_{j\in J} H_j \subset G$ and, if H', $H'' \in \mathscr{A}$ verify $\bigcap_{j\in J} H_j \subset H'$ and $H' - G \subset H'' \subset H' - \bigcap_{j\in J} H_j$, then we get

$$p(\lambda(H') - \lambda(H'') - \lambda(H)) \leq \epsilon/2.$$

Therefore, if for every $j \in J$ we introduce $H_j' = H_j - G$, then we get

$$p(\lambda(H_j) - \lambda(H_j') - \lambda(\bigcap_{r\in J} H_r)) \leq \epsilon/2.$$

Moreover, since $\bigcap_{j\in J} H_j' = \emptyset$, there exists $j_o \in J$ such that $p(\lambda(H_j')) \leq \epsilon/2$ for every $j \geq j_o (j \in J)$; and therefore,

$$p(\lambda(H_j) - \lambda(\bigcap_{r\in J} H_r)) \leq p(\lambda(H_j) - \lambda(\bigcap_{r\in J} H_r) - \lambda(H_j')) + p(\lambda(H_j')) \leq \epsilon$$

for every $j \geq j_o$; and

$$\lambda(\bigcap_{j\in J} H_j) = \lim_j \lambda(H_j).$$

iv) For every $B \in \mathscr{B}$, the net $(\lambda(H))_{H\in P(B)\cap\mathscr{A}}$ is convergent.

In fact, for every $p \in \mathscr{P}$ and $\epsilon > 0$, there exists $\eta > 0$ and $i_o \in I$ such that $||\mu_i||_p(B_i) \leq \epsilon/2$ for every $i \geq i_o$ $(i \in I)$ and $B_i \in \mathscr{B}_i$ with $\nu_i^p(B_i) \leq \eta$.

Set

$$\hat{\lambda}_p(H) = \inf\{\nu_i^p(\Pi_i(H)) : i \geq i_H\}$$

for every $H \in \mathscr{A}$; then

$$r_B = \sup\{\hat{\lambda}_p(H) : B \supset H \in \mathscr{A}\} < +\infty$$

for every $B \in \mathscr{B}$, and there exists $H_o \in \mathscr{A}_B$ (with $\mathscr{A}_B = P(B) \cap \mathscr{A}$) such that $r_B - \eta/2 \leq \hat{\lambda}_p(H_o) \leq r_B$; and so,

$$0 \leq \hat{\lambda}_p(H) - \hat{\lambda}_p(H_o) \leq \eta/2$$

holds for every $H \in \mathscr{A}$ with $H_o \subset H \subset B$.

Moreover, there exists $r \in I$ such that $r \geq i_o$, i_H, i_{H_o} and

$$\hat{\lambda}_p(H) \leq \nu_j^p(\Pi_j(H)) \leq \hat{\lambda}_p(H) + \eta/2$$

for all $j \geq r$ $(j \in J)$. Therefore,

$$\nu_j^p(\Pi_j(H) - \Pi_j(H_o)) = \nu_j^p(\Pi_j(H)) - \nu_j^p(\Pi_j(H_o)) \leq$$

$\leq \hat{\lambda}_p(H)+\eta/2-\hat{\lambda}_p(H_o)\leq \eta$

for all $j\geq r$. Then,

$p(\lambda(H)-\lambda(H_o))=\lim_{j\geq r, i_H, i_{H_o}} p(\mu_j(\Pi_j(H)-\Pi_j(H_o)))\leq$

$\leq \sup_{j\geq r, i_H, i_{H_o}} ||\mu_j||_p(\Pi_j(H)-\Pi_j(H_o))\leq \epsilon/2,$

and

$$p(\lambda(H)-\lambda(H'))\leq \epsilon$$

if $H, H' \epsilon \mathscr{U}_B$, $H_o\subset H\subset B$ and $H_o\subset H'\subset B$. The result now follows immediately since the space X is complete.

v) The mapping

$$\mu:\mathscr{B}\longrightarrow X$$
$$B\longrightarrow \mu(B)=\lim_{H\epsilon \mathscr{U}_B} \lambda(H)$$

is well defined, and clearly $\mu(H)=\lambda(H)$ for every $H\epsilon \mathscr{U}$.

vi) For every $B\epsilon \mathscr{B}$, the equality

$$\mu(B)=\mu(B\cap F)+\mu(B-F) \qquad [5.2]$$

holds for every closed subset $F\subset E$.

In fact, let $B\epsilon \mathscr{B}$, F be a closed subset of E, $p\epsilon \mathscr{P}$ and $\epsilon>0$; then there exist H, H_1, $H_2\epsilon \mathscr{U}$ such that $H\subset B$,

$H_1\subset B\cap F$, $H_2\subset B-F$, $p(\lambda(K)-\mu(B))\leq \epsilon/4$,

$p(\lambda(K')-\mu(B\cap F))\leq \epsilon/4$ and $p(\lambda(K")-\mu(B-F))\leq \epsilon/4$ for every K, K',

$K"\epsilon \mathscr{U}$ such that $H\subset K\subset B$, $H_1\subset K'\subset B\cap F$ and $H_2\subset K"\subset B-F$.

Moreover, as we have proved in ii), there exists an open subset $G\subset E$ such that $H_1\cup(H\cap F)\subset G$ and

$$p(\lambda(H')-\lambda(H")-\lambda(H_1\cup(H\cap F)))\leq \epsilon/4$$

if H', $H"\epsilon \mathscr{U}$ are such that $H_1\cup(H\cap F)\subset H'$ and $H'-G\subset H"\subset H'-(H_1\cup(H\cap F))$.

Let be $H' = H \cup H_1 \cup H_2$ and $H'' = H_2 \cup (H - G)$. Then H', $H'' \in \mathscr{A}$, $H_1 \cup (H \cap F) \subset H'$, $H' - G \subset H'' \subset H' - (H_1 \cup (H \cap F))$, $H \subset H' \subset B$, $H_1 \subset H_1 \cup (H \cap F) \subset B \cap F$ and $H_2 \subset H'' \subset B - F$; and therefore,

$p(\mu(B) - \mu(B \cap F) - \mu(B - F)) \leq$

$\leq p(\mu(B) - \lambda(H')) + p(\lambda(H') - \lambda(H'') - \lambda(H_1 \cup (H \cap F))) +$

$+ p(\lambda(H'') - \mu(B - F)) + p(\lambda(H_1 \cup (H \cap F)) - \mu(B \cap F)) \leq \epsilon$, and [5.2] holds trivially.

vii) If $(A_n)_{n \in N} \subset \mathscr{B}$ is a decreasing sequence then

$$\mu(\bigcap_{n \in N} A_n) = \lim_{n \to +\infty} \mu(A_n). \qquad [5.3]$$

Let $p \in \mathscr{P}$ and $\epsilon > 0$; then there exists $H_1 \in \mathscr{A}$ such that $H_1 \subset A_1$ and

$$p(\lambda(H) - \lambda(H_1)) \leq \epsilon/8$$

if $H \in \mathscr{A}$ and $H_1 \subset H \subset A_1$. Thus, if $H \in \mathscr{A}$ verifies $H \subset A_1 - H_1$, we get

$$p(\lambda(H)) = p(\lambda(H \cup H_1) - \lambda(H_1)) \leq \epsilon/8;$$

and therefore, $p(\mu(B)) \leq \epsilon/8$ for every $B \in \mathscr{B}$ with $B \subset A_1 - H_1$.

Moreover, there exists $H_2 \in \mathscr{A}$ such that $H_2 \subset A_2 \cap H_1$ and $p(\lambda(H) - \lambda(H_2)) \leq \epsilon/16$ if $H \in \mathscr{A}$ and $H_2 \subset H \subset A_2 \cap H_1$. Consequently, if $H \in \mathscr{A}$ and $H_2 \subset H \subset A_2$ we have

$p(\lambda(H) - \lambda(H_2)) \leq p(\lambda(H) - \lambda(H \cap H_1)) + p(\lambda(H \cap H_1) - \lambda(H_2)) \leq$

$\leq p(\lambda(H) - \lambda(H \cap H_1)) + \epsilon/16 = p(\mu(H - H_1)) + \epsilon/16 \leq$

$\leq \epsilon/8\,(1 + 1/2)$.

So, if $H \in \mathscr{A}$ and $H \subset A_2 - H_2$ then

$p(\lambda(H)) = p(\lambda(H \cup H_2) - \lambda(H_2)) \leq \epsilon/8\ (1 + 1/2)$;

and $p(\mu(B)) \leq \epsilon/8\ (1 + 1/2)$ for every $B \in \mathscr{B}$ with $B \subset A_2 - H_2$. In particular,

$$p(\mu(A_2) - \lambda(H_2)) = p(\mu(A_2 - H_2)) < \epsilon/4.$$

Proceeding in this way, we construct a sequence $(H_n)_{n \in N} \subset \mathscr{A}$ such that $H_n \subset A_n \cap H_{n-1}$ for $n \geq 2$ and

$$p(\mu(B)) \leq \epsilon/8\left(\sum_{j=0}^{n-1} 1/2^{j}\right) \qquad (<\epsilon/4)$$

for every $B \in \mathscr{B}$ such that $B \subset A_n - H_n$ (and in particular,

$$p(\mu(A_n) - \lambda(H_n)) = p(\mu(A_n - H_n)) < \epsilon/4) \qquad (n \in N^*).$$

Since $(H_n)_{n \in N}$ is a decreasing sequence, we get

$$\lambda(\bigcap_{n \in N} H_n) = \lim_{n \to +\infty} \lambda(H_n),$$

and there exists $n_o \in N$ such that

$$p(\lambda(H_m) - \lambda(\bigcap_{n \in N} H_n)) \leq \epsilon/6$$

for every $m \geq n_o$.

Moreover, there exists $H \in \mathscr{A}$ such that $\bigcap_{n \in N} H_n \subset H \subset \bigcap_{n \in N} A_n$ and

$$p(\lambda(H) - \mu(\bigcap_{n \in N} A_n)) \leq \epsilon/6;$$

the sequence $(H \cap H_n)_{n \in N} \subset \mathscr{A}$ is decreasing, and there exists $n_1 \in N$ such that

$$p(\lambda(H \cap H_m) - \lambda(\bigcap_{n \in N} H_n)) \leq \epsilon/6$$

for all $n \geq n_1$.

So, if $m \geq \max(n_0, n_1)$, then

$$p(\mu(A_m) - \mu(\bigcap_{n \in N} A_n)) \leq$$
$$\leq p(\mu(A_m) - \lambda(H_m)) + p(\lambda(H_m) - \lambda(\bigcap_{n \in N} H_n)) +$$
$$+ p(\lambda(\bigcap_{n \in N} H_n) - \lambda(H \cap H_m)) + p(\lambda(H \cap H_m) - \lambda(H)) +$$
$$+ p(\lambda(H) - \mu(\bigcap_{n \in N} A_n)) < \epsilon;$$

and [5.3] holds.

viii) As it is easily proved, the set

$$\mathscr{S} = \{A \in \mathscr{B} : \mu(B) = \mu(B \cap A) + \mu(B - A) \text{ for all } B \in \mathscr{B}\}$$

is an algebra.

ix) If $(A_n)_{n\in N}\subset \mathscr{L}$ is an increasing sequence, then $\bigcup_{n\in N} A_n \in \mathscr{L}$ and

$$\mu(B\cap(\bigcup_{n\in N} A_n))=\lim_{n\to+\infty} \mu(B\cap A_n)$$

holds for every $B\in\mathscr{B}$.

In fact, if $(A_n)_{n\in N}\subset \mathscr{L}$ is an increasing sequence and $B\in\mathscr{B}$,

then $\bigcap_{m\in N}((B\cap(\bigcup_{n\in N} A_n))-A_m)=\emptyset$ and

$0=\lim_m \mu(B\cap(\bigcup_{n\in N} A_n))-A_m)=$

$=\mu(B\cap(\bigcup_{n\in N} A_n))-\lim_m \mu(B\cap A_m)$;

and therefore,

$\mu(B\cap(\bigcup_{n\in N} A_n))+\mu(B-\bigcup_{n\in N} A_n)=$

$=\lim_m \mu(B\cap A_m)+\lim_m \mu(B-A_m)=\mu(B)$,

and $\bigcup_{n\in N} A_n\in \mathscr{L}$.

x) Evidently, μ is a finitely additive vector measure and $\mathscr{L}$ is a σ-algebra.

xi) μ is a Radon measure of type $(\mathscr{H})$.

Let $A\in\mathscr{B}$, $p\in\mathscr{P}$ and $\epsilon>0$; then there exists $H\in\mathscr{H}$ such that $H\subset A$ and

$$p(\lambda(F)-\lambda(H))\leq\epsilon/2$$

if $F\in\mathscr{H}$ is such that $H\subset F\subset A$. Thus, if $F\in\mathscr{H}$ is such that $F\subset A-H$ then

$$p(\lambda(F))=p(\lambda(F\cup H)-\lambda(H))\leq\epsilon/2,$$

and $p(\mu(B))\leq\epsilon/2$ for every $B\in\mathscr{B}$ with $B\subset A-H$. Therefore,

$||\mu||_p(A-H)\leq 2\sup\{p(\mu(B)): B\in\mathscr{B},\ B\subset A-H\}\leq\epsilon$.

Moreover, every $H\in\mathscr{H}$ is μ-compact, since for every open cover $\{G_j\}_{j\in J}$ of H, $p\in\mathscr{P}$ and $\epsilon>0$, we get from condition 5.1 the existence of a finite subset $J'\subset J$ and $i_o\in I$ with $i_o\geq i_{H_1}$, with $H_1=H-\bigcup_{j\in J'} G_j$, such that $||\mu_i||_p(\Pi_i(H-\bigcup_{j\in J'} G_j))\leq\epsilon/2$ for every $i\geq i_o$.

So, if $H'\in\mathscr{H}$ and $H'\subset H-\bigcup_{j\in J'} G_j$, then

$$p(\mu(H'))=\lim_{i\geq i_o, i_H} p(\mu_i(\Pi_i(H')))\leq \sup_{i\geq i_o, i_H} ||\mu_i||_p(\Pi_i(H'))\leq \epsilon/2;$$

and

$$||\mu||_p(H-\bigcup_{j\epsilon J'} G_j)\leq 2 \sup\{p(\mu(H')): H'\epsilon \mathscr{H}, H'\subset H-\bigcup_{j\epsilon J'} G_j\}\leq \epsilon.$$

xii) It follows from the last results that μ is σ-additive.

xiii)

$$\mu(\Pi_i^{-1}(H_i))=\mu_i(H_i) \qquad [5.4]$$

for every $i\epsilon I$ and $H_i \epsilon \mathscr{H}_i$.

In fact, it follows from the condition 5.2 that for every $i\epsilon I$, $H_i\epsilon \mathscr{H}_i$, $p\epsilon \mathscr{P}$ and $\epsilon>0$, there exists $H\epsilon \mathscr{H}$ such that $H\subset \Pi_i^{-1}(H_i)$ and

$$||\mu_j||_p(\Pi_{ij}^{-1}(H_i)-\Pi_j(H))\leq \epsilon/2$$

for every $j\geq i$, i_H. Moreover, there exists $H'\epsilon \mathscr{H}$ such that $H\subset H'\subset \Pi_i^{-1}(H_i)$ and

$$p(\mu(\Pi_i^{-1}(H_i))-\lambda(H'))\leq \epsilon/2.$$

Therefore,

$$p(\mu(\Pi_i^{-1}(H_i))-\mu_i(H_i))\leq$$

$$\leq p(\mu(\Pi_i^{-1}(H_i))-\lambda(H'))+p(\lambda(H')-\mu_i(H_i))\leq$$

$$\leq \epsilon/2+\lim_{j\geq i, i_H, i_{H'}} p(\mu_j(\Pi_j(H'))-\mu_i(H_i))=$$

$$=\epsilon/2+\lim_{j\geq i, i_H, i_{H'}} p(\mu_j(\Pi_j(H'))-\mu_j(\Pi_{ij}^{-1}(H_i)))\leq$$

$$\leq \epsilon/2+\sup_{j\geq i, i_H, i_{H'}} ||\mu_j||_p(\Pi_{ij}^{-1}(H_i)-\Pi_j(H'))\leq$$

$$\leq \epsilon/2+\sup_{j\geq i, i_H} ||\mu_j||_p(\Pi_{ij}^{-1}(H_i)-\Pi_j(H))\leq \epsilon;$$

and $\mu_i(H_i)=\mu(\Pi_i^{-1}(H_i))$.

xiv) Let us prove that μ is the projective limit of $(\mu_i)_{i\epsilon I}$.

For this, we consider the family

$$\mathscr{S}_i=\{A_i\epsilon \mathscr{B}_i: \mu_i(A_i)=\mu(\Pi_i^{-1}(A_i))\} \qquad (i\epsilon I).$$

Then, if F_i is a closed subset of E_i, then for every $p \in \mathscr{P}$ and $\epsilon > 0$, there exists $H_i \in \mathscr{A}_i$ and $H \in \mathscr{A}$ such that $H_i \subset F_i$, $H \subset \Pi_i^{-1}(F_i)$, $||\mu_i||_p(F_i - H_i) \leq \epsilon/2$ and $||\mu||_p(\Pi_i^{-1}(F_i) - H) \leq \epsilon/2$. Therefore,

$$p(\mu_i(F_i) - \mu(\Pi_i^{-1}(F_i))) \leq$$

$$\leq p(\mu_i(F_i) - \mu_i(H_i \cup \overline{\Pi_i(H)})) + p(\mu_i(H_i \cup \overline{\Pi_i(H)}) - \mu(\Pi_i^{-1}(F_i))) \leq$$

$$\leq ||\mu_i||_p(F_i - (H_i \cup \overline{\Pi_i(H)})) + ||\mu||_p(\Pi_i^{-1}(F_i) - H) \leq \epsilon;$$

and $F_i \in \mathscr{S}_i$.

Moreover, if A_i, $B_i \in \mathscr{S}_i$ and $A_i \subset B_i$ then $B_i - A_i \in \mathscr{S}_i$; and $\bigcup_{n \in N} A_n^i \in \mathscr{S}_i$ for every increasing sequence $(A_n^i)_{n \in N} \subset \mathscr{S}_i$.

Now it follows immediatley that $\mathscr{S}_i = \mathscr{B}_i$, and that $\mu_i(A_i) = \mu(\Pi_i^{-1}(A_i))$ for every $i \in I$ and $A_i \in \mathscr{B}_i$, as we wanted to prove.

This ends the proof of Theorem 5.

Let us consider now a Hausdorff and complete locally convex space X which is the projective limit of a projective system $(X_i, f_{ij})_{i,j \in I}$ of Hausdorff and complete locally convex spaces and denote by $\mathscr{P}_i$ a generating and saturated family of seminorms on X_i (we will assume that if $i, j \in I$, $p_i \in \mathscr{P}_i$ and $i \leq j$, then $p_i f_{ij} \in \mathscr{P}_j$), and by $f_i : X \longrightarrow X_i (i \in I)$ the canonical projection.

As we have made before, let $((E_i, \mathscr{G}_i), \Pi_{ij})_{i,j \in I}$ be a projective system of topological spaces, with $\mathscr{B}_i$ the Borel σ-algebra of E_i, $\mathscr{A}_i$ a family of closed subsets of E_i, which is closed under finite unions, and $\mu_i : \mathscr{B}_i \longrightarrow X_i$ a Radon measure of type $(\mathscr{A}_i)$.

Let $E = \varprojlim E_i$ and denote by $\mathscr{B}$ the Borel σ-algebra of E and by $\Pi_i : E \longrightarrow E_i (i \in I)$ the canonical projection.

We will assume that

$$\mu_i(A_i) = f_{ij}(\mu_j(\Pi_{ij}^{-1}(A_i)))$$

for every $A_i \in \mathscr{B}_i$, $i, j \in I$ with $i \leq j$.

Definition 6. *We say that a measure $\mu : \mathscr{B} \longrightarrow X$ is the projective limit (it is easily proved that if the projective limit measure exists then it is unique) of the last system of measures $(\mu_i)_{i \in I}$ if*

$$f_i(\mu(\Pi_i^{-1}(A_i))) = \mu_i(A_i)$$

for every $A_i \in \mathscr{B}_i$ and $i \in I$.

Let us assume that $\mathscr{A}$ is a class of closed subsets of E which verifies the condition 3.1, and that for every $i \in I$ and every $p_i \in \mathscr{P}_i$ t' re exists a non negative and finite measure

$\nu_i^{p_i}: \mathscr{B}_i \longrightarrow \mathbb{R}^+$ such that:

a) The equality

$\nu_i^{p_i}(A_i) = \inf\{\nu_i^{p_i}(G): A_i \subset G,\ G$ open subset of $E_i\}$ holds for every $i \in I$, $p_i \in \mathscr{P}_i$ and $A_i \in \mathscr{B}_i$.

b) For every $i \in I$, $p_i \in \mathscr{P}_i$ and $\epsilon > 0$ there exists $i_o \geq i$ and $\eta > 0$ such that if $j \in I$ and $A_j \in \mathscr{B}_j$ are such that $i_o \leq j$ and $\nu_j^{p_i f_{ij}}(A_j) \leq \eta$, then

$$||\mu_j||_{p_i f_{ij}}(A_j) < \epsilon.$$

c) $\Pi_{ij}(\nu_j^{p_i f_{ij}}) = \nu_i^{p_i}$ holds for every $i, j \in I$ with $i \leq j$ and every seminorm $p_i \in \mathscr{P}_i$.

Then proceeding like in last proofs, the following results are obtained:

Proposition 7. *For every $H \in \mathscr{A}$ and $i \in I$, $(f_{ij}(\mu_j \Pi_j(H)))_{j \geq i_H}$ is a convergent net in X_i; and the mapping $\lambda: \mathscr{A} \longrightarrow X$ such that*

$$\lambda(H) = \varprojlim_{i} (\lim_{j \geq i_H} f_{ij}(\mu_j \Pi_j(H))) \qquad [7.1]$$

is well defined.

Theorem 8. *The projective limit μ of the (last) projective system of measures $(\mu_i)_{i \in I}$ exits and $\mu: \mathscr{B} \longrightarrow X$ is a Radon measure of type ($\mathscr{A}$), if and only if the following conditions are fulfilled:*

8.1. For every $H \in \mathscr{A}$, $i \in I$, $p_i \in \mathscr{P}_i$, $\epsilon > 0$ and every open cover $\{G_r\}_{r \in L}$ of H, there exists a finite subset $L' \subset L$ and $i_o \in I$ such that $i_o \geq i$, i_{H_1}, with $H_1 = H - \bigcup_{r \in L'} G_r$, and

$$||\mu_j||_{p_i f_{ij}}(\Pi_j(H - \bigcup_{r \in L'} G_r)) \leq \epsilon$$

holds for every $j \geq i_o$.

8.2. For every $i \in I$, $p_i \in \mathscr{P}_i$, $H_i \in \mathscr{A}_i$ and $\epsilon > 0$, there exists $H \in \mathscr{A}$ such that $H \subset \Pi_i^{-1}(H_i)$ and

$$||\mu_j||_{p_i f_{ij}}(\Pi_{ij}^{-1}(H_i) - \Pi_j(H)) \leq \epsilon$$

for every $j \geq i$, i_H.

Moreover, if $\mu=\varprojlim \mu_i$ exists and is a Radon measure of type $(\mathscr{U})$, then it is unique and $\mu(H)=\lambda(H)$ for every $H\in\mathscr{U}$, λ being the set function defined in Proposition 7. Also, the measure μ is of bounded semivariation if and only if the semivariations $\{\,||\mu_j||_{p_i f_{ij}} : j\geq i\}$ are uniformly bounded for every $i\in I$ and every $p_i\in\mathscr{P}_i$.

3. ON THE PROJECTIVE LIMIT OF PRODUCT MEASURES

Let us continue with the notations of last section and consider two projective systems of Hausdorff and complete locally convex spaces $(Y_i, g_{ij})_{i,j\in I}$ and $(Z_i, h_{ij})_{i,j\in I}$, with Q_i (resp., $\mathscr{R}_i$) a generating (and saturated) family of seminorms on Y_i (resp. on Z_i) $(i\in I)$, and suppose that $q_i g_{ij}\in Q_j$ and $r_i h_{ij}\in\mathscr{R}_j$ for every pair of seminorms $q_i\in Q_i$, $r_i\in\mathscr{R}_i$ and every $i,j\in I$ with $i\leq j$. We shall write $Y=\varprojlim Y_i$, $Z=\varprojlim Z_i$, $\mathscr{P}=\{p_i f_i : p_i\in\mathscr{P}_i,\ i\in I\}$, $Q=\{q_i g_i : q_i\in Q_i,\ i\in I\}$ and $\mathscr{R}=\{r_i h_i : r_i\in\mathscr{R}_i,\ i\in I\}$ ($g_i : Y\longrightarrow Y_i$ and $h_i : Z\longrightarrow Z_i$, $i\in I$, will be the natural projections as usual).

Let us consider another projective system of Radon measures of type $(\mathscr{F}_i)$, $((F_i, \mathscr{G}_i'), \Pi_{ij}', \Pi_i', \nu_i)_{i,j\in I}$, where $\mathscr{F}_i$ is a family of closed subsets of F_i, closed under finite unions, $\mathscr{B}_i'$ is the Borel σ-algebra of F_i and $\nu_i : \mathscr{B}_i'\longrightarrow Y_i$ is a Radon measure of type $(\mathscr{F}_i)$, $i\in I$ (see definition 3).

Let $\mathscr{B}'$ be the Borel σ-algebra of F, where $F=\varprojlim F_i$; we shall use the following notation: $\Pi_i' : F\longrightarrow F_i$ is the natural projection $(i\in I)$, $\hat{\Pi}_{ij}=(\Pi_{ij}, \Pi_{ij}')$, $l_{ij}=(f_{ij}, g_{ij})$, $\hat{\Pi}_i=(\Pi_i, \Pi_i')$ and $l_i=(f_i, g_i)$ for all $i,j\in I$ with $i\leq j$.

Suppose that, for every $i\in I$, there exists a bilinear and continuous mapping $\delta_i : X_i\times Y_i\longrightarrow Z_i$ such that the following diagram is commutative for $i\leq j$ $(i,j\in I)$:

$$\begin{array}{ccc} X_j\times Y_j & \xrightarrow{\delta_j} & Z_j \\ {\scriptstyle l_{ij}}\big\downarrow & & \big\downarrow{\scriptstyle h_{ij}} \\ X_i\times Y_i & \xrightarrow[\delta_i]{} & Z_i \end{array}$$

and let $\delta : X\times Y\longrightarrow Z$ be the function $(\delta_i)_{i\in I}$.

Theorem 9. *If the projective limits μ and ν of the systems of measures $(\mu_i)_{i\in I}$ and $(\nu_i)_{i\in I}$ ($\mu=\varprojlim \mu_i$, $\nu=\varprojlim \nu_i$) and the product measures $\mu_i\otimes\nu_i$ $(i\in I)$*

exist, then the product measure $\mu \otimes \nu$ exists if and only if the projective limit measure ($\varprojlim \mu_i \otimes \nu_i$) of the system of measures $(\mu_i \otimes \nu_i)_{i \in I}$ exists, and in this case they coincide.

Proof. Let us suppose that the measure $\gamma = \varprojlim \mu_i \otimes \nu_i$ exists; then it is easily proved that for every $A_i \in \mathscr{B}_i$ and $i \in I$ the equality

$$\gamma(\Pi_i^{-1}(A_i) \times B) = \delta(\mu(\Pi_i^{-1}(A_i)), \nu(B))$$

holds for every $B \in \mathscr{B}'$, from where it is deduced that

$$\gamma(A \times B) = \delta(\mu(A), \nu(B))$$

is verified for every $A \in \mathscr{B}$ and every $B \in \mathscr{B}'$ and consequently $\gamma = \mu \otimes \nu$.

The other implication is trivial.

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